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ELECTRONIC ANALOG COMPUTERS

ELECTRONIC ANALOG COMPUTERS

(D-c Analog Computers)

by

GRANINO A. KORN, Ph.D.

and

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Industrial Consultants

SECOND EDITION

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ELECTRONIC ANALOG COMPUTERS

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To the memory of Dean Roland George Dwight Richardson, who has contributed so much to the development of applied mathematics in the United States, this book is respectfully dedicated.

PREFACE

D-c analog computers are relatively simple electronic devices now commonly in use to solve a variety of problems in applied mathematics and engineering design. The application of such computers to the simulation of modern automatic control systems has been particularly successful; but some acquaintance with d-c analog techniques can benefit almost any engineer or research worker, no matter what his special field may be. *A d-c analog representation of a problem does not merely furnish needed numerical data. It often serves as a working model which helps to close the gap between physical intuition and exact analysis.*

The continuing remarkable progress in the related fields of electronic computers, instruments, and controls has led naturally to the preparation of a new, completely revised edition of *Electronic Analog Computers.* The greater part of the book has been entirely rewritten. We have attempted to carry out our original purposes, namely

1. To acquaint research and development workers with tried methods for the application of d-c analog computers as computing aids and simulators, and with the possibilities and limitations of such equipment;
2. To present a comprehensive body of up-to-date design information on computer components and systems.

We believe that such information is of particular interest to scientists and engineers engaged in the development of instruments and industrial control devices.

This volume is intended as a textbook with supplementary reference material and should be useful for self-instruction. An effort has been made to keep the text technically simple. No mathematics beyond that taught in an elementary course on differential equations is required. The large amount of material which had to be covered has made it impossible to add a chapter on elementary electronics; the reader is, instead, referred to good texts on this subject. We have attempted to incorporate sufficient circuit information, including many examples of actual computer circuits, to enable a reasonably qualified electronic engineer to design d-c analog-computing equipment.

Chapter 1 is an introduction to d-c analog computers. The reader will find the following technical chapters more useful if he is first given an opportunity to see the picture as a whole. Accordingly, Chapter 1 briefly treats the applications of computing devices, the mathematical relations involved, and the techniques used, with the objective of giving an over-all view of the subject. Both the classical differential-analyzer technique and direct-analog operational amplifier methods are introduced.

Chapter 2 attempts to teach a simple, standard procedure for setting up a d-c analog computer to solve a given set of differential equations. Emphasis is placed on a methodical approach to computer scale factors; every effort has been made to present this subject in detail. The procedure described has proved useful in several years of practical work. A section describing alternative methods for handling scale factors has been added, and the material on equation solvers has been transferred to this chapter.

Chapter 3 presents computer setups for a number of practical problems which serve as illustrations of modern industrial d-c analog-computer applications. Revised sections dealing with the simulation of dynamical systems include much new material on the representation of constraints and limit stops. The treatment of servo analysis and synthesis has been broadened to include applications of direct-analog operational amplifiers, new diode circuits to simulate nonlinearities, and statistical design criteria. Entirely new sections describe the application of d-c analog computers to automobile-suspension systems, to industrial process control, and to a problem in linear programming. Another new section discusses the d-c analog-computer solution of partial differential equations. The preparation of useful records of each computation is particularly stressed. It is hoped that the samples of problem-preparation records shown will be useful for reference.

Chapters 4 to 6 discuss the design of d-c analog computing elements according to the best modern engineering practice. Specifically, Chapter 4 describes the design of the networks and feedback circuits used to perform the mathematical operations of addition, multiplication by constant coefficients, differentiation, and integration. Two new sections discuss direct-analog operational amplifiers and the error analysis of linear differential-equation solvers.

Chapter 5 deals with the design and operation of modern d-c amplifier circuits for computer applications. A discussion of the stabilization of high-gain feedback amplifiers is included. Automatic balancing circuits are described, and the actual circuit diagrams of the latest d-c amplifiers employed in modern d-c analog computers are presented. Three revised sections introduce commutator stabilization and photoelectric choppers, and dwell briefly on transistor d-c amplifiers.

Chapter 6 discusses analog multiplication and function generation and describes the remarkable strides made in this field in recent years. Included are descriptions of new time-division multipliers, AM-FM multipliers, amplitude selection multipliers, thermal-transfer multipliers, and a changed treatment of potentiometer-loading errors. Diode function generators, including "fabricated" diode circuits, are treated in special detail. A new section deals with the generation of functions of two or more variables.

Chapter 7 is concerned with the design of the auxiliary components, such as power supplies and recorders; three complete sections are devoted to a discussion of improved control circuits for d-c analog computers. The new material includes a description of modern automatic control features, such as keyboard-set potentiometers, digital readout, automatic programming, and automatic problem-check circuits.

Chapter 8 again deals with complete d-c analog-computer installations from a systems point of view and has been brought up to date to review the design philosophy of typical modern computers. Descriptions of several modern d-c analog-computer installations are presented in the light of the design principles introduced in the earlier chapters.

A considerable amount of reference material has been added in the new edition. Six reference tables present various "trick" computer setups. Many new illustrations are arranged in groups which serve as additional reference tables and permit the reader to compare related circuits and computer setups. An enlarged bibliography refers the reader to over 260 original articles and reports.

The Revised Second Edition of *Electronic Analog Computers* is based on a careful survey of both manufacturers and users of electronic analog computers for up-to-date design and application data. We are deeply indebted to our friends in the computer industry for their cooperation. In a very real sense, this Revised Second Edition is the result of *their* design experience and *their* application know-how. We are particularly grateful to the following individuals and organizations for contributing reports, circuits, illustrations, and many valuable suggestions:

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Borg Equipment Division The George W. Borg Corporation Janesville, Wisc.	Mr. N. W. Alexander

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CHAPTER 1

INTRODUCTION TO D-C ANALOG COMPUTERS

1.1. INTRODUCTION

An *analog computer* represents the variables of a given problem by corresponding physical quantities, such as continuously variable voltages or shaft rotations. These *machine variables* are made to obey mathematical relations analogous to those of the original problem. The desired relations are established by a set of *computing elements* capable of enforcing suitable physical relationships. The conveniently measurable machine variables will, then, vary so that records of their values or behavior constitute solutions of the given problem.

Analog computers may be classified according to the nature of the physical quantities used to represent mathematical variables and according to the computing elements used. Slide rules, network calculators, and Bush differential analyzers are, respectively, examples of mechanical, electrical, and electromechanical analog computers.

D-c Analog Computers. In d-c analog computers, the variables are represented by d-c voltages which may vary with time; time is generally used as the independent variable in problems involving differential equations. The principal computing elements are d-c amplifiers with resistive and capacitive feedback networks. In multipurpose computers, such computing elements permit flexible electrical interconnections by means of patch cords. Parameter values and coefficients can usually be changed conveniently by means of adjustable potentiometers or rheostats.

D-c analog computers can solve many complicated problems by means of essentially simple electrical circuits and require relatively little maintenance. The electronic computing elements are usually constructed from readily available components by ordinary radio assembly techniques. A small d-c analog computer can be built with precision resistors and chrome-plated dials to occupy a place of honor in a laboratory or industrial control center, or it may be wired up from scrap parts on a breadboard to furnish quick answers with fair accuracy. Larger d-c analog computers range from *multipurpose electronic differential analyzers* (Figs. 1.1 and 1.2) to *specialized calculators and control devices,* such as

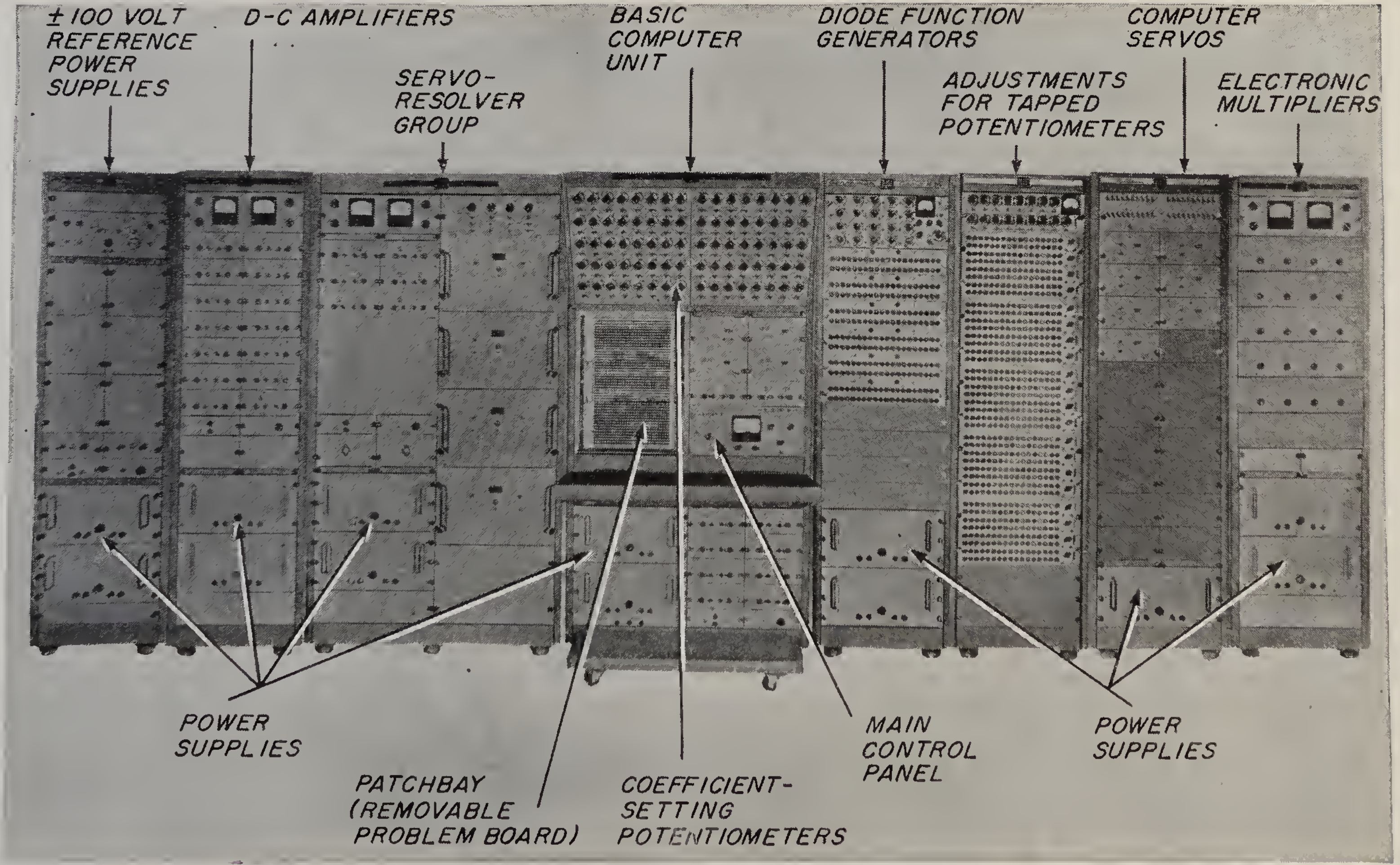
±100 VOLT REFERENCE POWER SUPPLIES
D-C AMPLIFIERS
SERVO-RESOLVER GROUP
BASIC COMPUTER UNIT
DIODE FUNCTION GENERATORS
ADJUSTMENTS FOR TAPPED POTENTIOMETERS
COMPUTER SERVOS
ELECTRONIC MULTIPLIERS
POWER SUPPLIES
PATCHBAY (REMOVABLE PROBLEM BOARD)
COEFFICIENT-SETTING POTENTIOMETERS
MAIN CONTROL PANEL
POWER SUPPLIES

Fig. 1.1. A practical and flexible d-c analog-computer installation for an industrial research department (see also Sec. 8.4). The main console (center) is a complete basic computer unit comprising a control panel, a patch bay, 20 d-c amplifiers, 80 coefficient-setting potentiometers, 4 computer servos, and the associated power supplies. The photograph shows how such a basic computer can be expanded through the addition of computing elements and power supplies. All units connect into the central patch bay. No plotting equipment is shown (Electronic Associates, Inc.).

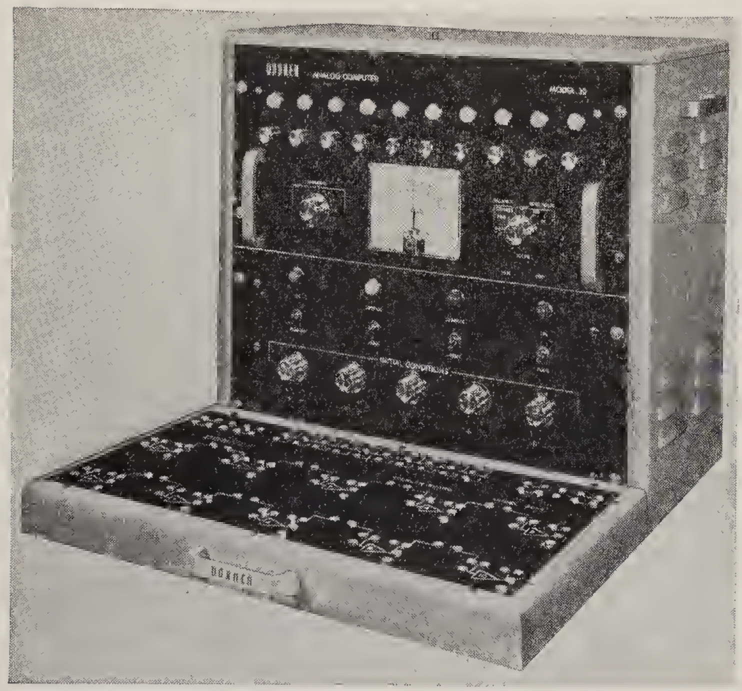

Fig. 1.2*a*. A complete table-top d-c analog computer priced just below $1,000 (see also Sec. 8.4). The unit comprises ten operational amplifiers, power supplies, all necessary controls, and a removable problem board. Multipliers and function generators are available as accessories (Donner Scientific Co.).

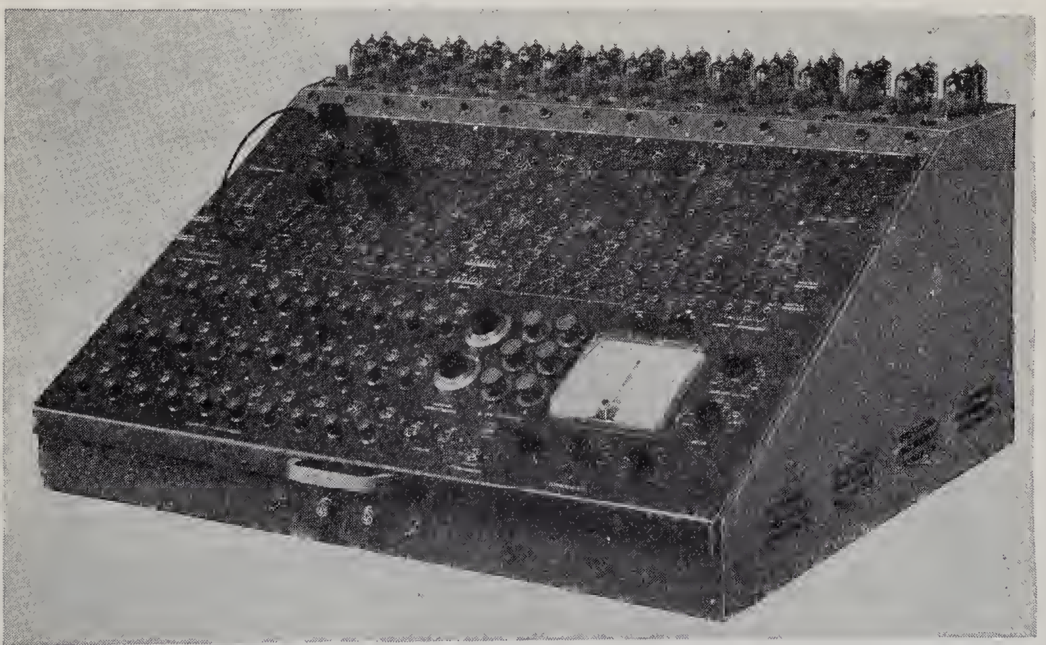

Fig. 1.2*b*. This complete 15-amplifier machine is available as a do-it-yourself kit for less than $1,000. The amplifier tubes are top-mounted to minimize heating of components. The 30 coefficient-setting potentiometers are set by comparison with a built-in precision voltage divider which is also available for gain measurements (Heath Co.).

the fire-control computers which defeated the German V-1 missiles over Antwerp.

Operation of "Slow" D-c Analog Computers. D-c analog computers are particularly useful as differential-equation solvers in many scientific and engineering applications requiring slide-rule accuracy. The solution of a set of differential equations—or the computer representation of a physical process described in terms of differential equations—proceeds as follows:

1. With the machine connected up to solve the given problem, the machine variables (voltages) are set to the correct initial conditions prescribed by the problem.
2. The computing elements are then made operative and force the voltages in the machine to vary in a manner prescribed by the given differential equations. The voltage variations with time are recorded and constitute solutions of the given problem.
3. The machine is stopped at a time chosen by the operator. The maximum allowable computing time is usually determined by limitations of the computing elements. The machine is at once reset to its initial condition and is ready for the next run with changed coefficients, initial conditions, etc.

The recorders used for such applications are usually direct-inking recording milliammeters which afford an instantly available, permanent record of the computation.

Repetitive Computers. In an interesting alternative scheme, the steps of the solution outlined above are repeated automatically at a rapid rate (10 to 60 cps) through the action of an electromechanical or electronic switching system. During each cycle, each machine variable varies in the prescribed manner and is then reset to its initial value. The results can now be presented on an ordinary cathode-ray oscillograph. The sweep frequency of the oscillograph must be equal to the repetition rate of the computer.

The rapid operation of such "repetitive" differential analyzers constitutes a striking advantage in some applications. In particular, one may observe the results of varying coefficient settings and initial conditions at once without having to wait for the completion of a computer "run" as in ordinary computers. On the other hand, the necessity for rapid switching and computation on a fast time scale makes the design of accurate computing elements quite difficult. Accordingly, repetitive operation is seldom considered sufficiently reliable for large-scale computations. This ingenious scheme of operation is most useful for quick qualitative estimates of solutions and has been employed very successfully for exploratory runs to avoid unnecessary computations on a larger differential analyzer (see also Sec. 8.5).

An Appraisal of D-c Analog Computers. Unlike digital computers, analog computers employ distinct computing elements for each mathematical operation required to solve a given problem (parallel operation).[1] Parallel operation is an essential reason for the *very high computing speeds* possible with d-c analog computers: most problems, regardless of complication, are solved within two minutes by "slow" d-c analog computers and within fractions of a second by repetitive machines. On the other hand, parallel operation imposes *practical limits on the complexity of problems* which can be solved on analog computers.[2] Also, the cost of analog computers with component accuracies better than 0.1 per cent rises sharply for even small improvements in accuracy. For computations requiring very high accuracy, the use of digital computers is thus indicated. Figure 1.3 shows a qualitative comparison of the cost of analog and digital computers for different accuracies and sizes of installation.

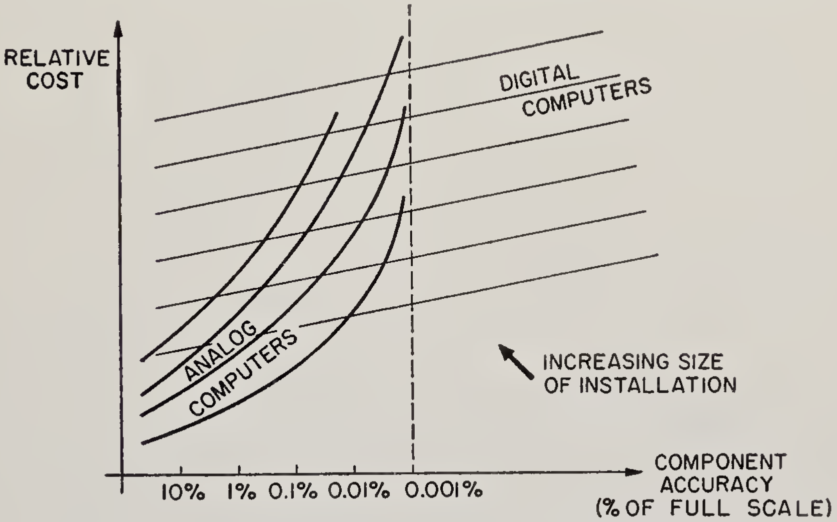

FIG. 1.3. Relative cost vs. accuracy of analog and digital computers for different sizes of installations.

The relatively low cost of some d-c analog computers is of interest in certain applications. Practical and useful low-cost d-c analog differential-equation solvers have been constructed and are particularly well suited for instruction purposes and for industrial organizations wishing to provide individual engineers with desk-top analog computers (Fig. 1.2). In the case of larger, more flexible, and more generally applicable d-c analog-computer installations, the gap between the costs of analog and digital computing equipment has closed; many research and development groups own d-c analog-computer installations whose cost easily rivals that of their large digital computers.

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[1] Rubinoff, M., Analog vs. Digital Computers—A Comparison, *Proc. IRE*, **41**: 1254, 1953.

[2] The definition of this limit is subject to controversy and depends strongly on the nature of the problems in question. Many types of d-c analog-computer setups involving well over a hundred d-c amplifiers work well. The simpler and somewhat less accurate *network-type analog computers* permit high-speed solutions of special extremely complicated problems (simulation of elastic structures) with excellent results (see also Sec. 3.10).

What are the concrete reasons for such investments in comparatively inaccurate computing equipment? In the writers' opinion, d-c analog computers should *not* be regarded as low-cost substitutes for digital computers. Analog computers can only rarely compete with digital machines in the mass production of numerical data required as *end products* of scientific or engineering analyses. In such applications, the high computing speeds of analog equipment are often outweighed by the more advanced automatic programming of digital computers, even if only slide-rule accuracy is required.

The really important contributions of d-c analog computers to modern research and development techniques go beyond mere numerical computations. In many applications, the analog approach functions as a direct aid to a research worker's or engineer's thinking process: an analog-computer setup serves as a *model* which helps to bridge the gap between mathematical symbolism and physical reality. In this sense, analog-computer representation of a process may be useful to both engineers and scientists even in cases where analytical methods apply.

Simple patch-cord connections and potentiometer settings on a d-c analog computer enable the systems designer to create a scale model which permits convenient investigation of the performance and interaction of components and systems. New ideas can be tested at once: blocks of computing elements represent system components and are readily replaced or assembled into different system designs. Design parameters may be changed instantly by corresponding resistance changes, and high computing speeds permit optimization of system performance by successive parameter adjustments.

The time scale of such d-c analog computations may be slowed down or speeded up within wide limits at the convenience of the experimenter. In particular, d-c analog-computing speeds are sufficiently high to permit representation of most dynamical and electromechanical processes on a 1:1 time scale (*simulation*).

With the computer running on a 1:1 time scale, physical components of a system under investigation can actually take the place of a corresponding block of computing elements. The operation of these system components may then be observed in the laboratory (use of simulation for *partial system testing*). Thus a computer can simulate the load of an automatic-control system under test, or the computer may serve as a simulated controller operating an actual motor and load. Such applications often yield very concrete design information: quite frequently, d-c analog-computer simulation actually replaces cumbersome breadboard equipment and permits a design to progress directly to the prototype model stage.

It is only fair to remind the reader that digital computers as well as analog computers can be set up as mathematical models representing physical processes. The inherent flexibility of digital-computer programming makes such machines particularly suitable for the representation of discrete random processes and problems involving logical decisions. The writers believe that such applications of digital computers will become even more useful as computing speeds and memory capacities increase. On the other hand, the reformulation of problems for digital-computer programming is difficult for many present-day engineers; as a rule, the services of applied mathematicians are required. A significant reason for the remarkable successes of d-c analog computers is their ready acceptance by engineering personnel.

Experience has shown that personnel with routine training in engineering or science will learn the effective use of d-c analog-computing equipment quite rapidly. As with digital computers, the time required for problem preparation will depend on the problem and on the programmer's experience with similar computer setups.

1.2. HOW AN ELECTRONIC DIFFERENTIAL ANALYZER REPRESENTS MATHEMATICAL RELATIONS

The problems solved by analog computers usually deal with the behavior of a number of variables $x_1, x_2, \ldots, x_n$. One such problem may be to find the values of these variables under certain conditions by solving a set of equations. Another problem may be to determine how these variables change with time, or with another independent variable t, under given conditions. The particular conditions imposed by each problem are usually given as a set of mathematical relations or equations.

The variables $x_1, x_2, \ldots, x_n$ are represented in the d-c analog computers by corresponding voltages, or *machine variables*, $X_1, X_2, \ldots, X_n$. In general, these machine variables are simply proportional to the corresponding original variables on a convenient scale. The given relations between the original variables are now expressed by an analogous set of relations between the machine variables. A straightforward, generally applicable method of obtaining convenient scale factors and the corresponding correct machine equations is discussed in detail in Chap. 2.

Block Representation. Each problem will, then, require certain relations between machine variables to be set up in the computer. Consider first some examples of relations between *two* machine variables:

$$\left.\begin{aligned} X_o &= X_1 + 15X_1^2 \\ X_o &= \sin X_1 \end{aligned}\right\} \tag{1.1}$$

In each case, X_o and X_1 are voltages; and the voltage X_o is a *function* of the voltage X_1. If X_1 changes, X_o must also change, in accordance with the given relation.

Figure 1.4*a* shows schematically how such relations are represented in a d-c analog computer. The voltage X_1 will appear as the *input voltage* of a *block of computing elements* which produces X_o as an *output voltage*. This output voltage will be the desired function of the input voltage X_1 if the correct combination of computing elements has been used. Note that, in general, the relation produced by the block in Fig. 1.4*a* is "unilateral": whereas changes in the input voltage X_1 will produce the correct changes in the output voltage X_o, the converse is not true unless additional connections and computing elements are used.

Several Variables. More often than not, a voltage X_o will appear as a function of *several* other machine variables $X_1, X_2, X_3, \ldots$:

$$\left.\begin{aligned} X_o &= X_1 + 19X_2 + \cos X_3 \\ X_o &= X_1X_2^2 \end{aligned}\right\} \tag{1.2}$$

The block of computing elements producing X_o as its output voltage will, then, have more than one input voltage. The effects of the different input voltages $X_1, X_2, \ldots$ are "mixed" to produce the correct functional relation (see Fig. 1.4*b*).

Time Dependence of Variables. In many problems, it is desired to observe the variations of the machine variables as *functions of the time* τ, which plays the part of the independent variable in the computer. For example, the relations

$$X_1 = X_2 \qquad X_2 = \sin \tau \tag{1.3}$$

describe the variations of two machine variables, X_1 and X_2, with the time τ. Physical time τ here serves as an input to certain computing elements which produce voltages as functions of time. Examples of such devices are potentiometers driven by a clock motor or circuits containing charging and discharging capacitors. The machine variables can thus be made to vary with time in the prescribed manner and may be recorded by means of a recording milliammeter or oscillograph.

Calculus. It is easily seen that a majority of problems in applied mathematics require a description of *changes* of quantities (variables) involved in physical processes. The *differential calculus* is the branch of mathematics concerned with the description of such changes. A *time derivative* like $dX/d\tau$ (derivative of the machine variable X) is a quantitative measure of a change dX: the latter is compared with the corresponding change of a known physical quantity, the *time* τ indicated by a clock.

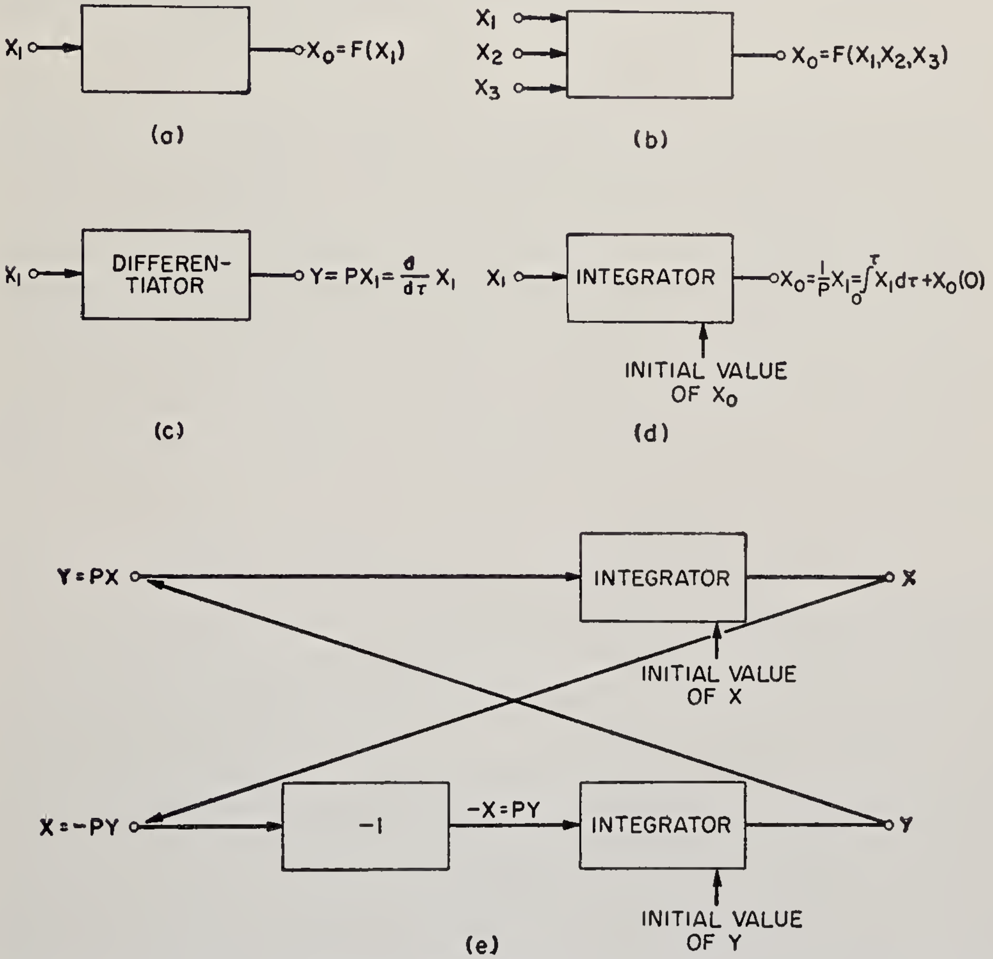

FIG. 1.4. How an electronic differential analyzer works (block representation of relations between machine variables): generation of a function (Fig. 1.4*a*), a function of several variables (Fig. 1.4*b*), differentiation with respect to the time τ (Fig. 1.4*c*), integration with respect to the time τ (Fig. 1.4*d*), and solution of differential equations by implicit computation (Fig. 1.4*e*). As a rule each block shown will require several actual computing elements (see also Figs. 1.5 and 2.1). All computing elements are assumed to act unilaterally; note the feedback connections in Fig. 1.4*e*.

It is practical to write

$$\frac{d}{d\tau} = P \qquad \text{thus} \qquad \frac{dX}{d\tau} = PX \tag{1.4}$$

This notation will be used throughout this book. For example, if $X = \cos \omega\tau$, then

$$PX = P \cos \omega\tau = -\omega \sin \omega\tau \tag{1.5}$$

Derivatives of machine variables, like PX, are themselves functions of time and may change with time. It is thus usually possible to define the derivative of PX. This is called the *second derivative* of X, $d^2X/d\tau^2 = P^2X$. Higher derivatives are defined accordingly.

A computing element whose output voltage X_o is proportional to the time derivative of its input voltage X_1 is called a *differentiator* (Fig. 1.4*c*); an *integrator* produces an output voltage proportional to its input voltage (Fig. 1.4*d*).

Solution of Ordinary Differential Equations: Classical Differential-analyzer Technique. Mathematical equations relating measures of change (derivatives) of this type are called *ordinary differential equations.* In many problems a set of such differential equations is given, and it is desired to find the corresponding time dependence of the variables involved. The set of differential equations is *solved* when all the variables are computed as functions of time. Thus the set of differential equations

$$PX = Y \qquad PY = -X \tag{1.6}$$

is satisfied by the functions

$$X = C \sin (\tau + \varphi) \qquad Y = C \cos (\tau + \varphi) \tag{1.7}$$

where C and φ are arbitrary constants.

Figure 1.4*e* shows how relations of this type may be represented on d-c analog computers. All derivatives of machine variables are represented by voltages in the machine and are, thus, themselves machine variables. These voltages, $PX = Y$ and $PY = -X$, are treated as if they were known and are fed to integrators which produce the machine variables X and Y. *The resulting feedback scheme forces the machine variables* X, Y, PX, *and* PY *to vary as prescribed by the given mathematical relations.* Records of X and Y will, then, constitute a solution of the set of differential equations (1.6). Analog computers capable of solving sets of ordinary differential equations are often referred to as *differential analyzers.*[1] Solutions of differential equations are often needed and may be very laborious if performed by conventional numerical methods. Differential-analyzer setups for many practical problems are described in Chaps. 2 and 3. An alternative method of solving differential equations will be described in Sec. 1.5.

Initial Conditions. In order to obtain definite information from a mathematical relation specifying changes of variables, it is necessary to know the values of the variables at the start of the changes. Because of this simple fact a set of ordinary differential equations cannot be definitely solved unless the *initial conditions* of variables and derivatives corresponding to the start of the computer "run" are known. Specifically, the output voltage of each integrator at the start of the computation must be specified if a definite solution is to be obtained. In the computer setup illustrated by Figs. 1.4*d* and 1.4*e*, it will be necessary to specify the initial conditions for X_0, X, and Y. The output voltages of the corresponding integrators must be set to the correct voltages. This will determine the constants C and φ left arbitrary in Eq. (1.7), and a definite solution will be obtained. Every differential analyzer must contain provisions for setting such initial conditions into the computing elements.

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[1] Caldwell, S. H., and V. Bush, A New Type of Differential Analyzer, *J. Franklin Inst.*, October, 1945, p. 255.

Implicit Computation. The blocks of computing elements shown in Figs. 1.4*a* to 1.4*d* compute X_o *directly* from the given input voltages. The differential-analyzer setup of Fig. 1.4*e* is an example of an *implicit computation:* the unknown input voltages $PX = Y$ and $-PY = X$ are treated as if they were known, and the desired differential equations are enforced by a set of *feedback connections.* Under certain restrictive conditions, similar techniques are useful for enforcing not only differential equations but also relations like

$$X_o = X_2^2 + 2X_o^3 - X_o^7 \tag{1.8}$$

which cannot be solved directly for X_o; note that Eq. (1.8) cannot be represented in the manner of Fig. 1.4*a*. Implicit computation techniques permit the representation of constraints on dynamical systems (Sec. 3.3), the solution of equations (Sec. 2.11), and the computation of inverse functions (Sec. 6.12).

Computing Elements Needed. It will, in general, be desirable to make a computing machine easy to construct and suitable for a wide variety of applications. Accordingly, it is customary to obtain more complicated mathematical operations on the computer voltages (machine variables) through *combinations of a limited number of simple operations, performed by basic computing elements. It has been proved explicitly*[1] *that a very wide range of problems can be solved conveniently by the application of only the following computing elements:*

1. Devices which will multiply a machine variable by a (positive or negative) constant coefficient
2. Devices which will generate the sum of two (or more) machine variables
3. Devices which will generate the product of two machine variables
4. Devices which will generate functions of machine variables
5. Devices which will generate the time integral of a machine variable

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[1] Murray, Francis J., *The Theory of Mathematical Machines*, King's Crown Press, New York, 1947; Shannon, C. E., Mathematical Theory of the Differential Analyzer, *J. Math. and Phys.*, **20**: 337, 1941.

Some of the more commonly used devices capable of performing such operations with d-c voltages will be discussed, one by one, in the following sections; in addition, high-gain amplifiers are often used for implicit computation (Secs. 2.11, 3.3, and 6.12). The detailed design of computing elements will be treated in Chaps. 4, 5, and 6.

1.3. BASIC COMPUTING ELEMENTS: ALGEBRAIC OPERATIONS

Compared with other computing devices, d-c analog computers are often strikingly simple. The reason for this is in the relative ease with which the fundamental operations of multiplication by constants, addition, differentiation, and integration can be performed by utilizing simple basic laws of current flow.

Coefficient-setting Potentiometers. Figure 1.5*a* shows how a simple voltage divider or potentiometer is used to multiply a voltage by a constant positive coefficient $a (0 \leq a \leq 1)$:

$$X_o = aX_1 \tag{1.9}$$

The coefficient setting may be read on the potentiometer dial. The dial scale will be linear unless the potentiometer is loaded appreciably by another computing element connected to its output terminals (Sec. 4.1).

Phase-inverting Amplifiers. *D-c amplifiers with feedback* make it possible to multiply a d-c voltage by a constant, which may be greater than 1. In most practical cases, the sign of the output voltage will be the opposite of that of the input voltage. The feedback amplifier thus multiplies by a *negative* constant; it acts as a *phase inverter* or *sign changer*.

D-c amplifiers with various feedback networks (*operational amplifiers*) are used extensively in d-c analog computers to perform operations on voltages. The transfer functions of such amplifiers are determined mainly by the feedback networks; the amplifier forward gain is usually made as high as possible in order to make the transfer function stable. The amplifiers and their associated power supplies are designed for low drift.[1] All input and output voltages are measured with respect to a ground reference. It is clear that the output voltages of operational amplifiers must not exceed certain maximum absolute values (usually 100 volts) corresponding to the range of linear amplifier operation.

Consider the arrangement of Fig. 1.5*b*. Let A be the amplifier forward

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[1] For an elementary discussion of amplifier terminology, the reader is referred to D. G. Fink, *Engineering Electronics*, McGraw-Hill, New York, 1938. More advanced discussions will be found in Vol. 18 of the MIT Radiation Laboratory Series. Properties desirable in computer amplifiers as well as typical circuits are discussed in Chap. 5 of the present work.

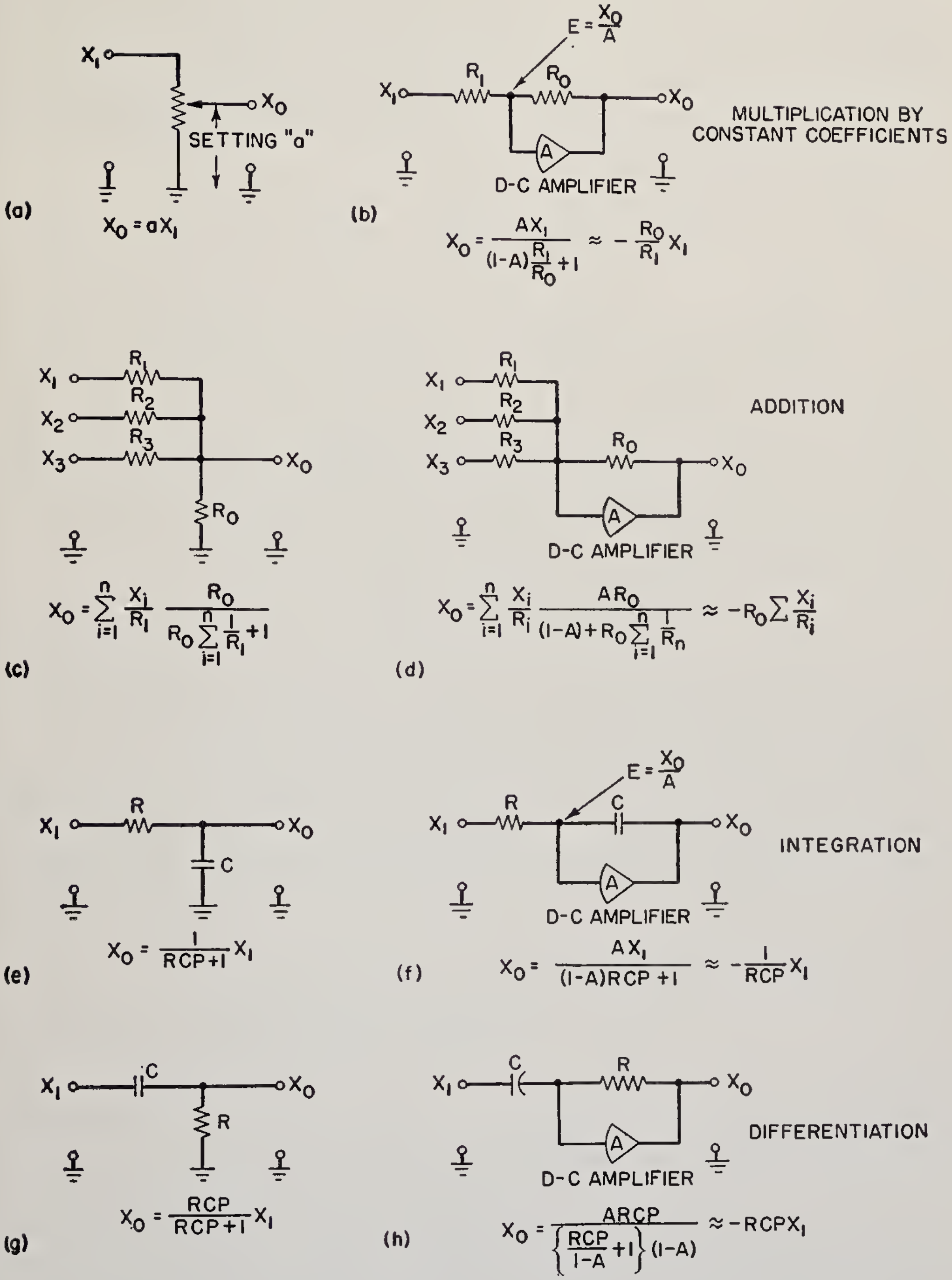
(a)
X1
X0
SETTING "a"
X0 = aX1
(b)
E = X0/A
R1
R0
X1
X0
D-C AMPLIFIER
A
MULTIPLICATION BY CONSTANT COEFFICIENTS
X0 = AX1 / ((1-A)R1/R0 + 1) ≈ -R0/R1 X1
(c)
X1
X2
X3
R1
R2
R3
R0
X0
X0 = Σ_{i=1}^{n} Xi/Ri · R0 / (R0 Σ_{i=1}^{n} 1/Ri + 1)
(d)
ADDITION
D-C AMPLIFIER
X0 = Σ_{i=1}^{n} Xi/Ri · AR0 / ((1-A) + R0 Σ_{i=1}^{n} 1/Rn) ≈ -R0 Σ Xi/Ri
(e)
R
C
X0 = 1/(RCP+1) X1
(f)
E = X0/A
INTEGRATION
D-C AMPLIFIER
X0 = AX1 / ((1-A)RCP + 1) ≈ -1/RCP X1
(g)
X0 = RCP/(RCP+1) X1
(h)
DIFFERENTIATION
D-C AMPLIFIER
X0 = ARCP / ({RCP/(1-A) + 1}(1-A)) ≈ -RCPX1

FIG. 1.5. Summary of simple computing networks and operational amplifiers.

gain which will be large (500 to 300 million) over the range of operating frequencies. The voltage at the amplifier input grid is denoted by E, so that

$$X_o = AE \qquad \text{or} \qquad E = \frac{X_o}{A} \tag{1.10}$$

One might note in passing that in such high-gain amplifiers the input grid is practically at ground potential while the output voltage has a finite value. According to Kirchhoff's first law,[1] the sum of all currents flowing to the grid must be zero. If the grid current is neglected for the present, the "nodal" equation is

$$\left(\frac{X_o}{A} - X_o\right)\frac{1}{R_o} + \left(\frac{X_o}{A} - X_1\right)\frac{1}{R_1} = 0$$

so that

$$X_o = X_1 \frac{A}{(1 - A)\dfrac{R_1}{R_o} + 1} = -\left(1 - \frac{1}{1 - A\beta}\right)\frac{R_o}{R_1} X_1 \tag{1.11}$$

with

$$\beta = \frac{R_1}{R_o + R_1} = \left(1 + \frac{R_o}{R_1}\right)^{-1} \tag{1.12}$$

In most practical cases, the amplifier gain is so high that $|A\beta|$ is very large compared to unity, and Eq. (1.11) reduces to

$$X_o = -\frac{R_o}{R_1} X_1 \tag{1.13}$$

The significance of the quantity $A\beta$ (idealized loop gain) and the validity of the approximation (1.13) are discussed in detail in Sec. 4.2.

Summing Networks. *Summing or averaging networks* of the type shown in Fig. 1.5*c* may be used to obtain a voltage proportional to the sum of several input voltages. In Fig. 1.5*c*, the sum of all currents flowing to the output terminal must be zero, thus

$$(X_o - X_1)\frac{1}{R_1} + (X_o - X_2)\frac{1}{R_2} + \cdots + (X_o - X_n)\frac{1}{R_n} + X_o\frac{1}{R_o} = 0 \tag{1.14}$$

so that

$$X_o = \left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right)\frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \cdots + \dfrac{1}{R_n} + \dfrac{1}{R_o}} \tag{1.15}$$

Such networks may be used for addition, but it is seen from Eq. (1.15) that the performance changes appreciably with the load resistance R_o and that there is considerable attenuation in this circuit.

Summing Amplifiers. The *summing amplifier* circuit of Fig. 1.5*d* permits the addition of several voltages or machine variables as well as multiplication by constants and phase inversion. The voltage at the input grid is, again, equal to X_o/A. The nodal equation for this point becomes

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[1] Smythe, W. R., *Static and Dynamic Electricity*, McGraw-Hill, New York, 1939.

$$\left(\frac{X_o}{A} - X_1\right)\frac{1}{R_1} + \left(\frac{X_o}{A} - X_2\right)\frac{1}{R_2} + \cdots + \left(\frac{X_o}{A} - X_n\right)\frac{1}{R_n} + \left(\frac{X_o}{A} - X_o\right)\frac{1}{R_o} = 0$$

so that

$$X_o = -\left(1 - \frac{1}{1 - A\beta}\right) R_o \left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right) \qquad (1.16)$$

with

$$\beta = \left(1 + \frac{R_o}{R_1} + \frac{R_o}{R_2} + \cdots + \frac{R_o}{R_n}\right)^{-1} \qquad (1.17)$$

In most applications, the amplifier gain is, again, so high that $|A\beta|$ is very large compared to unity, and Eq. (1.16) reduces to

$$X_o = -R_o\left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right) \qquad (1.18)$$

The validity of this approximation is discussed in Sec. 4.2. Equation (1.18) is particularly useful for establishing relations of the form

$$X_o = -(a_1X_1 + a_2X_2 + \cdots + a_nX_n) \qquad (1.19)$$

with

$$a_1 = \frac{R_o}{R_1} \qquad a_2 = \frac{R_o}{R_2} \qquad \cdots \qquad a_n = \frac{R_o}{R_n}$$

A typical summing amplifier employs the arrangement of Fig. 1.5*d* with the resistance values

$$R_o = 1 \text{ megohm}$$
$$R_1 = R_2 = R_3 = R_4 = 1 \text{ megohm}$$
$$R_5 = R_6 = 0.25 \text{ megohm}$$
$$R_7 = 0.1 \text{ megohm}$$

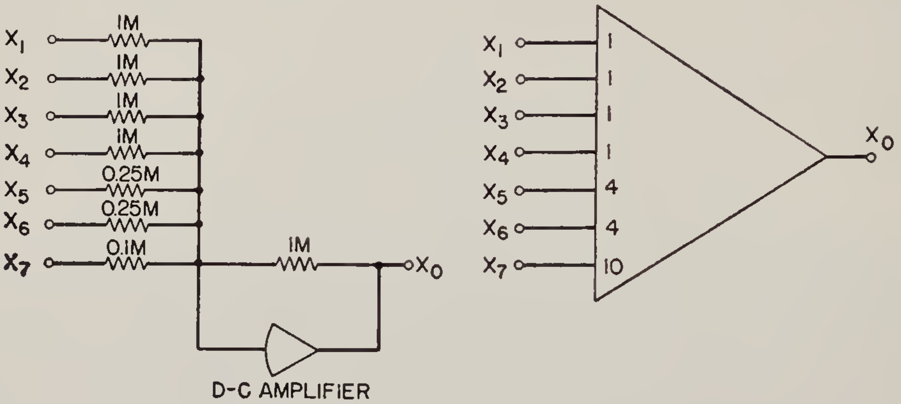
so that (1.19) becomes

$$X_o = -(X_1 + X_2 + X_3 + X_4 + 4X_5 + 4X_6 + 10X_7) \qquad (1.20)$$

Seven input terminals corresponding to the coefficients 1, 1, 1, 1, 4, 4, and 10 are available. Figure 1.6 shows this arrangement and the commonly used block-diagram symbol for such a summing amplifier.

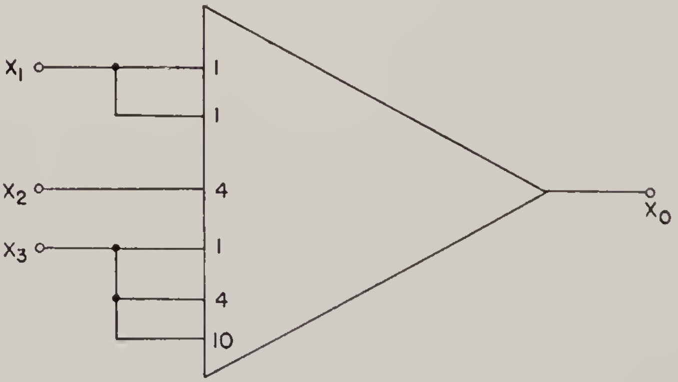
A summing amplifier may be used for multiplying a variable by a constant coefficient in yet another manner. It is possible to connect the same input voltage to two or more input terminals of the summing amplifier; the variable is thus added to itself (or several products of the variable by constant coefficients are added). Figure 1.7 illustrates this use of "parallel inputs" with a summing amplifier. Summing amplifiers and potentiometers can be combined to form more complicated sums and differences (see also Fig. 2.2).

Multiplication. The design of computing elements whose output voltage is the *product* of two input voltages is somewhat more difficult. Chapter 6 lists a number of electromechanical and electronic devices capable of performing this function. It will suffice here to mention the most commonly used of these multipliers. Consider an ordinary linear potentiometer of the type described above with an input voltage X_1. If the potentiometer setting a is made proportional to a second input voltage X_2, then the output voltage aX_1 must be proportional to the product of X_1 and X_2. A *servomechanism* consisting of an amplifier or relay and an electric motor can be made to position the potentiometer shaft in the desired manner. The voltage X_2 is applied to the input terminal of the servomechanism, and the servomotor will turn the potentiometer shaft continuously to the correct setting (Fig. 1.8*a*).



$$X_0 = -(X_1 + X_2 + X_3 + X_4 + 4X_5 + 4X_6 + 10X_7)$$

FIG. 1.6. Summing amplifier and block-diagram symbol.



$$X_0 = -(2X_1 + 4X_2 + 15X_3)$$

FIG. 1.7. Summing amplifier, illustrating use of parallel inputs for multiplication by constant coefficients.

This arrangement permits multiplication by positive voltages X_2 only. In order to permit X_2 to assume negative as well as positive values, it is customary to apply the voltages X_1 and $-X_1$ to two ends of the potentiometer. The output voltage X_o is then still aX_1, and a varies from -1 through 0 to $+1$ as the potentiometer shaft is turned. It is thus possible for a to be proportional to negative as well as positive values of X_2, and the output voltage will be proportional to the product X_1X_2 for all combinations of signs (Fig. 1.8*b*).

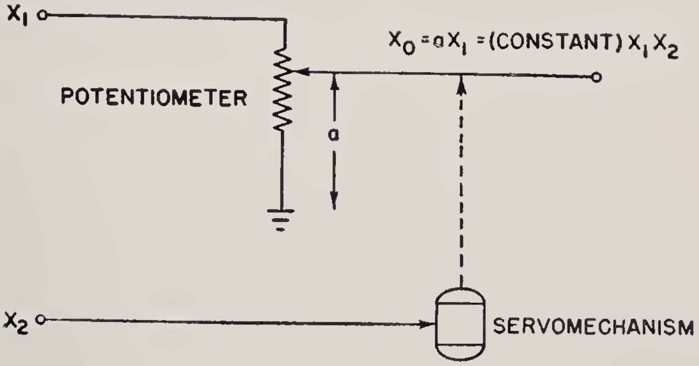


FIG. 1.8*a*. Servomultiplier.

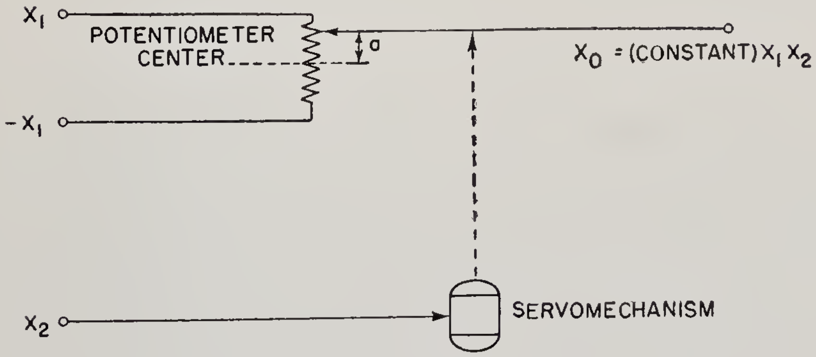


FIG. 1.8*b*. Servomultiplier.

Function Generation. With properly tapered and tapped potentiometers, the output voltage can be made equal to a nonlinear function of the shaft setting a and thus of the second input voltage X_2:

$$X_o = X_1a = X_1a(X_2) \qquad (-1 \leq a \leq 1)$$

Such servo-driven *function generators* permit the multiplication of a voltage (constant or variable) by a function of another voltage. For example, it is possible to *divide* a voltage X_1 by another voltage X_2 by performing a multiplication by the function $1/X_2$. A number of other circuits for division are discussed in Sec. 6.12. With all servo-driven devices, the speed of the computation is limited by the speed with which the motor can follow the variations of the voltage X_2. A number of all-electronic function generators are described in Chap. 6.

1.4. ELECTRICAL INTEGRATION AND DIFFERENTIATION

Integrations and differentiations of d-c voltages are usually performed with respect to the time τ as the independent variable.

The integrators described in this book permit integration with respect to the time τ. If integration of a machine variable X with respect to another machine variable, Y, say, is desired, use must be made of the relation

$$\int X\,dY = \int X\frac{dY}{d\tau}\,d\tau = \int XPY\,d\tau \qquad (1.21)$$

which can be implemented through the use of an integrator and a multiplier (Fig. 2.2).

The physical basis of most electrical integrating and differentiating devices is a simple property of capacitors. The current through such a capacitor (Fig. 1.9) is always proportional to the derivative of the voltage E across the capacitor:

$$i = CPE \qquad (1.22)$$

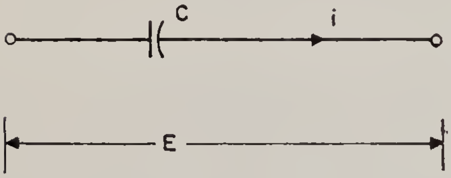


FIG. 1.9. Illustrating the basic property of capacitors.

In the simple *integrating network* shown in Fig. 1.5*e*, an input voltage X_1 will start to charge the capacitor C through the resistor R in a manner amounting approximately to integration. Unfortunately, the voltage built up on C during this process can oppose the input voltage and prevent accurate computation. Quantitatively speaking, the sum of all currents flowing to the output terminal must be zero, so that

$$(X_o - X_1)\frac{1}{R} - X_oCP = 0$$

and

$$X_o = \frac{1}{RCP + 1}X_1 \qquad (1.23)$$

The *transfer function* X_o/X_1 of a true integrator would be

$$\frac{X_o}{X_1} = \frac{const}{RCP} \qquad (1.24)$$

Figure 1.10 shows the response of an integrating network to an input voltage step (a constant-voltage source suddenly turned on). The integrating network will, in all cases, approximate the performance of a true integrator reasonably well for a short time after the computation is started. The transfer function of (1.23) approximates that of a true integrator more closely if the values of C and R are increased, but this also increases the attenuation.

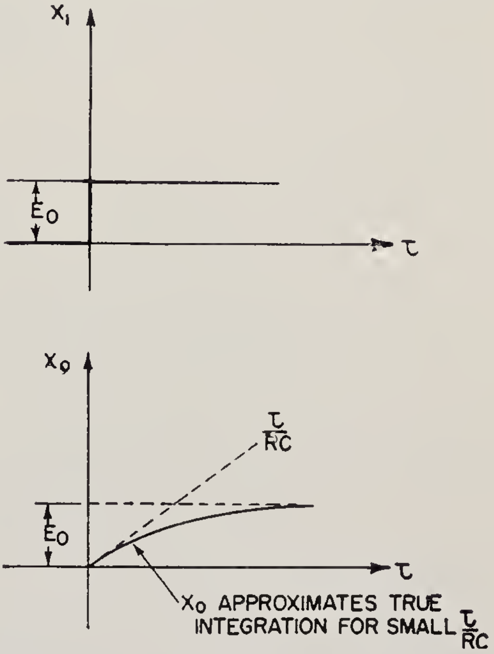


Fig. 1.10. Integration of step-function voltage by a simple integrating network.

Electronic Integration. The *electronic integrator* shown in Fig. 1.5*f* is more accurate and practical than the simple integrating network. Good integration is obtained with little or no attenuation and smaller values of resistance and capacitance. Moreover, the output can be made to match a low-impedance load, such as a potentiometer, without impairing the quality of the integration.

With high-gain feedback amplifiers the input grid voltage

$$E = X_o/A$$

will be quite small for high values of the amplifier gain A. Since now only this small voltage can oppose the input voltage X_1, the amplifier will charge the capacitor at a rate quite closely proportional to X_1. The output voltage X_o will then be closely proportional to the time integral of input voltage X_1. The transfer function of the electronic integrator is derived precisely in the same manner as that of other operational amplifiers. The nodal equation for the input grid is

$$\left(\frac{X_o}{A} - X_1\right)\frac{1}{R} + \left(\frac{X_o}{A} - X_o\right)CP = 0 \tag{1.25}$$

if capacitor leakage can be neglected (see also Sec. 4.5). Thus

$$X_o = \frac{A}{(1 - A)RCP + 1} X_1 \tag{1.26}$$

For large values of A and reasonably short computing times, this is approximated by

$$X_o = -\frac{1}{RCP} X_1 \tag{1.27}$$

which corresponds to integration.

Since Eq. (1.27) is only approximately true, there is still a limitation on the allowable computing time. A comparison of (1.26) and (1.23) shows, however, that a considerable improvement has been effected. Because of the nature of integration, it is possible for initial errors in integration to increase with time. The design of accurate integrators is a crucial part of the entire computer design; detailed design considerations as well as estimates of maximum allowable computing times are discussed in Secs. 4.3 and 4.5.

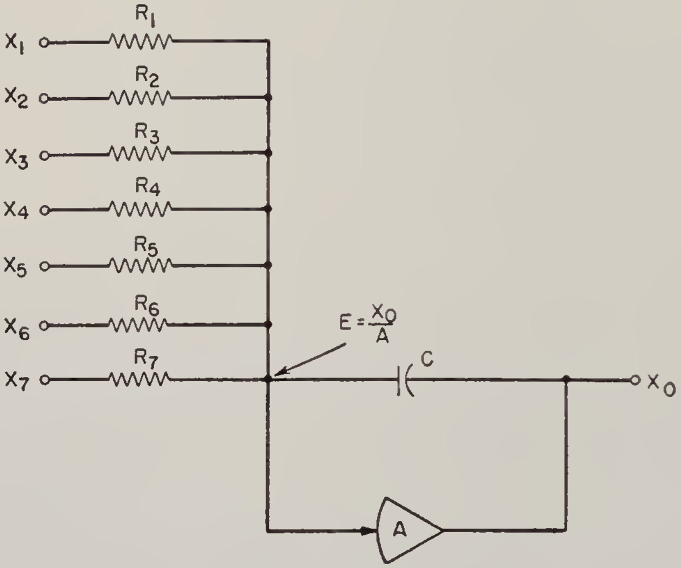


FIG. 1.11. Summing integrator.

Figure 1.11 shows an electronic *summing integrator.* This operational amplifier is capable of summing as well as integration. The output voltage of the summing integrator is, to a similar approximation as (1.27),

$$X_o = -\frac{1}{CP}\left(\frac{X_1}{R_1} + \frac{X_2}{R_2} + \cdots + \frac{X_n}{R_n}\right) \tag{1.28}$$

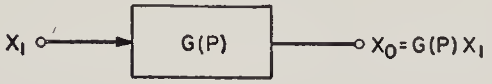
Electrical Differentiation. Networks and operational amplifiers capable of approximate differentiation with respect to time may be constructed by reversing the positions of resistors and capacitors in the corresponding integrators, as illustrated in Figs. 1.5*g* and *h*, which also show the respective transfer functions. Although differentiators can replace integrators in *some* computer setups for the solution of differential equations, the use of differentiators is avoided whenever possible because their frequency-response characteristics tend to produce noisy and unstable computer setups.

1.5. DIRECT-ANALOG REPRESENTATION OF MATHEMATICAL OR PHYSICAL RELATIONS. NETWORKS AND OPERATIONAL AMPLIFIERS

In many important applications of d-c analog computers, two machine variables X_1 and X_o represent input and output variables of a physical or logical system (*e.g.*, stimulus and response, force and displacement, motor current and torque) and are related by a linear differential equation

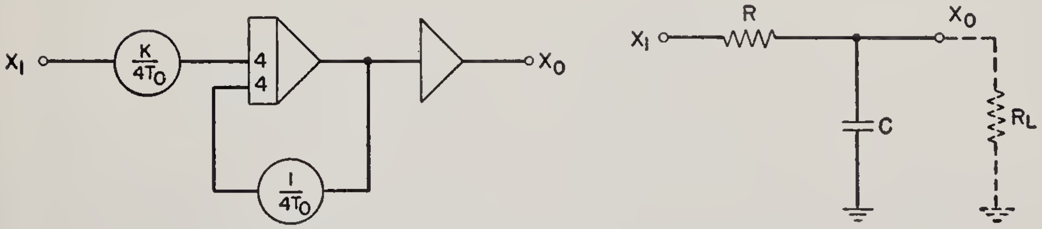
$$\left.\begin{aligned}(a_0P^n + a_1P^{n-1} + \cdots + a_n)X_o &= (b_0P^m + b_1P^{m-1} + \cdots + b_m)X_1\\ \text{or}\qquad X_o &= G(P)X_1\end{aligned}\right\} \tag{1.29}$$

The techniques described in Secs. 1.2 to 1.4 permit the representation of such a relation in terms of simple computing elements, each associated with an *elementary mathematical operation.* Thus the computation of the function $X_o(\tau)$ for arbitrarily given $X_1(\tau)$ by classical differential-analyzer methods requires the synthesis of a "block" of computing elements (potentiometers, summing amplifiers, integrators, and differentiators) which establishes the relation (1.29) between its input voltage X_1 and its output voltage X_o.



(a)

FIG. 1.12*a*. Block representation (transfer-function representation) of a linear differential equation relating two machine variables X_1 and X_o.



(b) (c)

FIG. 1.12*b*, *c*. Two possible computer setups for a relation of the form $X_1 = \frac{1}{k}(T_oP + 1)X_o$ with $T_o > 0$. In Fig. 1.12*c*, $T_o = RR_LC/(R_L + R)$, $k = R_L/(R + R_L)$.

An alternative d-c analog-computer technique is to find a relatively simple electrical circuit, such as a network or a feedback amplifier, whose input and output voltages X_1 and X_o are related by the desired differential equation (1.29). Such a circuit will constitute a *direct analog* of the system under consideration (Fig. 1.12*a*). Networks and feedback amplifiers can also represent "blocks" having more than one input variable. The networks and operational amplifiers shown in Fig. 1.5 may be regarded as *special cases* of direct analogs.

The physical analogies inherent in direct-analog representations appeal to many systems designers. Although direct-analog methods are not as systematic nor as generally applicable as the classical differential-analyzer approach, direct-analog techniques very often result in simpler and more accurate computer setups, particularly if the problem involves the interconnection of many "blocks" of the type described. *Both computing schemes can, and should, be used in combination.*

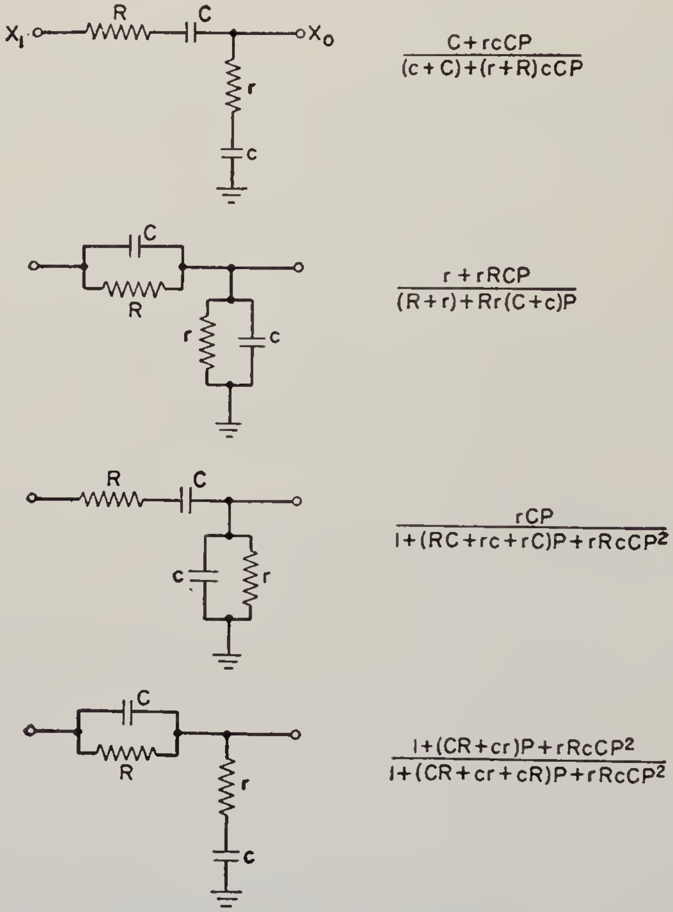


FIG. 1.13. Some computing networks and their transfer functions. The transfer functions of several simpler networks may be obtained from those given if one or more resistances and/or capacitances are set equal to zero or infinity.

Computing Networks. Figure 1.12*b* shows a differential-analyzer representation of the relation

$$X_1 = \frac{1}{k}(T_oP + 1)X_o \qquad \text{or} \qquad X_o = \frac{k}{T_oP + 1}X_1 \qquad (T_o > 0) \qquad (1.30)$$

which is of importance in many problems dealing with damping and decay phenomena. Figure 1.12*c* shows how a simple network with the transfer function $G(P) = k/(T_oP + 1)$ can save two amplifiers (and thus also drift, phase shift, and output range limitations). A number of other computing networks and their transfer functions are shown in Fig. 1.13; such circuits are particularly useful in special-purpose computers.

Operational Amplifiers. In spite of the attractive simplicity of passive computing networks, the use of d-c amplifiers with suitable input and feedback networks (operational amplifiers) is often preferable.[1] Operational amplifiers permit a more flexible choice of transfer characteristics; unlike networks, operational amplifiers can have high input impedance and low output impedance without attenuation (see also Secs. 1.3 and 1.4). The low output impedance of an operational amplifier operated within its voltage and current ratings makes its behavior essentially independent of the load.

The parallel-feedback phase-inverting amplifiers, integrators, and differentiators of Fig. 1.5 are all special cases of the more general operational amplifier circuits shown in Figs. 1.14*a* and *b*. The transfer function of the circuit of Fig. 1.14*a* is derived from the nodal equation

$$\frac{1}{Z_1(P)}\left(\frac{X_o}{A} - X_1\right) + \frac{1}{Z_o(P)}\left(\frac{X_o}{A} - X_o\right) = 0 \tag{1.31}$$

which leads to

$$X_o = -\left[1 - \frac{1}{1 - A\beta(P)}\right]\frac{Z_o(P)}{Z_1(P)}X_1 \tag{1.32}$$

with

$$\beta(P) = \frac{Z_1(P)}{Z_o(P) + Z_1(P)} = \left[1 + \frac{Z_o(P)}{Z_1(P)}\right]^{-1} \tag{1.33}$$

If the absolute value[2] of $A\beta(P)$ is large compared to unity for the working frequencies used, the performance equation (1.32) reduces to

$$X_o = -\frac{Z_o(P)}{Z_1(P)}X_1 \tag{1.34}$$

The circuit of Fig. 1.14*b* is a similar generalization of the summing amplifiers and summing integrators described in Secs. 1.3 and 1.4. The nodal equation yields the performance equation

$$X_o = -\left[1 - \frac{1}{1 - A\beta(P)}\right]Z_o(P)\left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right] \tag{1.35}$$

with

$$\beta(P) = \left[1 + \frac{Z_o(P)}{Z_1(P)} + \frac{Z_o(P)}{Z_2(P)} + \cdots\right]^{-1} \tag{1.36}$$

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[1] See also J. R. Ragazzini, R. H. Randall, and F. A. Russell, Analysis of Problems in Dynamics by Electric Circuits, *Proc. IRE*, **35**: 444, May, 1947.

[2] For sinusoidal input voltages of circular frequency ω, the *complex impedances* $Z_o = Z_o(j\omega)$, $Z_1 = Z_1(j\omega)$, $Z_2 = Z_2(j\omega)$, . . . correspond to the *operators* $Z_o(P)$, $Z_1(P)$, $Z_2(P)$, . . . , and the complex function $\beta(j\omega)$ corresponds to $\beta(P)$. See also Sec. 4.2.

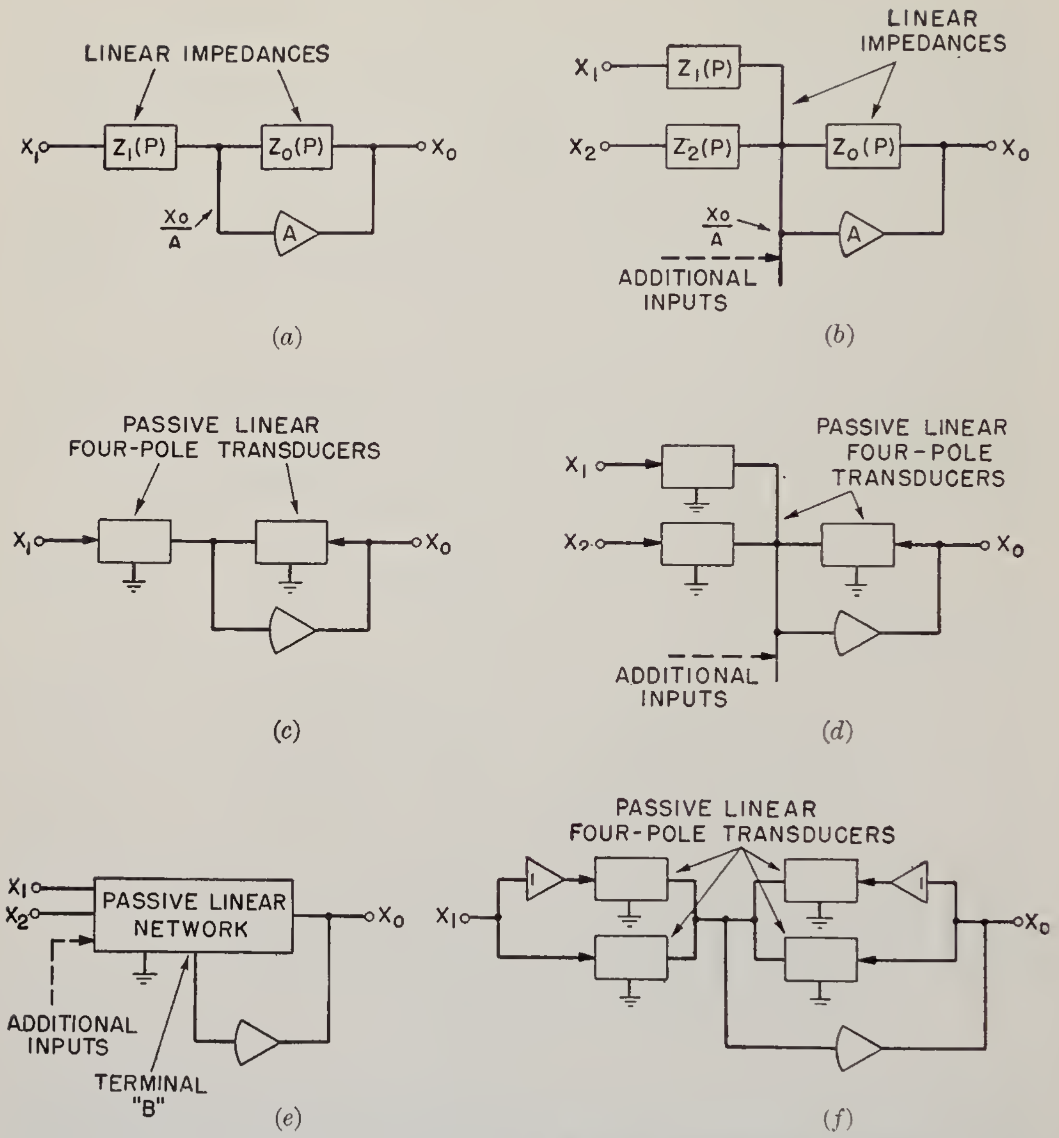

FIG. 1.14. Block diagrams of parallel-feedback operational amplifiers.

If $|A\beta(j\omega)| \gg 1$ for the working frequencies in question, Eq. (1.35) reduces to the simplified form

$$X_o = -Z_o(P)\left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right] \tag{1.37}$$

Since the amplifier forward gain A is itself a function $A = A(j\omega)$ of frequency, the simplified performance equations (1.34) and (1.37) will apply only for a restricted range of input frequencies. The validity of the approximations (1.34) and (1.37) is discussed in Sec. 4.2.

The operational amplifier technique can be generalized still further. In the operational-amplifier circuits of Figs. 1.14c and d, each input and feedback voltage is applied to the d-c amplifier input terminal through a passive linear four-pole transducer. The (complex) *short-circuit transfer impedance* of each transducer is defined as the ratio of its input voltage to its short-circuit output current (Fig. 1.15*a*); this definition includes that of an ordinary impedance as a special case. If the loop gain of the operational amplifier is sufficiently large, the respective performance equations corresponding to Figs. 1.14*c* and *d* are approximately[1]

$$X_o = -\frac{Z_o(P)}{Z_1(P)}X_i \tag{1.38}$$

and

$$X_o = -Z_o(P)\left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right] \tag{1.39}$$

where Z_o, Z_1, Z_2, . . . are the respective transfer impedances of the transducers transmitting the voltages X_o, X_1, X_2, ... Note that Eqs. (1.34), (1.37), and (1.38) are special cases of Eq. (1.39). Table 3 lists the transfer impedances of a number of networks as an aid in the design of such operational amplifiers (Fig. 1.15*b*).

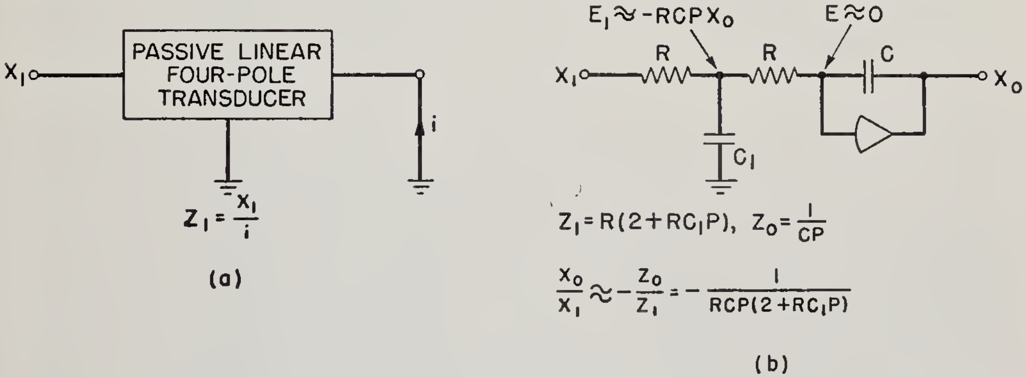
X_1
PASSIVE LINEAR FOUR-POLE TRANSDUCER
i
$Z_1 = \frac{X_1}{i}$
(a)
$E_1 \approx -RCPX_o$
$E \approx 0$
R
R
C
X_1
X_o
C_1
$Z_1 = R(2+RC_1P)$, $Z_o = \frac{1}{CP}$
$\frac{X_o}{X_1} \approx -\frac{Z_o}{Z_1} = -\frac{1}{RCP(2+RC_1P)}$
(b)

FIG. 1.15. Transfer impedance of a passive linear four-pole transducer (Fig. 1.15*a*), and design of an operational amplifier with the aid of Table 3 (Fig. 1.15*b*). The performance equation in Fig. 1.15*b* can also be derived directly from the nodal equations.

The infinite-gain performance equation (1.39) also applies to the more general operational amplifier circuit of Fig. 1.14*e* if one defines each (complex) transfer impedance Z_i (or Z_o) as the ratio of X_i (or X_o) to the corresponding short-circuit ground current into the network terminal "B" when all other network terminals are grounded. In general it is, however, easier to derive the performance equation directly from the nodal equations. Thus, the performance equation of each of the useful circuits of Table 2 follows at once from the nodal equation for the integrator input terminal, whose voltage must be $-RCPX_o$ for infinite amplifier gain.

The circuit of Fig. 1.14*f* permits the representation of complex transfer functions having zeros in the positive as well as in the negative half-plane.[1] The synthesis of even more general "electronic operators" involving two or more parallel-feedback amplifiers has been treated by Raymond.[2]

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[1] Bradley, F. R., and R. McCoy, Driftless D-c Amplifier, *Electronics*, April, 1952.

Operational amplifiers of the types shown in Fig. 1.14 *have many interesting and important applications.* They may save over one-third of the integrators needed for large differential-analyzer setups (*e.g.*, in the solution of partial differential equations by difference-equation methods, Sec. 3.10). In the simulation of automatic control systems (Sec. 3.4), operational amplifiers may serve as *direct analogies* of control devices or dynamic systems whose performance is approximated by Eq. (1.38) or (1.39). A number of particularly useful operational amplifier circuits are given in Tables 1 and 2; others may be designed with the aid of Table 3. Each operational-amplifier circuit approximately described by the transfer-impedance method may be subjected to a more accurate performance analysis in the manner of Sec. 4.2.

Nonlinear Relations. Direct-analog techniques utilizing operational amplifiers can be extended to nonlinear problems; servomechanisms or switching devices may change one or more circuit impedances as a function of a machine variable (Secs. 3.2 and 6.9, Tables 5 and 6). Additional uses of high-gain feedback amplifiers are described in Secs. 2.11 and 3.3.

1.6. COMPLETE D-C ANALOG-COMPUTER INSTALLATIONS

By judiciously combining basic computing elements of the types discussed in the last sections, one can design complete d-c analog computers for a wide range of applications. The technical details associated with the economical design of such installations will be discussed in Chap. 8.

A complete d-c analog computer will, in general, comprise the following components in one form or another (Fig. 1.16):

1. *Linear electronic computing elements* (operational amplifiers and associated networks). These will perform the operations of amplification, phase inversion, differentiation, and integration, as necessary. The cost and complication of these and other computing elements will depend on the required accuracy (Chaps. 4 and 5).
2. *Potentiometers or attenuators* to set constant coefficients (Sec. 4.1).
3. *"Nonlinear" electronic or electromechanical computing elements* for performing operations such as multiplication, division, function gen-

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[1] Unpublished lecture by Mathews and Seifert (Massachusetts Institute of Technology); see also Beck, lecture presented at *Project* CYCLONE *Symposium I*, 1950, Reeves Instrument Corporation, New York City.

[2] Raymond, F. H., Le Calcul analogique, booklet published by the Société d'Electronique et d'Automatisme, Courbevoie, France, 1952.

eration, and limiting. Such devices are needed if the problems to be solved call for nonlinear operations (Chap. 6).

4. Permanent wiring or flexible means for *interconnecting* these computing elements as needed for the problems in question. Patch cords and jacks used for this purpose are usually plugged into a removable *problem board,* which in turn is plugged into the machine proper (Sec. 8.3).

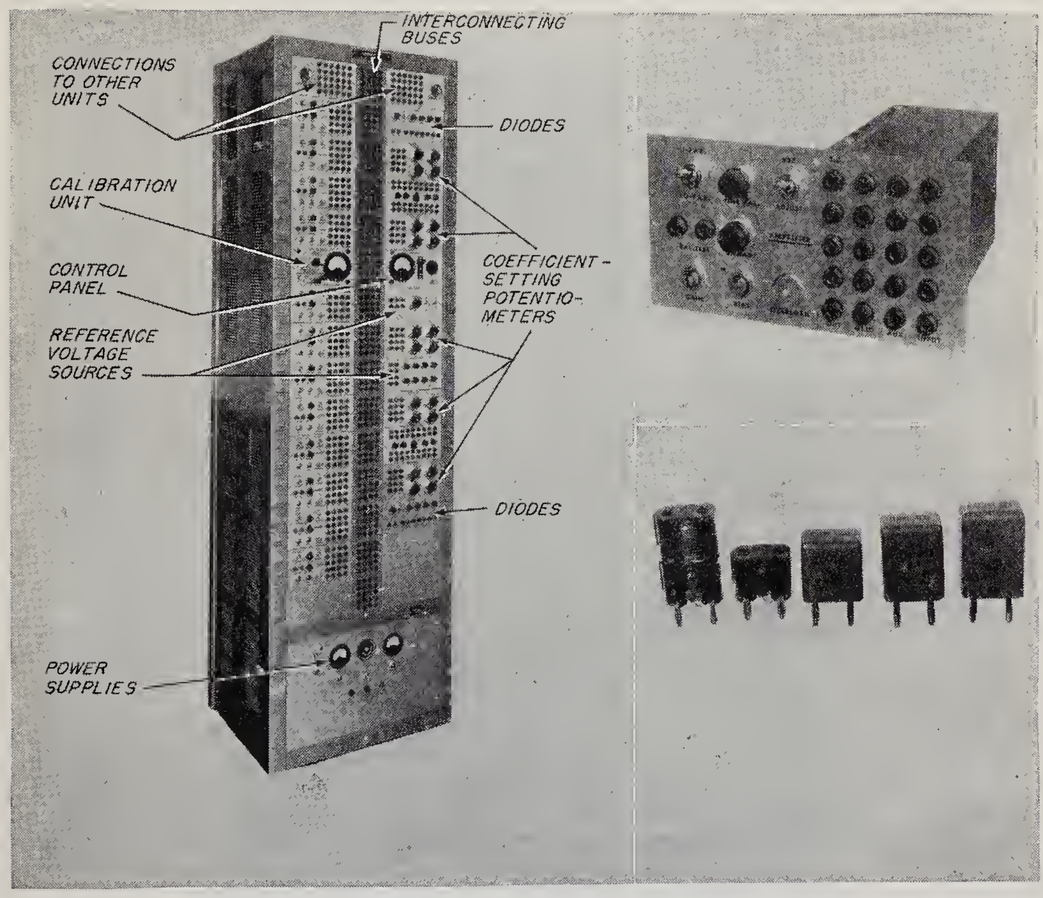
INTERCONNECTING BUSES
CONNECTIONS TO OTHER UNITS
DIODES
CALIBRATION UNIT
CONTROL PANEL
COEFFICIENT-SETTING POTENTIOMETERS
REFERENCE VOLTAGE SOURCES
DIODES
POWER SUPPLIES

FIG. 1.16. The basic components of the flexible and economical electronic analog computer shown at left (see also Sec. 8.4) are high-gain d-c amplifiers (upper right) and plug-in resistors and capacitors (lower right). Other computing elements include diode function generators, computer servos, and electronic multipliers (Boeing Airplane Company).

5. *Control circuits* for starting the computation with the correct initial-condition settings and for stopping the machine after the computation is completed (Sec. 7.2).
6. Means for *recording or measuring* the d-c voltages in the machine. Recording milliammeters or servo plotting boards (Fig. 1.17) are commonly used to furnish a permanent inked record of voltages against time (Sec. 7.3).

7. Regulated *power supplies* which furnish the voltages needed for the operation of the electronic components (Sec. 7.4).

In addition to special-purpose computers assembled permanently or on a breadboard for the solution of single problems, an institutional or industrial laboratory may utilize flexible electronic differential analyzers of small to moderate size (10 to 60 operational amplifiers) for the solution of general mathematical problems which present themselves in the course of research or development work. Such computers can be maintained and serviced by any alert radio technician without special preparation. Experience has shown that engineers and scientists with an average college background in mathematics can learn the use of an electronic differential analyzer without difficulty and will quickly appreciate the possibilities afforded by this powerful modern tool.

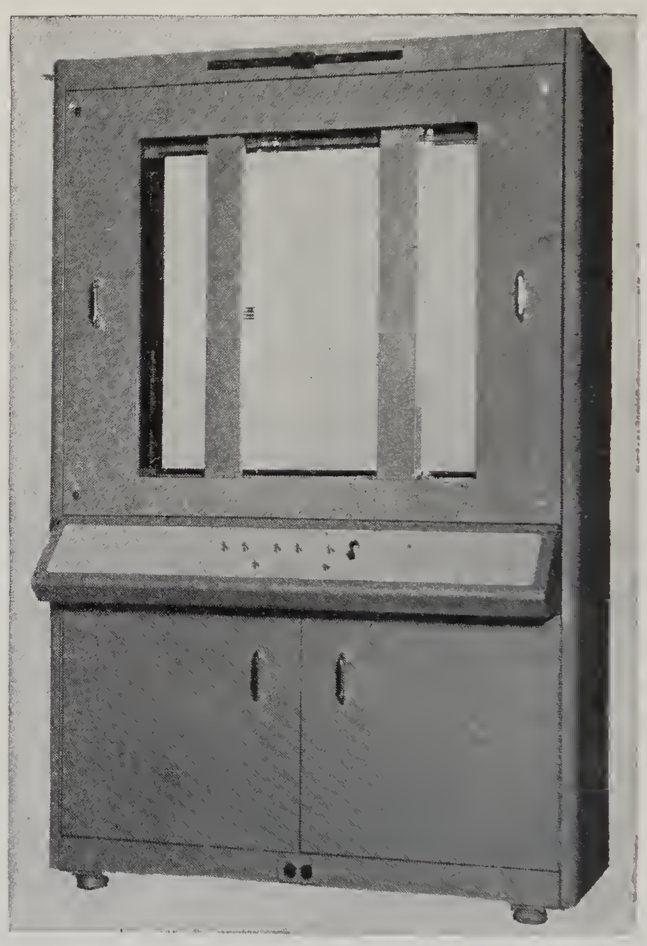

FIG. 1.17. A large servo plotting board (Electronic Associates, Inc.).

Many large analog-computer laboratories, staffed by mathematical experts as well as specialists in the principal fields of application, are now in existence in the United States and have an enviable record of achievement. Such laboratories are capable of rendering invaluable services to science and industry by performing computations too extensive in scope or volume to be handled by smaller facilities. The computer laboratories are usually started by government or industrial grants and may well be self-supporting.[1] An installation of this type may comprise two or more large electronic differential analyzers which can be interconnected and operated together for the solution of especially complicated problems. Test facilities for automatic control machinery to be tested with analog computers may adjoin the differential analyzers. Most analog-computer laboratories use electronic digital computers to check selected analog-computer solutions.

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[1] Herbert, E., A New Computation Center for Industry, *Automatic Control,* August, 1954.

CHAPTER 2

PRACTICAL SETUP PROCEDURE

2.1. INTRODUCTION

In order to solve a given set of differential equations on a d-c analog computer it is necessary to make certain voltages in the machine vary with time in a manner prescribed by the given differential equations. The computer is "set up" for the given problem if a suitable arrangement of computing elements ensures the existence of the correct mathematical relations between the voltages in the computer.

It is clearly of the greatest importance for the engineers or mathematicians in charge of a computer laboratory to acquire skill and facility in the technique of setting the machine up for a given problem.

The setup procedure described in the following sections has evolved during several years of practical work. It seems to be sufficiently simple and straightforward to enable even inexperienced personnel to obtain workable setups for a very large number of problems. On the other hand, there are usually many possible setups for a given problem. It will always be somewhat of an art to find the setup which will result in the greatest possible accuracy or in the most economical use of the computing elements. Experience as well as ingenuity will give the operator a feeling for the new mathematical medium, and one may say that an elegant setup will be aesthetically satisfying as well as practically useful.

2.2. MACHINE VARIABLES AND SCALE FACTORS

It is very important to remember that the machine will establish mathematical relations not between the original variables of a problem but between voltages "simulating" these variables. In order to keep this distinction clearly in mind it is useful to denote the voltage or *machine variable* representing a given mathematical variable in the computer by the corresponding capital letter. Thus an angle δ would be represented by a voltage Δ, a velocity v by a voltage V, a variable x by X, etc. Since in most d-c analog computers the machine variables can vary between the limits of plus and minus 100 volts, it has been found convenient to measure the machine variables in units of 100 volts, or *machine units.* These conventions will be adhered to in the following discussions.

The first step in any computer setup must be the establishment of relations describing the transformations from the given problem variables to machine variables and vice versa. In all ordinary cases these *transformation equations* are simply of the form

$$X = a_x x \qquad \Delta = a_\delta \delta \qquad V = a_v v, \text{ etc.} \tag{2.1}$$

The dimensional coefficients a_x, a_δ, a_v, etc., in Eq. (2.1) are called *scale factors.* The scale factor of any variable is numerically equal to the number of 100-volt machine units corresponding to 1 unit of the simulated variable. Each voltage is given, numerically and dimensionally, by the product of scale factor and variable. Conversely, the correct value of any variable can be determined from the corresponding voltage by dividing the latter by the scale factor.

Example. Let δ be an aircraft angle of attack. The transformation equation $\Delta = 4\delta$ indicates that the voltage Δ simulates the angle δ with a scale factor $a_\delta = 4$ (100 volts/radian), so that an angle of ¼ radian or 14.3 deg is represented by 1 machine unit or 100 volts in the machine.

The choice of scale factors is governed by two conflicting considerations between which some compromise must be made: (1) scale factors should be as large as possible in order to minimize percentage errors due to stray voltages, and (2) the machine variables must not exceed plus or minus 100 volts (1 machine unit) so that all tubes operate within the linear portions of their characteristics. Because of these considerations, it is necessary to possess some information as to the expected ranges of the variables in order to limit the scale factors accordingly. Estimates of these ranges may be obtained from a knowledge of the physical background of the problem, from approximate numerical computations, or from the results of trial runs. This information need not be accurate, since it is relatively easy to change the scale factors after one or more trial runs.

In general, a scale factor a_x associated with the variable x will be chosen according to the following rule:

$$a_x \leq \frac{1}{\text{maximum expected value of } |x|} \; \frac{\text{machine units}}{\text{unit}} \tag{2.2}$$

This ensures that the largest expected excursion of the machine variable X will never exceed 1 machine unit. In most cases a convenient round number, somewhat less than the numerical value given by Eq. (2.2), may be used for the scale factor. Scale factors smaller than those indicated by the foregoing must be used (1) if it is inevitable that *multiples* of the machine variables in question appear at an amplifier output terminal; (2) if the time scale (see Sec. 2.4) is fixed and the voltages representing *derivatives* of machine variables would overload one or more amplifiers.

Scale factors for different variables will, in general, differ from one another in magnitude and dimensions. As a rule, there will be one transformation equation and thus one scale factor associated with each variable of the problem. Quite frequently, however, a mathematical variable, x, say, may be represented by two or more different voltages or machine variables X, X', etc., in different parts of the computer. In such cases there may be more than one transformation equation and more than one scale factor for the variable in question.

Functions of Machine Variables. Functions of machine variables such as X^2, $\sin Y$, e^Y, etc., may be considered as new problem variables f_1, f_2, etc. The absolute values of the corresponding machine variables F_1, F_2, etc., should not exceed 1 machine unit.

Examples. If $f_1 = \sin Y$, the corresponding transformation equation can be simply

$$f_1 = F_1$$

If $f_2 = e^Y$ with $-1 \leq Y \leq 1$, the corresponding transformation may be

$$f_2 = eF_2$$

where

$$F_2 = \frac{1}{e} e^Y$$

Note that the new function F_2 can be obtained simply by means of a potentiometer function generator, whereas the original f_2 cannot (see also Secs. 2.8 and 6.9).

It is not really necessary to write such transformations explicitly in each case; it will usually suffice to rewrite the machine equation in the manner shown in the following example.

Example. The machine equation

$$PX = \frac{1.5}{X} \qquad \text{with } 0.1 < X < 1$$

can be rewritten in the form

$$PX = 15\left(\frac{1}{10X}\right)$$

The function in parentheses can be generated by means of a function potentiometer.

Recorder Scales. The adjustment and choice of the *recorder scales* before computation may be properly considered as part of the computer operation as such and are described in Sec. 7.3.

2.3. CHOICE OF TIME SCALE

If the transfer functions of all computing elements are given in terms of real time units, a d-c analog computer may be considered as capable of integrating and differentiating the machine variables *directly only with respect to real time* (note, however, Fig. 2.2). In other words, *real time, henceforth denoted also as machine time* τ, *is the independent machine variable.*[1] The *time scale* is established by writing a transformation equation relating the machine time τ to the mathematical independent variable, t, say,

$$\tau = \alpha_t t \tag{2.3}$$

From (2.3) it follows that

$$p = \frac{d}{dt} = \alpha_t \frac{d}{d\tau} = \alpha_t P \tag{2.4}$$

where the operator $P = d/d\tau$ denotes differentiation with respect to the real or machine time τ. The dimensional coefficient α_t is called the *time-scale factor* and is often referred to simply as the time scale. The time-scale factor is numerically equal to the number of seconds representing the unit of the independent variable under consideration. Again, the value of the independent variable at any instant of the computation is obtained by dividing the elapsed computing time by the time-scale factor.

Example. Let the time t be the independent variable in a physics problem. *Do not confuse* τ *and* t! The transformation equation $\tau = 10t$ indicates that the machine time "simulates" the problem time t with a time-scale factor of 10. Thus a problem time of $\frac{1}{10}$ sec is represented by a machine time of 1 sec. All rates of change in the problem have been *slowed down* in the ratio α_t:1 or 10:1.

The nature of a d-c analog computer makes it easy to change the time scale of a computer run within certain limits. When the time scale is changed, the voltage changes in the computer remain proportional to the corresponding changes of the mathematical variables, but the rates at which these voltage changes take place may be speeded up or slowed down in order to improve the accuracy or increase the convenience of the computation (see also Sec. 1.1).

The *choice of the time scale* is, again, governed by a number of conflicting considerations, which may be summarized as follows:

1. Almost all errors associated with d-c integrators tend to increase with time (see Sec. 4.3) and are thus minimized by short computing times (small α_t).
2. Computer servos, on the other hand, may become unreliable at frequencies higher than 2 cps (see Sec. 6.8). From this point of view, a slow time scale seems to be more favorable if servos are to be used.

---

[1] While the symbol t for real time might be preferable, its use is not practical because of the frequent use of t as the independent variable in physics problems. It is important to consider the meanings of the two symbols τ and t very carefully so that the two may not be confused.

3. With a small time-scale factor all rates of changes and therefore the coefficients in the machine equations are increased. This may make it necessary to cascade amplifiers to the point of instability and also to reduce the scale factors of certain variables whose derivatives would otherwise overload the computer.

Example. Consider the differential equation

$$\frac{dy}{dt} = py = 50$$

Suppose the scale factor for y has been chosen so that

$$a_y = \frac{1}{200}\left(\frac{\text{machine units}}{\text{unit}}\right)$$

or

$$y = 200Y$$

Then for $\tau = t$ (1:1 time scale), the machine equation is

$$Y = \frac{1}{P}\left(\frac{1}{4}\right)$$

For $\tau = 100t$ ("slow" time scale = 100:1)

$$Y = \frac{1}{P}\left(\frac{1}{400}\right)$$

But for $\tau = {}^1\!/_{100}t$ ("fast" time scale = ${}^1\!/_{100}$:1)

$$Y = \frac{1}{P}\,(25)$$

and the voltage of -25 machine units needed to drive the integrator is not available in the computer. The equation could be solved, however, with a reduced scale factor a_y.

On the basis of these conditions a compromise must be made. It is evident that a choice of the time scale can be made only after a study of the particular problem in question. Some examples will be presented in Chap. 3. In general, a total computing time of 1 to 4 min will be satisfactory for accurate integration. If computer servos are used, they should not be positioned by voltages changing at frequencies higher than 2 to 10 cps, depending on the machine used. If a d-c analog computer is to be used in a *partial system test* (see Sec. 3.7) in connection with other equipment, it must, of course, be operated on a 1:1 time scale.

In many problems, the independent variable, t, say, or some function of t, occurs explicitly in the differential equations.

Example.

$$\frac{dy}{dt} = -y + t - t^3 - \sin 2t$$

In such cases, t is not represented by the machine time τ alone. It is necessary to define an additional voltage or machine variable which will represent t in the machine. In accordance with previous practice, this machine variable will be denoted by T and will be defined by a transformation equation

$$T = a_t t \tag{2.5}$$

The scale factor a_t must not be confused with the time-scale factor α_t, from which it is in general numerically and dimensionally different.[1] The machine variable T is necessarily always generated as a function of the machine time τ, which is the real independent variable in the machine. It is, therefore, proper to consider T as a dependent and not as an independent machine variable.

2.4. MACHINE EQUATIONS AND BLOCK DIAGRAMS

As soon as all scale factors, for the independent variable as well as for the dependent variables, have been chosen (subject to later revisions), the transformation equations (2.1), (2.3), (2.4), and (2.5) should be collected and recorded on a sheet or card clearly labeled with the problem number or designation. Thus,

$$\begin{gathered} X = a_x x \qquad \Delta = a_\delta \delta \qquad V = a_v v \qquad \cdots \qquad T = a_t t \\ \tau = \alpha_t t \qquad P = \frac{1}{\alpha_t} p \end{gathered} \tag{2.6}$$

These equations are next solved for the original mathematical variables:

$$\begin{gathered} x = \frac{1}{a_x} X \qquad \delta = \frac{1}{a_\delta} \Delta \qquad v = \frac{1}{a_v} V \qquad \cdots \qquad t = \frac{1}{a_t} T \\ t = \frac{1}{\alpha_t} \tau \qquad p = \alpha_t P \end{gathered} \tag{2.7}$$

(2.7) is recorded on the same sheet or card as (2.6).

The given differential equations are, of course, expressed in terms of the mathematical variables $(x,\delta,v, \ldots ,t)$. It is easy to rewrite the given differential equations in terms of the machine variables $(X,\Delta,V, \ldots ,T,\tau)$. It is merely necessary to substitute the expressions (2.7) for the original variables in the given differential equations. The resulting mathematical relations between the machine variables are called the *machine equations;* they are the equations actually solved by the computing machine.

Example. Consider a given set of differential equations

$$\left.\begin{aligned} p^2x + 15.9py + 10px + 731y + 50x &= 200 \\ p^2y + 7.34py + 4.9px + 3.5y + 100x &= 0 \end{aligned}\right\} \tag{2.8}$$

---

[1] The numerical value of a_t is determined by simply applying Eq. (2.2) to the independent variable.

If the scale factors are chosen to be

$$a_x = a_y = \frac{1}{100}\left(\frac{\text{machine units}}{\text{unit}}\right) \qquad \alpha_t = 10\left(\frac{\text{seconds}}{\text{unit}}\right) \tag{2.9}$$

the transformation equations become

$$x = 100X \qquad y = 100Y \qquad t = \tfrac{1}{10}\tau \qquad p = 10P \tag{2.10}$$

These expressions are substituted into the original differential equations. The machine equations are

$$\left.\begin{aligned} P^2X + 1.59PY + PX + 7.31Y + 0.5X &= 0.02 \\ P^2Y + 0.734PY + 0.49PX + 0.035Y + X &= 0 \end{aligned}\right\} \tag{2.11}$$

The machine equations express mathematical relations between *voltages* in a form more or less readily set up on the machine. The arrangements of computing elements which will establish these relationships between the voltages in the computer are first drawn up on paper as *block diagrams*, one for each equation. Each block diagram will serve as a visual aid for connecting the computing elements proper and also constitutes an important record of the computer setup.

Just as a mathematical equation describes relations between variables and operators, the block diagram describes relations between voltages (machine variables) and computing elements (machine operators). The *basic computing elements* available for this purpose were discussed in Chap. 1. Some symbols commonly used in block diagrams are shown in Fig. 2.1, together with a summary of the operations performed by each of the basic computing elements. On all block diagrams, the machine variables at all amplifier output terminals should be clearly indicated so that the block diagram can be easily interpreted.[1]

The block-diagram symbols presented in Fig. 2.1 are intended as suggestions only and will frequently have to be supplemented by more complete diagrams (partial circuit diagrams). This is especially true for setups on computers which have no fixed computing and feedback networks built into the machine. Since block diagrams for simple relations are most easily understood, *it usually is best to draw separate block diagrams for each machine equation* or even several block diagrams for the various relations expressed by one machine equation.

Each block diagram should be drawn so that it can be "read" from one or more input terminals on the left to one or more output terminals on the right like a sentence or an equation in a book. It is usually not necessary or desirable to show interconnections between the output and input termi-

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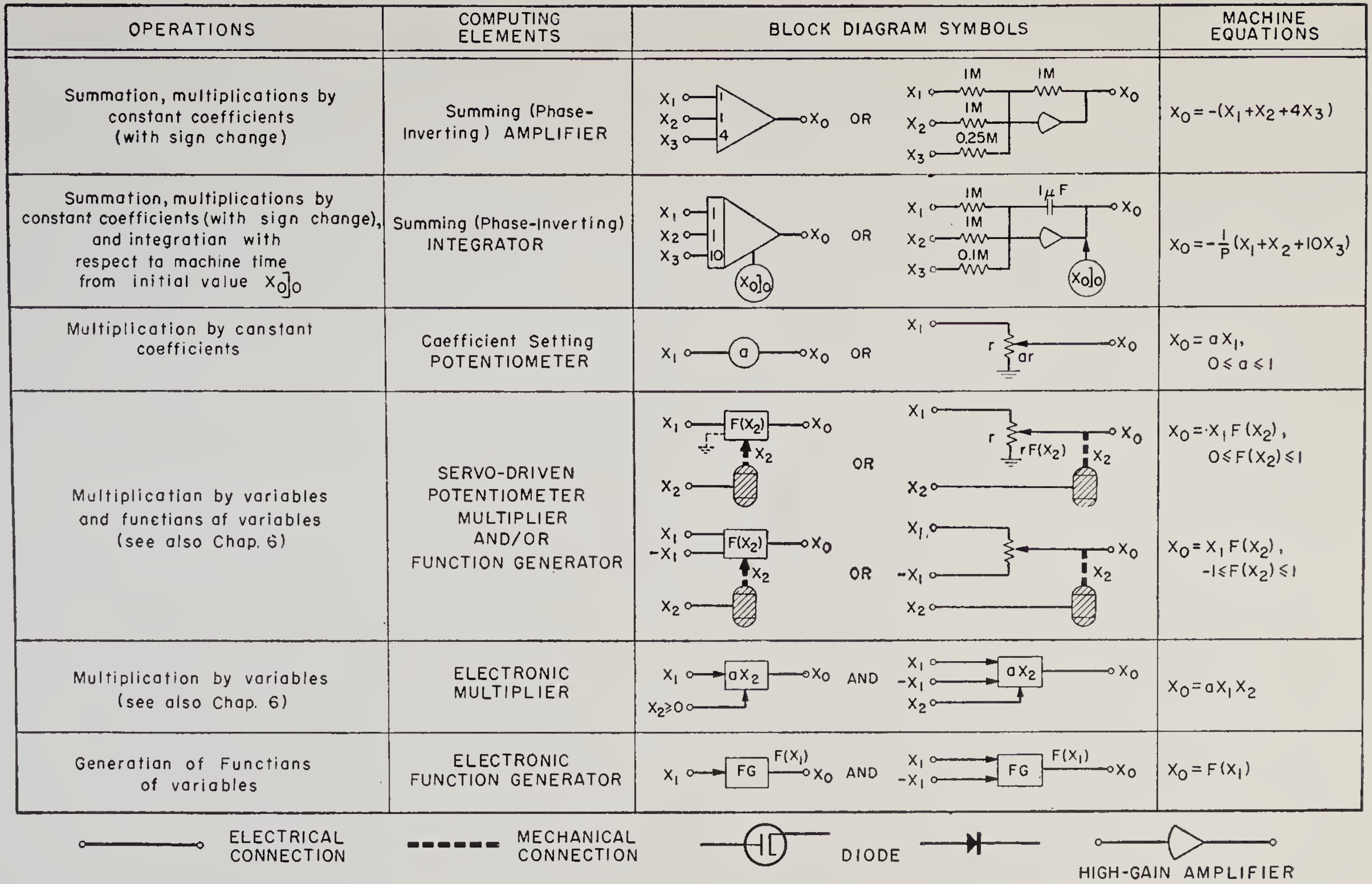
[1] In a number of computer laboratories, it is customary to label voltages in block diagrams with the products of scale factors and mathematical variables (such as $a_y y$ instead of Y). It is believed that this procedure will permit errors more easily than that described in the text.



OPERATIONS	COMPUTING ELEMENTS	BLOCK DIAGRAM SYMBOLS	MACHINE EQUATIONS
Summation, multiplications by constant coefficients (with sign change)	Summing (Phase-Inverting) AMPLIFIER	OR	$X_0 = -(X_1 + X_2 + 4X_3)$
Summation, multiplications by constant coefficients (with sign change), and integration with respect to machine time from initial value $X_0]_0$	Summing (Phase-Inverting) INTEGRATOR	OR	$X_0 = -\frac{1}{p}(X_1 + X_2 + 10X_3)$
Multiplication by constant coefficients	Coefficient Setting POTENTIOMETER	OR	$X_0 = aX_1$, $0 \leq a \leq 1$
Multiplication by variables and functions of variables (see also Chap. 6)	SERVO-DRIVEN POTENTIOMETER MULTIPLIER AND/OR FUNCTION GENERATOR	OR OR	$X_0 = X_1 F(X_2)$, $0 \leq F(X_2) \leq 1$ $X_0 = X_1 F(X_2)$, $-1 \leq F(X_2) \leq 1$
Multiplication by variables (see also Chap. 6)	ELECTRONIC MULTIPLIER	AND	$X_0 = aX_1X_2$
Generation of Functions of variables	ELECTRONIC FUNCTION GENERATOR	AND	$X_0 = F(X_1)$

FIG. 2.1. Simple computing elements are the building blocks of block diagrams and computer setups. Ordinary electrical circuit symbols are often used in connection with the symbols shown.

nals of the block diagrams corresponding to two different machine equations. Such interconnections are, in any case, clearly seen from the fact that the respective terminals are labeled with the same symbols.

The computer will be set up by using wires or patch cords to interconnect the physical components of the machine in the manner shown by the block diagram. *The code or serial number of each actual computing element must be noted carefully on the block diagram as the interconnections are made, and an accurate record must be made of any changes in the computer setup or in the scale factors used.*

Several examples of simple block diagrams, together with the corresponding machine equations, are shown in Fig. 2.2 which also illustrates a number of ways of combining the basic computing elements to perform various operations.

While the number of possible interconnections in a d-c analog computer is very large, several precautions have to be observed, which will be brought out more clearly when the design of the individual computing elements is discussed in Chaps. 4 to 6. A practical limit exists on the amount of amplification and thus on the magnitude of a coefficient which can be obtained in most d-c analog computers. Gains higher than plus or minus 200 obtained by means of cascaded amplifiers usually lead to instability.

Again, many amplifiers or integrators should not be loaded by more than two simple potentiometers without special "booster" connections (see Sec. 5.5).

Another point to be remembered is that voltages like $10Y$, $4V$, etc., and also derivatives like PY, P^2Y (see Fig. 2.4) may overload an amplifier output even if the voltages Y, V, etc., do not. Setups involving such situations must be avoided unless the respective scale factors are reduced accordingly.

The simple potentiometers in the computer are usually *loaded* by amplifier input networks. As a result, the potentiometer dial readings are not accurately equal to the respective potentiometer transfer functions. The subject of potentiometer loading is discussed in detail in Sec. 4.1. It suffices to say here that the loading errors can be corrected accurately by reference to a comparison potentiometer or to the correction charts given in Sec. 4.1. *The potentiometer transfer functions indicated in all block diagrams should be the actual transfer functions, including the effects of loading;* otherwise, the machine equations could not be easily derived from the block diagram.

Because of the very great variety of problems which can be solved by means of d-c analog computers, the technique of setting up a machine is best taught and learned by working as many examples as possible. The material in Secs. 2.5, 2.7, and 2.8 as well as Chap. 3 has been prepared with this idea in mind. A number of specific setup techniques will be discussed in Secs. 2.5, 2.7, and 2.8. Chapter 3 will describe the solution of several complete problems and will also serve to illustrate a method of keeping convenient records of the computations.

Setup Sheets. A number of computer laboratories have successfully used so-called *setup sheets* as an intermediate step between the preparation of the block diagram and the actual computer setup. *A setup sheet is a form listing the following information for each computing element used in the computer setup in question:*

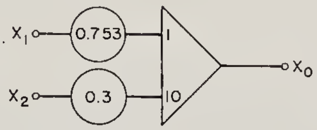
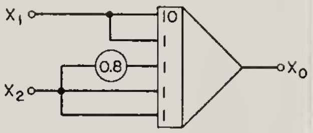
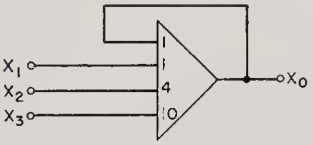
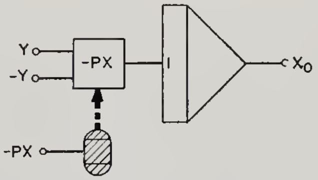
OPERATIONS	COMPUTING ELEMENTS	BLOCK DIAGRAM SYMBOLS	MACHINE EQUATIONS
Combinations of multiplications by constant coefficients, summation, sign change, and integration	Coefficient-setting potentiometers with summing amplifier (or integrator)		$X_0 = -(0.753X_1 + 3X_2)$
	Use of parallel inputs (see also Fig. 1.7)		$X_0 = -\frac{1}{p}(11X_1 + 2.8X_2)$
	Summing amplifier with feedback connection		$X_0 = -\frac{1}{2}(X_1 + 4X_2 + 10X_3)$
Integration with respect to a dependent variable	Integrator and servo-driven potentiometer		$X_0 = \int Y dX$ $= \frac{1}{p}(YPX)$

FIG. 2.2. Examples of combined operations with basic computing elements.

1. *Input machine variables—from which point in the machine*
2. *Output machine variables—to which point(s) in the machine*
3. Mathematical relation enforced
4. Scale factors used
5. Special settings or remarks

Setup sheets may be convenient particularly if the computer setup proper is to be made by relatively unskilled personnel. Setup sheets may also be helpful for checking completed computer setups. On the other hand, it may be difficult to keep the setup sheets up to date during changes in the computer setup, and the additional intermediate step may provide an added opportunity to commit errors.

2.5. D-C ANALOG-COMPUTER SOLUTION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A very large class of engineering problems concerned with oscillations and automatic controls leads to *sets of linear ordinary differential equations.* D-c analog computers are especially well adapted to the solution of such equations. Two typical cases will serve to illustrate the manner in which such problems are handled.

Suppose that the dependent machine variable Y in a certain problem is to vary with the machine time τ as prescribed by the machine equation

$$\frac{d^2Y}{d\tau^2} + r\frac{dY}{d\tau} + Y = 0 \qquad \text{or} \qquad P^2Y + rPY + Y = 0 \qquad (0 \le r \le 1) \qquad (2.12)$$

Equation (2.12) is first solved for the highest derivative of Y, resulting in

$$P^2Y = -(rPY + Y) \qquad (0 \le r \le 1) \qquad (2.13)$$

Each of the terms on the left and right of Eq. (2.13) will be a voltage. The voltage P^2Y must appear as a *sum,* multiplied by -1, of the voltages rPY and Y.

One obtains this relationship between voltages by applying the voltages rPY and Y to the input terminals of a summing amplifier (Fig. 2.3*a*) which will perform the required operations of addition and multiplication by -1.

The voltages rPY and Y, which must appear at the summing-amplifier input terminals, are not independent of P^2Y but must be obtained from P^2Y through successive integrations and multiplications by constants. The voltage P^2Y is treated as if it were already known and is applied to the input of an *integrator* whose output must then be equal to $-PY$. Integration of this voltage in turn yields the voltage Y; each electronic integration results in a change of sign (Fig. 2.3*b*). The voltage $-PY$ is, finally, multiplied by $-r$ by means of a phase-inverting amplifier and a simple potentiometer, so that the required voltage rPY is obtained. Figure 2.3*c* shows the block diagram corresponding to the machine equation (2.12), except for the initial-condition settings. The latter will be discussed in Sec. 2.6.

Experience has shown that there is usually more than one correct computer setup for any given problem. One or more of these may be preferable to others for reasons of economy in the use of computing elements, because of better accuracy, or because of scale-factor advantages. For this reason, every block diagram should be inspected with a view to possible revisions. Unless it is necessary to measure or record the voltage P^2Y itself, the operations of summing and integrating in Fig. 2.3*c* may

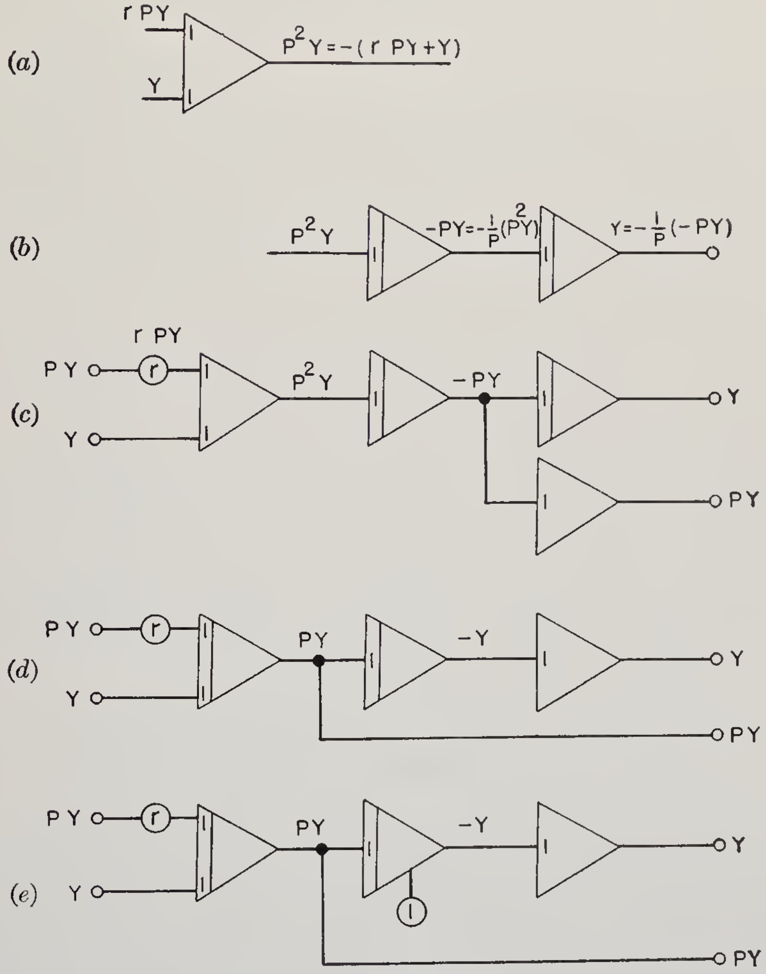


FIG. 2.3. Solution of the differential equation $P^2Y + rPY + Y = 0$ with $0 \leq r \leq 1$. Initial conditions: $Y_0 = 1$, $(PY)_0 = 0$.

both be performed by a summing integrator. This results in the saving of one amplifier. The *revised block diagram* is shown in Fig. 2.3*d*. A check shows that the revised block diagram corresponds to the relation

$$PY = -\frac{1}{P}\left[rPY - \frac{1}{P}(-PY)\right]$$

which is equivalent to Eq. (2.12). It is seen that *the machine variable Y* *will have no choice but to vary as prescribed by the given machine equation* (2.12).[1]

As a second example, suppose that two dependent machine variables X and Y are to vary with the machine time τ as prescribed by the set of machine equations derived in the example of Sec. 2.3, *viz.*,

$$\left.\begin{aligned} P^2X + 1.59PY + PX + 7.31Y + 0.5X &= 0.02 \\ P^2Y + 0.734PY + 0.49PX + 0.035Y + X &= 0 \end{aligned}\right\} \qquad (2.14)$$

In such problems, one machine equation is solved for the highest occurring derivative of each of the machine variables. This procedure is applicable even if the highest derivatives of two or more machine variables do not occur in more than one of the equations. It is always possible to differentiate one of the other equations until the desired derivative is obtained.

In the present case, one of the machine equations (2.14) is solved for each of the variables P^2X and P^2Y:

$$\left.\begin{aligned} P^2X &= -(1.59PY + PX + 7.31Y + 0.5X - 0.02) \\ P^2Y &= -(0.734PY + 0.49PX + 0.035Y + X) \end{aligned}\right\} \qquad (2.15)$$

Once the desired relations between the voltages in the machine have been expressed in the form (2.15), they can be easily established physically. Each Eq. (2.15) is taken up in turn.

Each of the terms in (2.15) will be a voltage or machine variable. Starting with the first equation (2.15), the voltage P^2X must appear as the *sum*, multiplied by -1, of the voltages

$$1.59PY,\ PX,\ 7.31Y,\ 0.5X,\ -0.02 \text{ machine unit} \qquad (2.16$$

This end is achieved by applying the voltages

$$\frac{1.59}{4}PY,\ PX,\ \frac{7.31}{10}Y,\ 0.5X,\ -0.02 \text{ machine unit} \qquad (2.17)$$

to the correct input terminals of a *summing amplifier* (see Fig. 2.4*a*), which now performs the required multiplications by -1, -4, and -10 as well as the addition. The voltages (2.17) are not independent of the voltages P^2X and P^2Y but must be obtained from them through successive integrations and multiplications by constants. As before, the voltage P^2X is treated as if it were already known and is applied to the input of an

---

[1] Equation (2.12) is the equation of damped harmonic motion. If $Y_0 = 1$, $(PY)_0 = 0$, the solution is found to be

$$Y = Y_0e^{-b\tau}\cos\omega\tau \qquad b = \frac{r}{2} \qquad \omega = \sqrt{1 - \frac{r^2}{4}}$$

This solution is easily recorded and verified for a number of values of r. For this reason, this simple problem is often used to check a new computer.

integrator whose output must then be equal to $-PX$. In this manner, *successive integrations* by means of cascaded integrators (Fig. 2.4*b*) yield the voltages $-PX$ and X at the output terminals of successive integrators. Similarly, successive integrations of P^2Y yield the voltages $-PY$ and Y. Each electronic integration results in a change of sign. It should be noted that an integrator may be used to effect a multiplication by a constant factor as well as an integration.

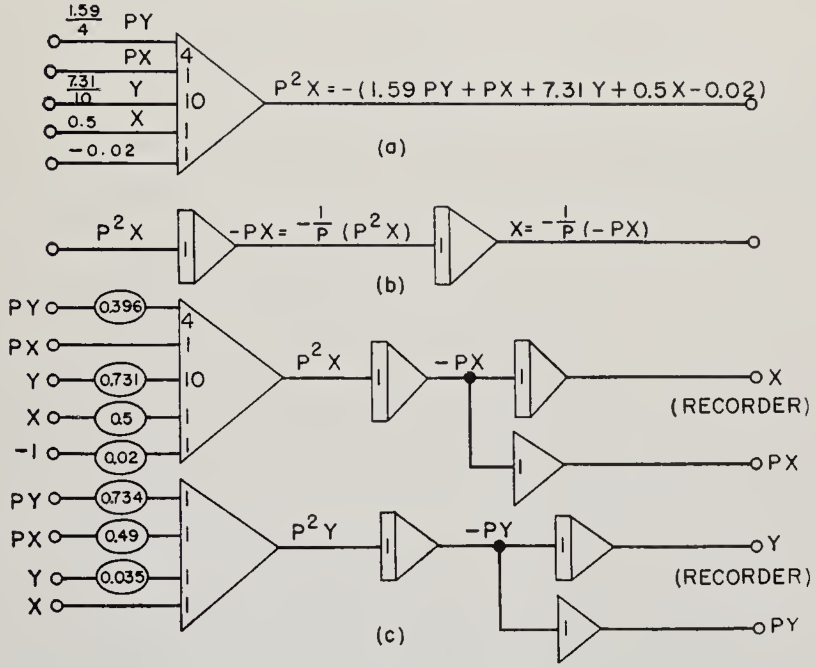


FIG. 2.4. Solution of the set of differential equations

$$P^2X + 1.59PY + PX + 7.31Y + 0.5X = 0.02$$
$$P^2Y + 0.734PY + 0.49PX + 0.035Y + X = 0$$

It is interesting to note in the above computer setup how all multiplications by factors greater than 1 have been performed at amplifier inputs. The voltages $1.59PY$ and $7.31Y$ do not occur at amplifier output terminals where they might cause overloading.

It remains to *multiply* the voltages $-PX$, X, $-PY$, and Y by the proper positive and negative constants by means of phase-inverting *amplifiers* and *potentiometers* to obtain the voltages (2.17), which are the terms to be summed by the first summing amplifier.

This completes the simulation of the first equation (2.15). A block diagram for the second equation is obtained in essentially the same manner. The machine variable P^2Y appears at the output of a second summing amplifier which sums and inverts the voltages $0.734PY$, $0.49PX$, $0.035Y$, X (machine units), obtained by successive integrations and multiplications of P^2X and P^2Y, as before. A complete, *composite* block diagram corresponding to the machine equations (2.15) is shown in Fig. 2.4*c*.

The composite block diagram should be reexamined with a view to possible improvements in the manner discussed further above.

Figure 2.4*c* shows the complete computer setup except for the initial-condition settings. The latter are discussed in Sec. 2.6, and a composite block diagram with the initial-condition settings indicated is shown in Fig. 2.5*a*.

Several remarks on such block diagrams in general are in order. It is seen that *a separate block diagram has been drawn for each of the machine equations.* Each machine variable appears at an output terminal on the right of one of the equation block diagrams and, usually, at one or more of the input terminals on the left. Several other voltages (auxiliary machine variables) also appear at the right and left and may or may not be recorded. The only input terminals will be those connected to a source of constant voltage or an external function generator "driving" the computer. *The interconnections between terminals on the right and left or the interconnections between block diagrams of different equations need not and should not be drawn.* This procedure will result in much greater clarity of the composite block diagram and will also aid in the handling of the records. This fact is illustrated by Fig. 2.5*b*, which shows the block diagram of Fig. 2.5*a* with the interconnections drawn in. It is readily seen that the arrangement in Fig. 2.5*a* is more easily interpreted and also permits changes to be made in one equation block diagram without making it necessary to redraw the entire figure.

Whenever a composite block diagram has been established, it should be carefully checked against the corresponding equations. An inspection of Fig. 2.5*a* shows that the machine variables X and Y will have to vary with time as prescribed by Eqs. (2.15) and thus by Eqs. (2.14). Records of the voltages X and Y will, then, constitute a solution of the problem in question.

2.6. INITIAL CONDITIONS

Before each computer "run" is started, it is necessary to establish at the output of each integrator in the computer a voltage equal to the given initial value of the machine variable which is to appear at that point. The machine variables will then be certain to start to vary from the correct initial values given by the problem.

Naturally, the machine variables could not in general be maintained constant at their initial values before each run while the computer is set up to solve a differential equation. Accordingly, the computer is made inoperative by disconnecting the integrator amplifier grids from the input networks while the initial conditions are set up. This is called the RESET condition of the computer. At the beginning of a computer run, a relay

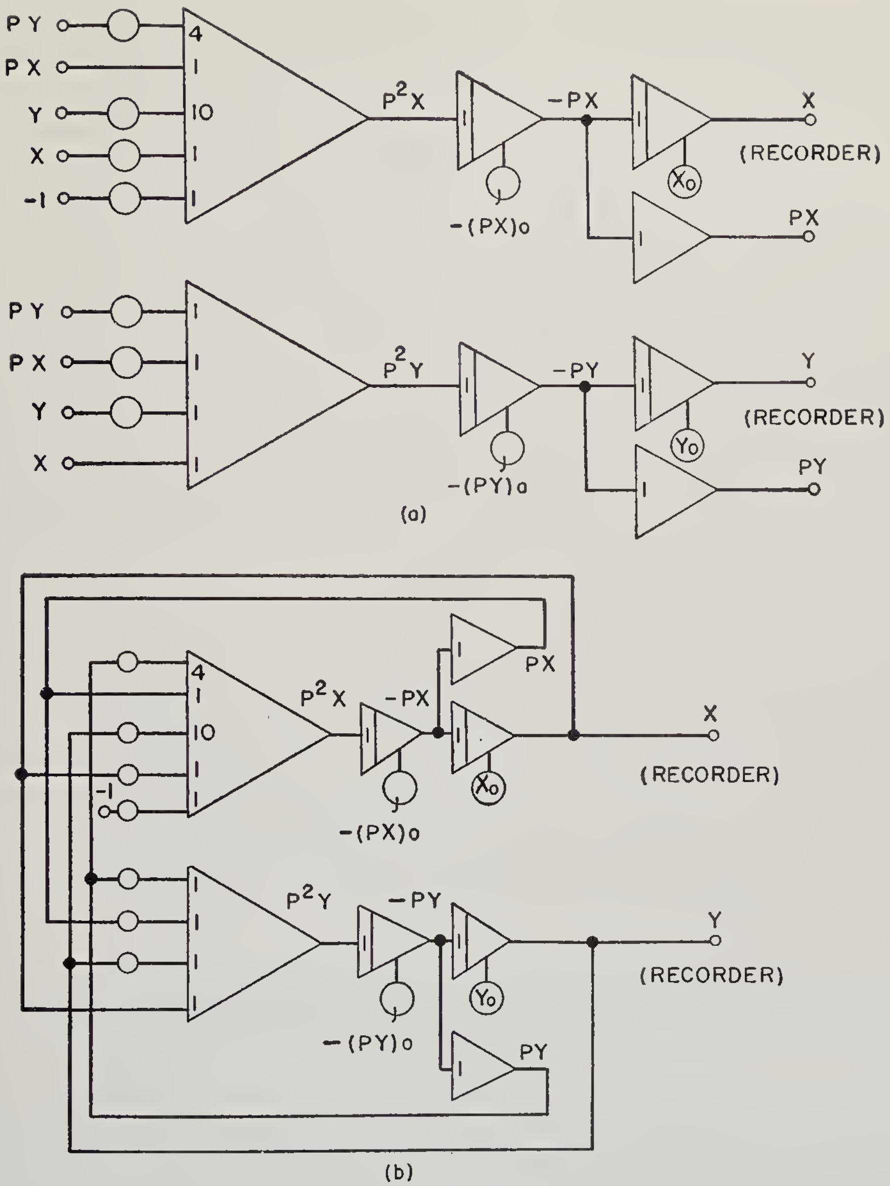


FIG. 2.5. Solution of the set of differential equations

$$P^2X + 1.59PY + PX + 7.31Y + 0.5X = 0.02$$
$$P^2Y + 0.734PY + 0.49PX + 0.035Y + X = 0$$

system connects the amplifier grids to the input networks and at the same time removes the initial-condition voltage sources. All integrating capacitors are now charged up to the correct voltages, and the machine variables are free to vary according to the differential equations set up. The exact circuits for the relay system and the initial-condition voltage sources are described in Sec. 7.1.

The number of initial conditions will usually equal the number of integrators used for the problem in question, and each integrator output must be set to the corresponding initial-condition voltage.

Summing amplifiers and phase inverters need not be made inoperative in the RESET conditon; their output voltages will, in any case, assume the correct initial values corresponding to the initial values of the voltages established at the amplifier input terminals by preceding integrators.

The correct initial values of the machine variables are easily found by substituting the given initial values of the original problem variables into the transformation equations. In this manner, initial values for the machine variables and their derivatives will be found. It happens frequently, however, that an integrator output voltage, which must be set to the correct initial value, is not simply equal to a machine variable or its derivative. The output of a summing integrator may well be a function of the machine variables and their derivatives. In such cases the correct initial value of this function must be computed and set into the integrator.

Example. Consider a summing integrator performing the operation

$$-\frac{1}{P}(-P^3Y + 4PX)$$

Its output voltage will be $P^2Y - 4X$. Given the initial conditions $(P^2Y)_0 = 0.5$, $X_0 = 0.25$, the correct initial condition setting is $0.5 - 1 = -0.5$.

Figure 2.1 shows the manner in which initial conditions are specified in block diagrams. Figures 2.3*e* and 2.5 show how initial conditions are introduced in the examples of Sec. 2.5. Whenever an integrator is shown without any initial-condition input, the initial value of the integrator output voltage is simply zero.

2.7. D-C ANALOG-COMPUTER SOLUTION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

D-c analog-computer setups for the solution of linear differential equations with variable coefficients and of nonlinear differential equations will require the use of computing elements capable of multiplying two machine variables and of generating functions of machine variables (see Chap. 6). Such computing elements will appear in addition to the "linear" computing elements used for the solution of linear differential equations with constant coefficients. No particular difficulties will be encountered once the limitations on the input and output voltages of the various devices used for multiplication and function generation are understood. Specifically, since the value of a function of a machine variable might exceed the dynamic range allowable in the computer even though the machine variable itself is well within that range, it is often necessary to rewrite the machine equation in the manner shown in Sec. 2.4 so as to keep the output of a function generator within its working range throughout a computation.

Note again that all initial choices of scale factors are preliminary in nature and that the scale factors, including the time scale, may have to be readjusted after a few trial runs in order to obtain the greatest possible accuracy and convenience in each case.

Solution of the Mathieu Equation. Solutions of Mathieu's differential equation

$$\frac{d^2y}{dt^2} + \omega_0^2(1 + \epsilon \cos \omega_m t)y = 0 \tag{2.18}$$

where ϵ, ω_0, and ω_m are constant parameters, are of interest in connection with the study of frequency-modulated oscillations. Inspection of Eq. (2.18) shows that Mathieu's equation is essentially the performance equation of a harmonic oscillator having a "stiffness coefficient" containing a time-variable term.

Assuming that

$$0 < \omega_0 \leq 50 \qquad 0 < \omega_m \leq 100 \qquad 0 < \epsilon \leq 1 \tag{2.19}$$

in Eq. (2.18) and that solutions will be of interest for $0 \leq t \leq 15$, the independent variable may be related to the machine time τ by the transformation equation

$$t = \frac{\tau}{10} \tag{2.20}$$

which leads to the relation

$$\frac{d}{dt} = 10P \tag{2.21}$$

The dependent variable y may be represented in the computer by a voltage of Y machine units such that

$$y = Y \tag{2.22}$$

The scale factor a_y is thus equal to 1, but this figure can be easily raised if an overload should occur in the computer for large values of the initial condition settings.

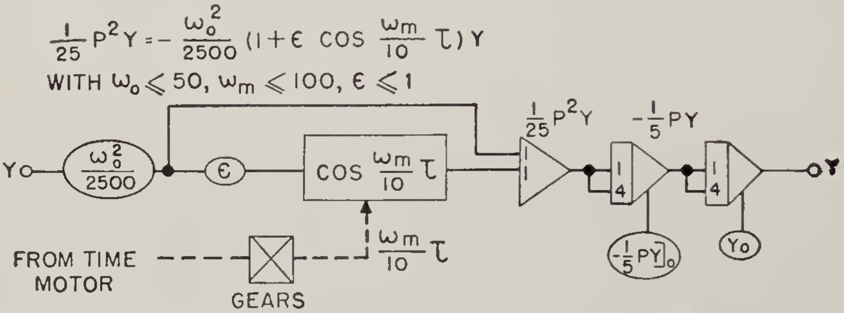
By substituting the transformation equations (2.21) and (2.22) into Eq. (2.18), one obtains the machine equation

$$100P^2Y = -\omega_0^2\left(1 + \epsilon \cos \frac{\omega_m}{10}\tau\right)Y \tag{2.23}$$

Since, according to (2.19), the value of ω_0^2 may be as large as 2,500, it is convenient to rewrite the machine equation (2.23) in the form

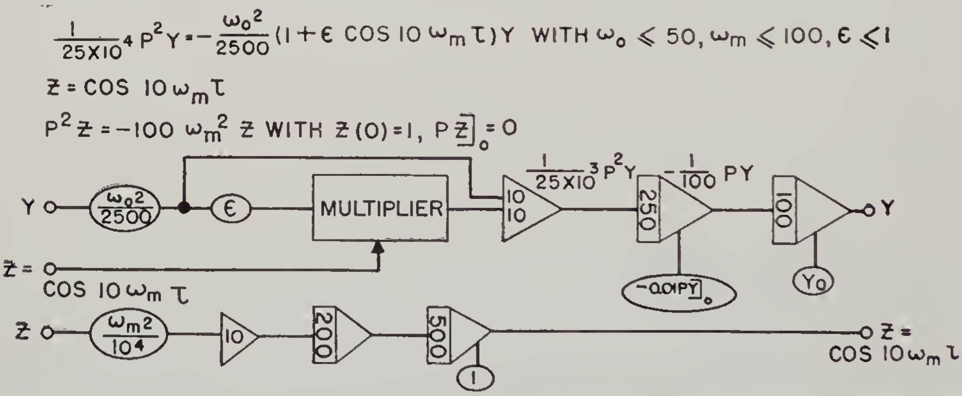
$$\frac{1}{25}P^2Y = -\frac{\omega_0^2}{2{,}500}\left(1 + \epsilon \cos \frac{\omega_m}{10}\tau\right)Y \tag{2.24}$$

so that a coefficient-setting potentiometer can be used for multiplication by $\omega_0^2/2{,}500$. The block diagram of Fig. 2.6*a* shows how a voltage equal to the expression on the right of Eq. (2.23) is formed by means of a summing amplifier having the input voltages $\omega_0^2/2{,}500$ and $(\omega_0^2/2{,}500)\ \epsilon \cos (\omega_m\tau/10)$. A cosine potentiometer capable of continuous 360-deg rotation (see Secs. 6.9 and 6.11) is driven by a clock motor at a speed $\omega_m/20\pi$ rps; this speed is adjusted in accordance with the "modulation circular frequency" ω_m by means of gears.



$\frac{1}{25}P^2Y = -\frac{\omega_0^2}{2500}(1+\epsilon \cos \frac{\omega_m}{10}\tau)Y$
WITH $\omega_0 \leqslant 50$, $\omega_m \leqslant 100$, $\epsilon \leqslant 1$
Y
$\frac{\omega_0^2}{2500}$
ϵ
$\cos \frac{\omega_m}{10}\tau$
$\frac{1}{25}P^2Y$
$-\frac{1}{5}PY$
1
1
1
4
1
4
Y
FROM TIME MOTOR
GEARS
$\frac{\omega_m}{10}\tau$
$-\frac{1}{5}PY]_0$
Y_0

FIG. 2.6*a*. Block diagram of a computer setup for the solution of Mathieu's differential equation (2.18) on a slow time scale. The cosine potentiometer driven by the clock motor is of a type capable of continuous 360-deg rotation (see Sec. 6.11). The potentiometer shaft makes a complete revolution every $20\pi/\omega_m$ seconds. Potentiometer-loading corrections must be observed.



$\frac{1}{25\times10^4}P^2Y = -\frac{\omega_0^2}{2500}(1+\epsilon \cos 10\,\omega_m\tau)Y$ WITH $\omega_0 \leqslant 50$, $\omega_m \leqslant 100$, $\epsilon \leqslant 1$
$z = \cos 10\,\omega_m\tau$
$P^2z = -100\,\omega_m^2\,z$ WITH $z(0) = 1$, $Pz]_0 = 0$
Y
$\frac{\omega_0^2}{2500}$
ϵ
MULTIPLIER
10
10
$\frac{1}{25\times10^3}P^2Y$
$-\frac{1}{100}PY$
250
100
Y
$-0.01PY]_0$
Y_0
$z =$ COS $10\,\omega_m\tau$
z
$\frac{\omega_m^2}{10^4}$
10
200
500
1
$z =$ COS $10\,\omega_m\tau$

FIG. 2.6*b*. Block diagram for the solution of Mathieu's differential equation (2.18) on a repetitive computer, such as the MacNee machine described in Sec. 8.5. The multiplier indicated is a crossed-fields electron-beam multiplier (see Sec. 6.2). Note how the function $Z = \cos 10\omega_m\tau$ is generated by implicit solution of the differential equation $P^2Z = -100\,\omega_m^2 Z$. Integrating capacitors of 0.1 μf are used.

The machine variables $-\frac{1}{5}PY$ and Y are obtained by successive integrations of the voltage $\frac{1}{25}P^2Y$; the correct initial conditions must be introduced in each integrator in the manner discussed in Sec. 2.6.

Repetitive-computer Setup for Mathieu's Equation. As an interesting alternative, Fig. 2.6*b* shows the block diagram of a repetitive-computer setup for the solution of Mathieu's equation (2.18). Since the repetitive display of the solutions requires the use of a fast time scale (see Secs. 1.1 and 8.5), the transformation equation (2.20) is replaced by

$$t = 10\tau \tag{2.25}$$

The machine equation becomes

$$\frac{1}{25 \times 10^4} P^2 Y = -\frac{\omega_0^2}{2{,}500}(1 + \epsilon \cos 10\omega_m \tau)Y \tag{2.26}$$

The computer setup of Fig. 2.6*b* is similar in principle to that for the "slow" d-c analog computer (Fig. 2.6*a*), but an electronic multiplier is used to multiply ϵ by a voltage

$$Z = \cos 10\omega_m \tau \tag{2.27}$$

This latter voltage is obtained as a function of the machine time τ as the solution of the differential equation

$$P^2 Z = -100\omega_m^2 Z \qquad \text{where } Z(0) = 1 \text{ and } PZ]_0 = 0 \tag{2.28}$$

This technique of obtaining functions of the machine time τ through implicit computation (see also Sec. 6.6) is widely usable, and its application is by no means restricted to repetitive computers. It will be an interesting exercise for the reader to modify the setup of Fig. 2.6*a* so as to apply this technique. The resulting computer setup is particularly useful if function potentiometers are not available.

Solution of Legendre's Equation. Solutions of Legendre's differential equation

$$(1 - t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + n(n + 1)y = 0 \tag{2.29}$$

where n is a real integer, are important for solving problems in potential theory and antenna theory in terms of spherical coordinates.[1] The initial conditions chosen for n between 0 and 6 are given by the following table:

$$\left.\begin{array}{rccccccc} n = & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ y(0) = & 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 & -\frac{5}{16} \\ \left.\frac{dy}{dt}\right]_0 = & 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{15}{8} & 0 \end{array}\right\} \tag{2.30}$$

This choice of initial conditions will result in[1]

$$y(1) = 1 \tag{2.31}$$

It will be convenient to represent the independent variable t in the computer by both the machine time τ *and* a voltage or shaft displacement

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[1] For a discussion of the solutions of Legendre's equation in terms of an eigenvalue problem, see L. Pauling and E. Wilson, *Introduction to Quantum Mechanics*, McGraw-Hill, New York, 1935, or H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry*, Van Nostrand, Princeton, N.J., 1946.

T (see also Sec. 2.3). The dependent variable y will be represented by a machine variable Y.

Assuming that the solutions are of interest only for

$$0 \leq t \leq 1 \qquad -1 \leq y \leq 1 \tag{2.32}$$

the problem variables may be related to the corresponding machine variables by means of the transformation equations

$$\left.\begin{array}{ll} y = \dfrac{1}{a_y} Y & t = T \\ t = \dfrac{\tau}{5} & \dfrac{d}{dt} = 5P \end{array}\right\} \tag{2.33}$$

The scale factor a_y of the dependent variable y is left undetermined for the time being, since estimates for the maximum value of dy/dt may not be available. The machine equation becomes

$$25P^2Y = \frac{1}{1 - T^2}[10TPY - n(n+1)Y] \qquad \text{where } T = \frac{\tau}{5} \tag{2.34}$$

It is seen that the machine equation (2.34) or, in fact, *any machine equation corresponding to a single linear and homogeneous ordinary differential equation (with either constant or variable coefficients) is not affected by the scale factor of the dependent variable. The corresponding computer setup is affected by the choice of this scale factor only as far as the initial-condition settings are concerned.* It is, then, particularly convenient to find scale factors like a_y in (2.33) through a trial-and-error process, since the computer setup proper will not be affected by changes in a_y. This is not true in the case of computer setups for nonlinear equations.

The block diagrams of Fig. 2.7 show two alternative d-c analog-computer setups for the solution of Legendre's equation for values of n up to 7. In each case, a voltage proportional to P^2Y is obtained as the sum of two voltages corresponding to the two terms on the right of Eq. (2.34). The machine variable Y is then obtained by successive integrations. Initial conditions must be introduced in each integrator as described in Sec. 2.6.

The computer setup of Fig. 2.7*a* is due to Hagelbarger, Howe, and Howe.[1] The machine equation (2.34) is established in a particularly straightforward manner through the use of function generators consisting of phase-inverting amplifiers whose feedback networks are varied stepwise in accordance with the desired function of the time τ by 40-step tap switches (see Sec. 6.9). The transfer functions of such function generators are always negative, but the absolute values of the transfer functions are not limited to the value 1 as in the case of simple potentiometer function generators.

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[1] Hagelbarger, D. W., C. E. Howe, and R. M. Howe, Investigation of the Utility of an Electronic Analog Computer in Engineering Problems, *UMM*-28, Engineering Research Institute, University of Michigan, Ann Arbor, Apr. 1, 1949.

Nevertheless, since the function $1/(1 - T^2)$ increases indefinitely for values of T approaching unity, it is necessary to replace the function by

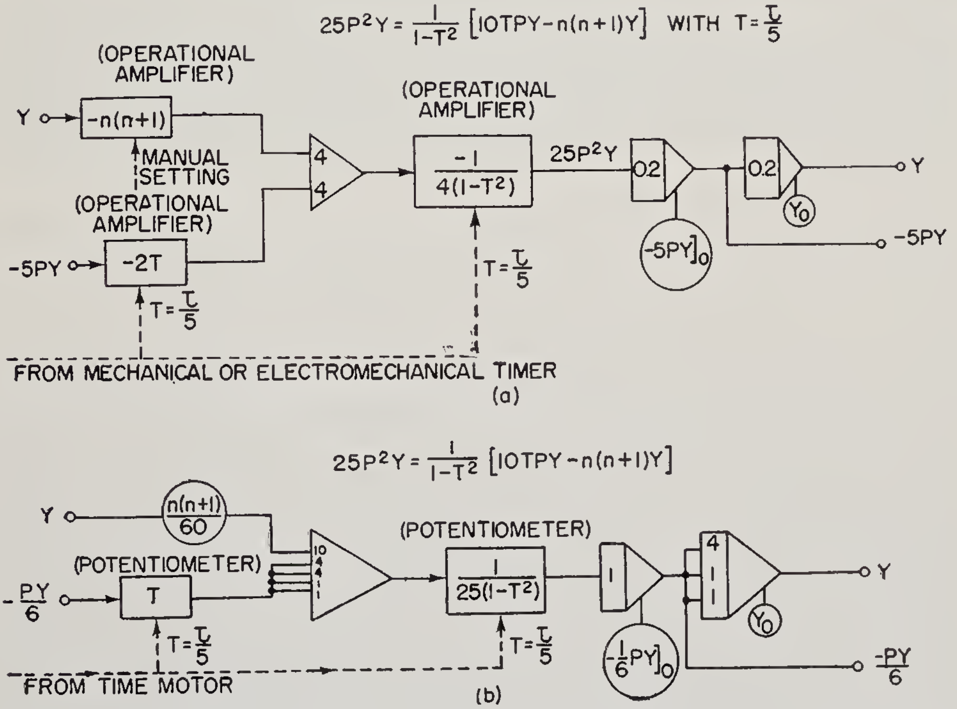

FIG. 2.7. Block diagrams of two d-c analog-computer setups for the solution of Legendre's differential equation (2.29). The function generators indicated in the setup of Fig. 2.7*a*, due to Hagelbarger, Howe, and Howe (see footnote, page 50), are operational amplifiers whose feedback networks are switched by a 40-step tap switch (see Sec. 6.9) in accordance with the desired functions. Figure 2.7*b* is an alternative setup using potentiometer function generators whose transfer functions must be less than unity.

its value for $T = 0.975$ for all larger values of T. This arrangement limits the absolute value of the transfer function of the function generator to a safe value.

The errors due to this limitation as well as those due to the stepwise generation of the functions $2T$ and $1/4(1 - T^2)$ are quite small, as shown by the results below.

Figure 2.7*b* shows a computer setup for the same machine equation based on the use of tapped potentiometer-type function generators driven by a clock motor. The tapped potentiometer-type universal function generators must be set up with the load connected, as described in Sec. 6.10.

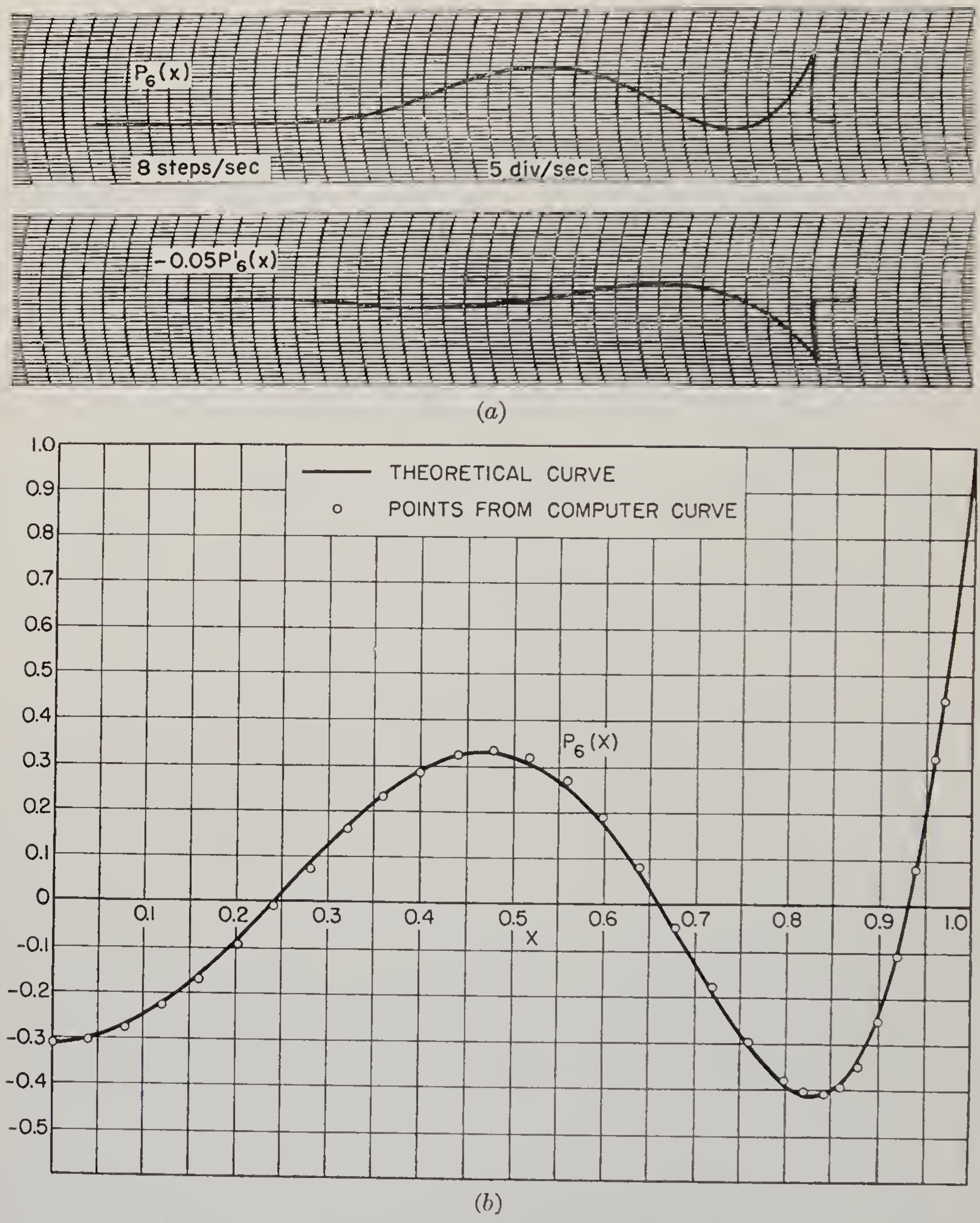

FIG. 2.8. D-c analog-computer solution of Legendre's differential equation for $n = 6$, and comparison with analytical results (from Hagelbarger, Howe, and Howe, *op. cit.*).

The function $1/(1 - T^2)$ is replaced by its value for $T = \sqrt{1 - 1/25}$ for all larger values of the argument T, so that the transfer function of the function generator will never exceed unity.

The computer setups of Fig. 2.7 are capable of solving Eq. (2.29) for values of n between 0 and 7. Trial solutions will show that for $n = 7$, the maximum value of the derivative dy/dt will be about 15, so that the scale factor a_y cannot be larger than $1/15$ machine unit per unit for the computer setup of Fig. 2.7*a* and 2 machine units per unit for the computer setup of Fig. 2.7*b*.

Figure 2.8*a* shows a sample computer record for $n = 6$ obtained by Hagelbarger, Howe, and Howe with the computer setup of Fig. 2.7*a*, and Fig. 2.8*b* shows a comparison between the d-c analog-computer solution for $n = 6$ and the analytically obtained result

$$y = \tfrac{1}{16}(231t^6 - 315t^4 + 105t^2 - 5) \qquad (2.35)$$

for $n = 6$.

2.8. D-C ANALOG-COMPUTER SOLUTION OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

The Van der Pol Equation. The solution of the Van der Pol equation

$$\frac{d^2y}{dt^2} + \mu(y^2 - 1)\frac{dy}{dt} + y = 0 \qquad (2.36)$$

where μ is a constant parameter, may be used, for instance, to describe the build-up of oscillations in certain nonlinear electrical circuits and furnishes an interesting example of the d-c analog-computer solution of a nonlinear differential equation. The Van der Pol equation (2.36) is seen to resemble the performance equation (2.12) of a damped harmonic oscillator but has a nonlinear damping term which will tend to build the amplitude of the oscillations up for small values of y and to decrease the amplitude for large values of y. If an oscillator of this type is given a small initial displacement y, periodic oscillations of constant amplitude will result after an initial transient.

Suppose that $\mu = 0.1$. From previous knowledge or through a trial-and-error process, it may be said that the variable y will not exceed the value 2 appreciably. It is reasonable to relate the problem variables y and t to the corresponding machine variables Y and τ through the transformation equations

$$y = 2.5Y \qquad t = \tau \qquad (2.37)$$

and the resulting machine equation may be written in the form

$$P^2Y = (0.1 - 0.625Y^2)PY - Y \qquad (2.38)$$

The block diagram of Fig. 2.9*a* shows how the machine variable P^2Y is formed through the processes of addition and successive multiplication, and how $-PY$ and Y are obtained through integration of P^2Y.

Only linear servo-driven potentiometers are required for the computer setup of Fig. 2.9*a*. A phase-inverting amplifier is used to avoid errors due to excessive loading of the first servo-driven potentiometer. If a function potentiometer is available to perform the required multiplication by Y^2, it is possible to save one operational amplifier, as shown in Fig.

2.9*b*. Errors due to amplifier drift, distortion, etc., may thus be reduced. Since Y^2 will be less than 1 throughout the computation, it is possible to use the function potentiometer without any necessity for rewriting the machine equation (see Sec. 2.2).

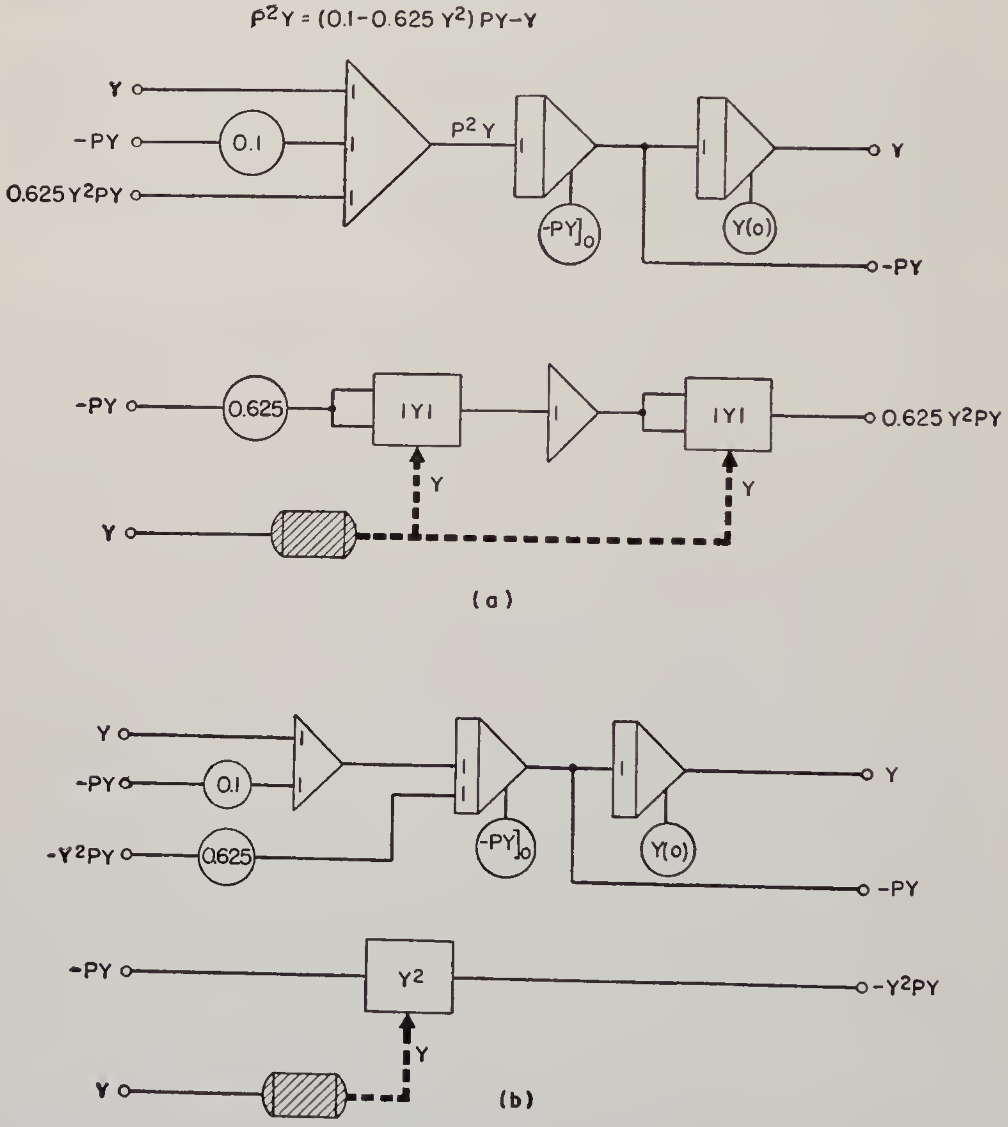

FIG. 2.9. Computer setups for the solution of the Van der Pol equation. Mechanical connections (servo shafts) are indicated as broken lines. In Fig. 2.9*a*, linear center-tapped potentiometers are used for multiplication by $|Y|$ (circuit 5.1, Table 5). Since all servo-driven potentiometers are equally loaded, one may eliminate potentiometer-loading effects by loading the servo follow-up potentiometer (Sec. 6.9).

Solution of the Rayleigh Equation. If a new dependent variable z is introduced in Eq. (2.36) through the transformation

$$\frac{dz}{dt} = y \tag{2.39}$$

one obtains the Rayleigh equation

$$\frac{d^2z}{dt^2} + \mu\left[\frac{1}{3}\left(\frac{dz}{dt}\right)^3 - \frac{dz}{dt}\right] + z = 0 \tag{2.40}$$

which was originally used for the description of certain acoustical phenomena.

The d-c analog-computer solution of the Rayleigh equation (2.40) is of interest not only for its own sake but will, in view of the transformation (2.39), also furnish the solution of the Van der Pol equation (2.36). This may be particularly useful in view of the fact that the d-c analog-computer solution of Eq. (2.40) requires only the use of a single function generator without the necessity for any multiplications of machine variables. This fact has been used by MacNee[1] for solving either Eq. (2.40) or (2.36) with a comparatively simple setup on his repetitive computer (see Sec. 8.5).

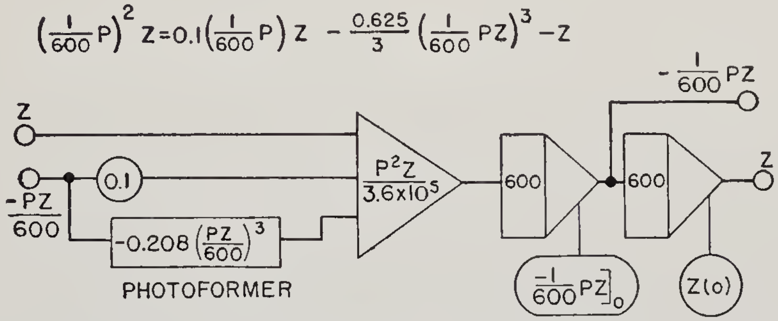
$\left(\frac{1}{600}P\right)^2 Z = 0.1\left(\frac{1}{600}P\right)Z - \frac{0.625}{3}\left(\frac{1}{600}PZ\right)^3 - Z$
$-\frac{1}{600}PZ$
Z
$-\frac{PZ}{600}$
0.1
$-0.208\left(\frac{PZ}{600}\right)^3$
PHOTOFORMER
$\frac{P^2Z}{3.6\times10^5}$
600
600
Z
$\frac{-1}{600}PZ]_0$
Z(0)

FIG. 2.10. Block diagram of a repetitive-computer setup for the solution of the Rayleigh equation (2.40). The same computer setup may be used for the solution of the Van der Pol equation (2.36) through the use of the relations (2.45) and (2.46) given in the text. Integrating capacitors of 0.1 μf may be used.

Assume again that $\mu = 0.1$. For a repetitive computer solution of Eq. (2.40), the machine variables Z and τ may be introduced through the transformation equations

$$z = 2.5Z \qquad t = 600\tau \tag{2.41}$$

The corresponding machine equation becomes

$$\frac{1}{3.6 \times 10^5}P^2Z = 0.1\left(\frac{1}{600}PZ\right) - \frac{0.625}{3}\left(\frac{1}{600}PZ\right)^3 - Z \tag{2.42}$$

Figure 2.10 shows the block diagram of the simple repetitive-computer setup for Eq. (2.42); a photoformer (see Sec. 6.6) is used to generate the function $0.208(\frac{1}{600}PZ)^3$.

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[1] MacNee, A. B., An Electronic Differential Analyzer, *Proc. IRE*, **37**: 1315, 1949.

The same computer setup may be used to solve the Van der Pol equation (2.36) through the use of the transformation equations

$$y = 2.5Y \qquad t = 600\tau \tag{2.43}$$

and the relation

$$\tfrac{1}{600}PZ = Y \tag{2.44}$$

corresponding to the transformation (2.39). From (2.42), (2.43), and (2.44) it follows that

$$PZ = 600Y = 240y \tag{2.45}$$

and

$$\left.\begin{aligned} Z &= 0.1Y - \frac{0.625}{3}Y^3 - \frac{1}{600}PY \\ z &= 0.1\left(y - \frac{1}{3}y^3\right) - \frac{dy}{dt} \end{aligned}\right\} \tag{2.46}$$

The last two relations will yield the correct initial conditions for Z and PZ from those given for y and dy/dt.

Remarks on the Intelligent Choice of Machine Variables. The use of transformations like (2.39) is often very helpful for simplifying the d-c analog-computer solution of a given problem, especially when only a limited number or choice of computing elements is available.

Again, an intelligent choice of scale factors including the time-scale factor will help one to realize the full accuracy capabilities of the d-c analog computer used. If, for instance, the value of the parameter μ in Eq. (2.36) or (2.40) is increased, the maximum value of the problem variable $z = dy/dt$ will tend to increase without a proportional increase in the maximum value of y. In such cases, the use of a slower time scale is indicated if the accuracy of the computation is to be maintained.

Plots of Solutions in the Phase Plane. The use of servo plotting boards (see Sec. 7.3) for recording d-c analog-computer solutions permits one to plot a problem variable, y, say, against its derivative dy/dt. Studies of such "phase plots" are of particular interest in connection with the theory of nonlinear oscillations.[1] In the case of repetitive computers, phase plots may be obtained conveniently by feeding voltages proportional to Y and PY to the horizontal and vertical deflection plates of a cathode-ray oscillograph.[2]

2.9. SUMMARY OF SETUP PROCEDURE

The following is a step-by-step review of the setup procedure presented in this chapter. Further examples of applications, together with sample records, will be given in the following sections.

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[1] Minorsky, N., *Introduction to Non-Linear Mechanics*, Edwards Bros., Inc., Ann Arbor, Mich., 1947.

[2] MacNee, *op. cit.*

Step 1. *Obtain a complete statement of the problem in question for the record (see Sec.* 3.1). *This statement should include, on separate sheets or cards, a clear indication of the object of the study; mathematical relations (differential equations, etc.); initial conditions; numerical values of parameters, given functions, etc.; parameters to be varied; any available solutions or checks; estimated ranges of variables (and their derivatives, if possible).*

Step 2. *Determine tentative scale factors for each variable and function, and write the corresponding transformation equations on a sheet or card.*

Step 3. *Choose the time scale, and write the transformation equation for the independent variable. Using the transformation equations, write the machine equations on another sheet or card. Solve one machine equation for the highest occurring derivative of each machine variable, differentiating some of the equations if necessary. Do not forget to transform initial values, limiting levels, etc., to machine variables.*

Step 4. *Using separate sheets or cards, draw a computer block diagram for each machine equation. Choose recorder speed and voltage scales.*

Step 5. *Review the combined equation block diagrams. Revise setup, scale factors, and circuit values where necessary to avoid excessive amplifier output currents (Sec.* 2.4), *to maintain the frequency response of summing amplifiers (Sec.* 4.2) *or computer servomechanisms (Sec.* 6.8), *etc.*[1] *Rewrite equations where necessary.*

Step 6. *Check whether the relations expressed by the revised block diagram reduce to the correct differential equations of the problem.*

Step 7. *Connect the various computing elements by means of the patch cords, following the revised block diagram, equation by equation. Do this with a view to short, convenient connections. Carefully note the number of each actual computing element on the block diagram for reference.*

Step 8. *Set potentiometers, initial values, limiting levels, and function generators according to the block diagram. Correct settings for potentiometer loading where necessary. Set or check recorder calibration.*

Step 9. *Check the entire setup. If at all possible, have a second person check steps* 2 *to* 8 *independently. A good check is also to rederive the machine equations from the computer setup.*

Step 10. *The machine is now ready for trial runs. Revise setup and scale factors where necessary. Keep a careful record of setup and scale-factor changes. Colored dummy plugs may be used to indicate temporarily removed patch cords.*

It is helpful to avoid, as far as possible, computer setups involving more integrations than absolutely needed and computer setups in which more than one amplifier output represents the same variable. In the latter case, it is advisable to test the sensitivity of the solution to a small gain change in one of the channels generating the variable. Similar precautions apply when a coefficient must be set by more than one coefficient-setting potentiometer or summing resistor.

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[1] Consult the manufacturer's frequency-response curves for various loads and feedback circuits.

A number of *methods for checking solutions* and *methods for checking the operating condition of the machine* are discussed in Sec. 7.6.

2.10. OTHER METHODS FOR HANDLING SCALE FACTORS

Introduction. Consider the d-c analog-computer solution of the differential equation

$$\frac{dy}{dt} = -y^2 \qquad \text{or} \qquad py = -y^2 \tag{2.47}$$

According to the procedure described in the foregoing sections, machine variables Y and τ representing the problem variables y and t, respectively, are introduced by the transformation equations

$$y = \frac{1}{a_y} Y \qquad t = \frac{1}{\alpha_t} \tau \tag{2.48}$$

where a preliminary choice of the scale factor a_y is made by observing the inequality

$$Y = a_y y \leq 1 \text{ machine unit} \tag{2.49}$$

in order to avoid overloading the computing elements. The resulting machine equation

$$PY = -\frac{1}{a_y \alpha_t} Y^2 \tag{2.50}$$

can be solved by the computer setup of Fig. 2.11*a* if the time scale chosen is sufficiently slow, so that $1/\alpha_t a_y \leq 1$. If a faster time scale (smaller α_t) is desired, the coefficient-setting potentiometer shown must be supplemented by a pair of amplifiers.

Procedure 1. The following alternative procedure for handling scale factors is of historical interest[1] and avoids the introduction of explicit symbols for the machine variables Y and τ:

1. A preliminary block diagram of a computer setup for the desired equation is drawn *without regard to scale-factor restrictions for the various computing elements* (see Fig. 2.11*b*). In general, the relations expressed by this block diagram cannot be enforced by the actual computing elements, since their dynamic ranges may be too small or too large.
2. A similar block diagram is now drawn (Fig. 2.11*c*) for machine

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[1] Caldwell, S. H., and V. Bush, A New Type of Differential Analyzer, *J. Franklin Inst.*, October, 1945, p. 255.

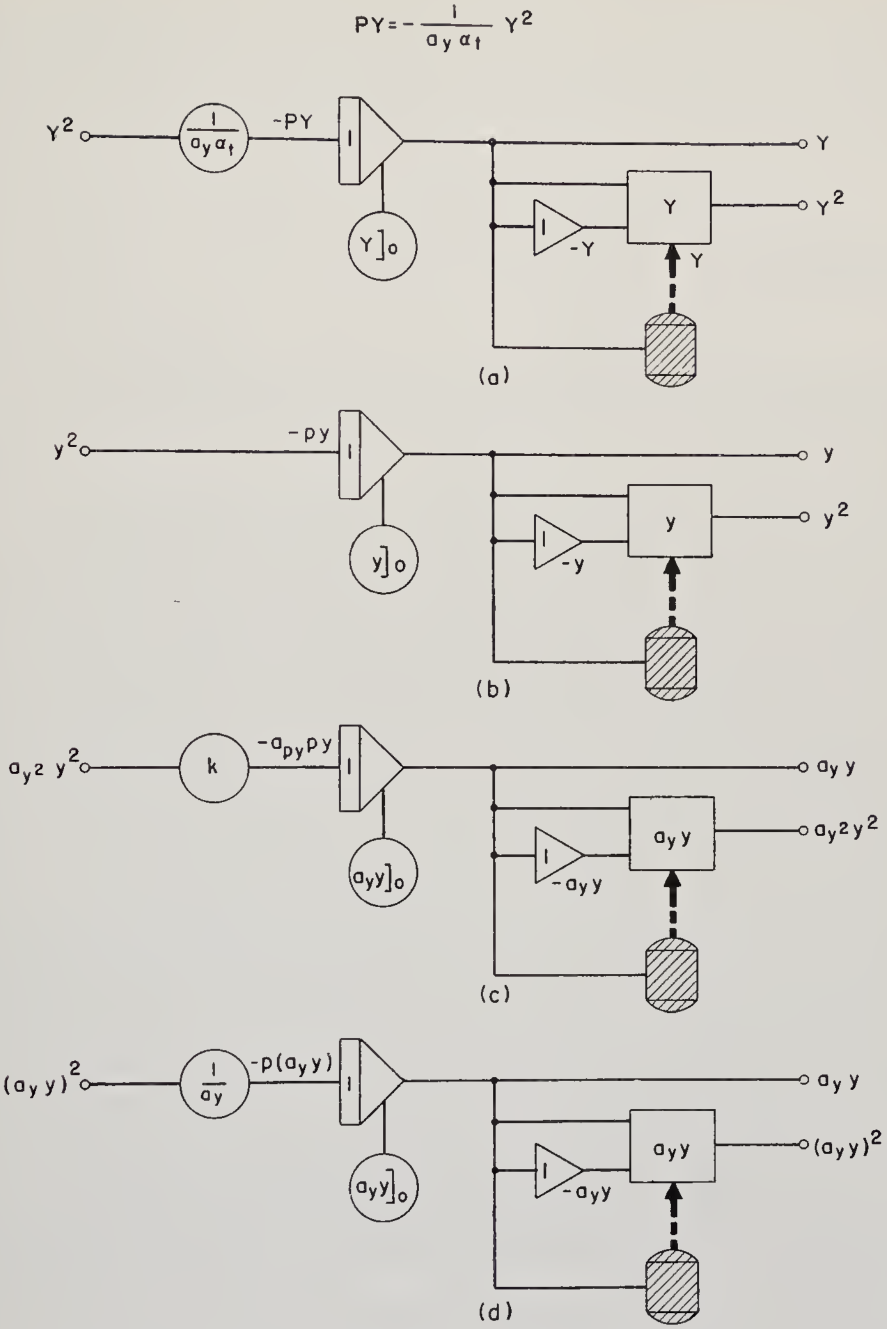

Fig. 2.11. Illustrating different methods of handling scale factors in the d-c analog-computer solution of the differential equation $dy/dt = -y^2$.

variables obtained by introducing (as yet unknown) scale factors for the voltages appearing at each terminal throughout the block diagram. In the notation chosen, $a_y y$ will be the machine variable representing y, $a_{y^2} y^2$ that representing y^2, $a_{py} py$ that representing py, etc. A coefficient-setting potentiometer (or one or more amplifiers) is provided in each feedback loop for adjusting the scale factors.

3. The scale factors to be chosen must satisfy the inequality (2.49). An inspection of the block diagram shows, however, that relations between the scale factors are enforced by the computing elements; thus, in the case of Fig. 2.11*c*, one has

$$a_{py} = \frac{a_y}{\alpha_t} \qquad \text{and} \qquad a_{y^2} = (a_y)^2 \tag{2.51}$$

If the resulting machine equation

$$-a_{py}py = ka_{y^2}y^2 \qquad \text{or} \qquad -\frac{a_y}{\alpha_t}\,py = k(a_y)^2y^2 \tag{2.52}$$

is to be equivalent to the given equation (2.47), one must have

$$k = \frac{1}{a_y\alpha_t} \tag{2.53}$$

which corresponds to the potentiometer setting in Fig. 2.11*a*.

The so-called *closing equations*[1] (2.51) and (2.53) together with the inequality (2.49) determine the potentiometer setting k as well as *all* the scale factors in Fig. 2.11*c* once the time scale α_t has been chosen; α_t is, again, assumed to satisfy $1/\alpha_t a_y \leq 1$. The values of the potentiometer setting and scale factors obtained in this manner are now entered in the block diagram (Fig. 2.11*c*), which is then used for the computer setup. It goes without saying that, with this method of scale-factor adjustment as with any other, the computer setup may require readjustments after trial runs indicate possible improvements in the choice of scale factors.

Procedure 2. The following method of handling scale factors is typical of several short-cut methods used instead of the procedure described in Secs. 2.2 to 2.4, to which it is essentially equivalent. The procedure to be described provides a rather convenient means for dealing with the troublesome time-scale factor.

1. For each given differential equation like (2.47), one finds a corresponding machine equation by multiplying and dividing each variable by the corresponding scale factor derived with the aid of Eq. (2.2). The resulting relation is the *machine equation for a one-to-one time scale*. Thus, if each y in Eq. (2.47) is multiplied and divided by the scale factor a_y, one obtains

$$\frac{1}{a_y}\,p(a_yy) = -\frac{1}{a_y^2}\,(a_yy)^2 \qquad \text{or} \qquad p(a_yy) = -\frac{1}{a_y}\,(a_yy)^2 \tag{2.54}$$

2. From each such machine equation, one may draw the *block diagram for a one-to-one time scale*, as in Fig. 2.11*d*.

3. *One may now introduce a time-scale factor* α_t (*time scale* α_t:1, *Sec.* 2.3) *simply by reducing the gain of each integrator in the block diagram in the*

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[1] Caldwell, S. H., and V. Bush, *ibid.* See also J. B. Resnick, Scale Factors for Analog Computers, *Product Eng.*, March, 1954.

ratio $1/\alpha_t$; this transformation will *not* affect the scale of any voltage or shaft rotation in the block diagram. In the case of the given example, the resulting block diagram is again that of Fig. 2.11*a*.

The last-described method of introducing changes of time scale is not as flexible as the basic method described in Secs. 2.3 and 2.4, but it is very convenient, especially with computers permitting changes of the integrator time constants R_1C, R_2C, . . . in Fig. 1.11. *If all integrator time constants are increased in the ratio* $\alpha_t:1$, *the time scale will also be increased in the ratio* $\alpha_t:1$.

Equivalence of Scaling Methods. As evidenced by the foregoing discussion, *the above methods for handling scale factors are equivalent.* Specifically, the determination of the potentiometer setting k from the closing equation (2.52) in the example corresponds to the computation of the machine equation (2.50).

For this reason, any of the block diagrams presented in Chaps. 2 and 3 are equivalent to those derived by one of the alternative setup methods if one remembers that the machine variables indicated (such as Y, $6X$, $P\Delta$, etc.) are equal to products of the corresponding problem variables by the respective scale factors (such as $a_y y$, $6a_x x$, $(a_\delta/\alpha_t)\,p\delta$, etc.).

Although no mathematical reasons exist for preferring any one of the setup procedures presented, *the methodical introduction of explicitly defined machine variables by a set of transformation equations appears to have certain striking advantages.* The authors' experience seems to indicate that this approach makes it easier to understand the operation of the machine, which, after all, handles voltages and shaft rotations rather than multiples of problem variables. The explicitly written transformation equations and machine equations are also particularly helpful in avoiding errors, especially errors associated with the effects of changing the time scale.

2.11. D-C ANALOG COMPUTERS AS EQUATION SOLVERS

The solution of a set of machine equations like

$$\left.\begin{aligned} X + Y &= 0 \\ X - Y &= 2 \end{aligned}\right\} \tag{2.55}$$

for the unknown voltages X and Y by means of an analog computer is a problem different from that of establishing the machine equation

$$X + Y = 0 \tag{2.56}$$

simultaneously with a differential equation like

$$\frac{dX}{d\tau} - Y = 0 \tag{2.57}$$

by means of an analog machine (solution of differential equations with constraints, Sec. 3.3). Since the differential equation (2.57) is essentially a relation involving only *changes* of the variable x, the machine equations (2.56) and (2.57) can often be satisfied *continuously* for any reasonable set of initial values of x. This is *not* true for the machine equations (2.55). Automatic equation solvers can enforce relations like (2.55) only *after an initial transient.* A machine of this type has actually solved a set of differential equations, *viz.*, the performance equations of the amplifiers, servomechanisms, and networks used for the computation. If these are correctly designed for the purpose, the machine variables X and Y will be made to approach the correct values corresponding to the solution of the given set of equations in the course of the "solution" of the performance equations. An *automatic equation solver* may thus be an analog computer comprising amplifiers and/or servomechanisms set up so that its performance equations cause the machine variables to approach the correct unknown values without any tendency toward *instability* or uncontrolled oscillations. *If* the machine variables converge to constant values within the dynamic range of the computing elements, these values will belong to the set of the correct unknowns. Generally applicable analytical conditions for the stable performance of such machines have been formulated for the case of linear equation solvers (p. 66), but more remains to be done in the general case.[1]

Ordinary equation solvers permit only the computation of real roots; an equation with complex roots must be transformed into a corresponding set of equations which can be solved for real unknowns, such as the absolute values and arguments of the desired roots.

D-c Analog Computers as Automatic Equation Solvers. Existing d-c analog computers are usually intended mainly for the solution of sets of differential equations. In machines specifically designed to solve problems involving neither differentiations nor integrations, the utilization of a-c amplitudes as machine variables may be preferable to d-c analog techniques; the use of a-c amplifiers will eliminate drift as an error source. Multipurpose d-c analog computers can, nevertheless, be very useful for the solution of certain types of equations and sets of equations. If very accurate results are required, approximate analog-computer solutions may still be useful for speeding up successive-approximation-type hand calculations.[2]

The solution of a single equation like

$$f(x) = 0$$

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[1] Murray, Francis J., *The Theory of Mathematical Machines*, King's Crown Press, New York, 1947.

[2] This applies, in particular, to the Gauss-Seidel method of solving linear simultaneous equations. See E. A. Goldberg and G. W. Brown, An Electronic Simultaneous Equation Solver, *J. Appl. Phys.*, **19**: 339, 1948.

for a real root x must be translated into the solution of a corresponding machine equation

$$F(X) = 0 \qquad \text{or} \qquad -F(X) = 0 \tag{2.58}$$

for an unknown voltage X within the operating range of the computer. Automatic equation solvers may be set up with ordinary d-c analog-computing elements by one of the following methods:

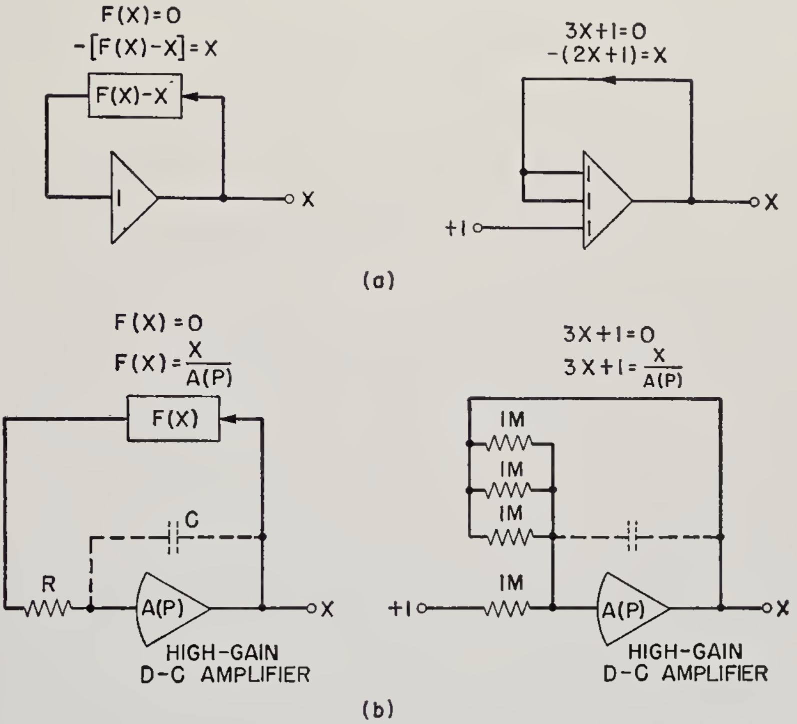

Fig. 2.12. D-c analog computers as automatic equation solvers. Stability tests are necessary in each case.

1. *Solution by implicit computation* (see also Sec. 1.2). The machine equation (2.58) is replaced by an equivalent equation

$$F(X) + X = X \qquad \text{or} \qquad -[F(X) - X] = X \tag{2.59}$$

and this relation is set up on the computer in the usual manner. This is shown in Fig. 2.12*a* for the general case as well as for a simple linear equation.

2. *Implicit computation by means of a high-gain d-c amplifier.* The computer setups of Fig. 2.12*b* imply the machine equation

$$F(X) = \frac{X}{A(P)} \tag{2.60}$$

which will tend to approximate the desired relation (2.58) if the amplifier d-c gain $A = A(0)$ is large, and the feedback loop is stable. The computer setups of Fig. 2.12*b* are essentially equivalent to those of Fig. 2.12*a*;

the error involved in the approximation (2.60) is also implicit in the performance equation (1.16) of the summing amplifiers used in Fig. 2.12*a*. Suitably connected function-generating servomechanisms may replace the d-c amplifiers shown in Fig. 2.12.

3. *Use of integrators.* If d-c integrators (Sec. 1.4) replace the high-gain amplifiers in the equation-solver setups of Fig. 2.12*b*, the machine equation (2.60) is replaced by

$$F(X) = \frac{[1 - A(P)]RCP + 1}{A(P)} X \qquad (\approx -RCPX) \tag{2.61}$$

which reduces to Eq. (2.60) under d-c conditions ($P = 0$). If the resulting computer setup is stable, X will, then, converge to the desired root of Eq. (2.58), as in Fig. 2.12*b*. The solution time (time required for convergence) increases with the integrator time constant RC, but the inherently bad high-frequency response of the integrator may help to prevent high-frequency instability. As a compromise, one may add small integrating capacitors (10 to 200 μμf) to the high-gain amplifier setups of Fig. 2.12*b* (dash lines), so that integration takes place at high frequencies. The use of integration is, however, not a guarantee of high-frequency stability.

Stability of Equation Solvers. If the variable X in any of the computer setups of Fig. 2.12 is to converge to the desired root of Eq. (2.58) from any initial value, the d-c loop gain $A(0) \dfrac{\partial F}{\partial X}$ must be negative for all possible values of X. If the function $F(X)$ is nonmonotonic—this is true, for instance, if Eq. (2.58) has more than one real root—the range of X can be suitably restricted by means of limiter circuits (Sec. 6.7), or by restrictions on the range of the function generator used to produce $F(X)$. The condition just given is not sufficient for stability; the high-frequency stability of each equation-solver setup requires a separate investigation.

Solution of Sets of Equations: Linear Simultaneous Equations. Quite similar analog-computer methods apply to the solution of sets of equations. In particular, the machine equations corresponding to a given set of n real *linear simultaneous equations* will have the form

$$\left.\begin{array}{l} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n + b_1 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n + b_n = 0 \end{array}\right\} \tag{2.62}$$

with $|a_{ik}| \leq 1$, and determinant $\{a_{ik}\} \neq 0$. The computer setup shown in Fig. 2.13*a* for the case of two unknowns ($n = 2$) is a generalization of the equation-solver setups of Fig. 2.12*b*. If stray coupling between amplifiers is neglected, the n relations actually enforced are[1]

$$(a_{i1}X_1 + a_{i2}X_2 + \cdots + a_{in}X_n + b_i) = \frac{n+1}{A(P)} X_i \qquad (i = 1, 2, \ldots, n) \quad (2.63)$$

so that the desired relations (2.62) will be approximated if the computer setup is stable.

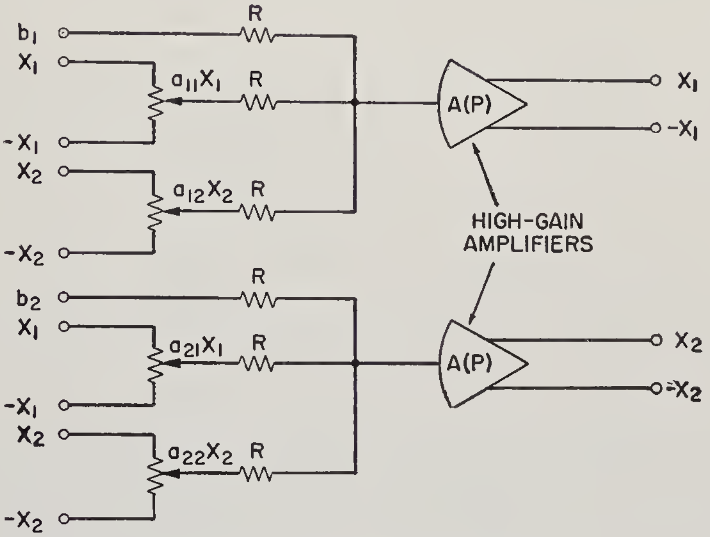

FIG. 2.13*a*. Simultaneous-equation solver using identical push-pull high-gain amplifiers. The resistances of the coefficient-setting potentiometers are considered as negligible.

Stability of Linear Simultaneous Equation Solvers.[2] As a first condition of stability, it will be assumed that all n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation

$$\text{determinant } \{a_{ik} - \lambda\delta_{ik}\} = 0 \qquad \delta_{ik} = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases} \quad (2.64)$$

(eigenvalues of the coefficient matrix) have positive real parts. If this condition is not satisfied by a given system (2.62), one may try to rear-range the equations (2.62),[1] or one solves the equivalent system[2]

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[1] Push-pull d-c amplifiers are indicated in Fig. 2.13*a*. The stability analysis is essentially unchanged if phase inverters are used to obtain negative coefficients a_{ik}, since a phase inverter will begin to have noticeable phase shift at a much higher frequency than a high-gain amplifier or integrator.

[2] Korn, G. A., Stabilization of Simultaneous Equation Solvers, *Proc. IRE*, **37**: 1000, 1949. The present treatment corrects some errors in the original paper. The analysis is here presented for d-c analog computers, but it applies also if the quantities b_i and X_k are represented by amplitudes of sinusoidal voltages of carrier frequency ν (a-c analog computer); see also E. A. Goldberg and G. W. Brown, *op. cit.*, and G. W. Brown, The Stability of Feedback Solution of Simultaneous Linear Equations, *Am. Math. Soc. Bull.*, part 1, **53**: 61, January, 1947 (abstract).

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji} a_{ik} X_k + \sum_{i=1}^{n} a_{ji} b_i = 0 \qquad (j = 1, 2, \ldots, n) \tag{2.65}$$

whose eigenvalues are necessarily real and positive.[3]

The system described by Eq. (2.63) will be stable if and only if all roots s of its characteristic equation

$$\text{determinant} \left\{ a_{ik} - \frac{n+1}{A(s)} \delta_{ik} \right\} = 0 \tag{2.66}$$

have negative real parts, *i.e., if and only if all roots s of the n equations*

$$1 - A(s) \frac{\lambda_j}{n+1} = 0 \qquad (j = 1, 2, \ldots, n) \tag{2.67}$$

have negative real parts. According to the Bode-Nyquist criterion,[4] this is true if and only if

$$\left. \begin{array}{l} \left| A(j\omega) \dfrac{\lambda_j}{n+1} \right| \geq 1 \text{ implies Arg} \left[A(j\omega) \dfrac{\lambda_j}{n+1} \right] < 2\pi \\ \qquad\qquad\qquad\qquad\qquad (j = 1, 2, \ldots, n) \end{array} \right\} \tag{2.68}$$

for all real positive ω.

Since $|a_{ik}| \leq 1$, $|\lambda_j|$ cannot exceed the value $n < n + 1$. If, in addition, all λ_j have positive real parts, *the computer will be stable if the amplifier phase shift* Arg $A(j\omega)$ *does not exceed* 90 *deg for any input frequency such that* $|A(j\omega)| \geq 1$. This condition is independent of the number n of unknowns; the theorem reduces the design of amplifiers for simultaneous equation solvers to the analytical or experimental analysis of a *single* amplifier.

Use of Integrators. As in the case of a single unknown, d-c integrators may be used instead of the high-gain amplifiers indicated in Fig. 2.13*a*; negative coefficients a_{ik} can be obtained by means of phase inverters. Figure 2.13*b* shows a computer setup for a simple numerical example.

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[1] Berry, C. E., D. E. Wilcox, S. M. Rock, and H. W. Washburn, A Computer for Solving Linear Simultaneous Equations, *J. Appl. Phys.*, **17**: 262–272, April, 1946.

[2] Adcock, W. A., An Automatic Simultaneous Equation Computer and Its Use in Solving Secular Equations, *Rev. Sci. Instr.*, **19**: 181–187, March, 1948.

[3] Refer to W. A. McCool, D-c Analog Solution of Simultaneous Linear Algebraic Equations, *NRL Rept.* 3533, U.S. Naval Research Laboratory, Washington, D.C., 1949, for an alternative stabilization method.

[4] Bode, H. W., *Network Analysis and Feedback Amplifiers*, Van Nostrand, Princeton, N.J., 1945.

The unknown voltages X_1 and X_2 satisfying the set of equations

$$\left.\begin{aligned} 0.4X_1 - 0.1X_2 - 0.5 &= 0 \\ 0.15X_1 + 0.9X_2 + 0.4 &= 0 \end{aligned}\right\} \qquad (2.69)$$

are found as the "equilibrium voltages" approached by the solutions $X_1(\tau)$ and $X_2(\tau)$ of the set of differential equations.

$$\left.\begin{aligned} 0.4X_1 - 0.1X_2 - 0.5 &= -PX_1 \\ 0.15X_1 + 0.9X_2 + 0.4 &= -PX_2 \end{aligned}\right\} \qquad (2.70)$$

after the initial transients have damped out. The solutions are stable, and in the equilibrium condition $PX_1 = PX_2 = 0$, so that the differential equations (2.70) reduce to Eqs. (2.69).

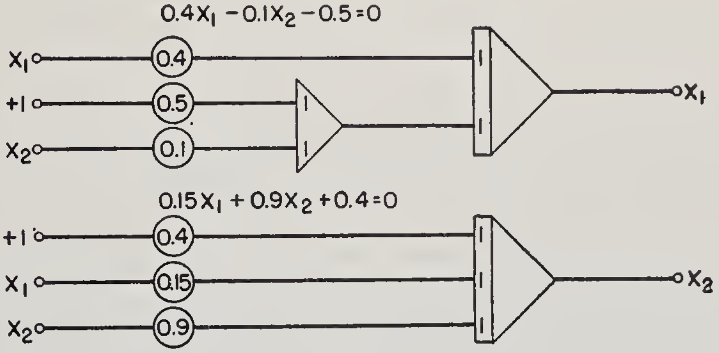

FIG. 2.13b. Solution of a set of linear simultaneous equations; d-c integrators are used as high-gain amplifiers.

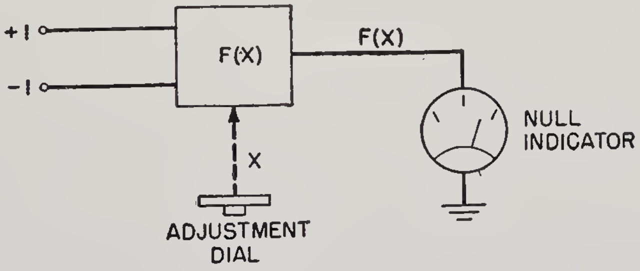

FIG. 2.14. Simple adjuster-type equation solver.

Adjuster-type Equation Solvers. The high computing speeds obtainable with d-c analog machines make it possible to solve equations with very simple equipment by a trial-and-error process. Figure 2.14 shows a computer setup forming a voltage $F(X)$ as a function of a dial rotation X by means of function potentiometers and summing networks. In order to solve the equation

$$F(X) = 0$$

for the unknown value or values of X, it is only necessary to adjust the dial until a meter or null indicator measuring the output voltage $F(X)$ reads zero. The value of the unknown may then be read on the adjustment dial. Devices of this type may be quite accurate (0.2 to 0.5 per cent), since the result is not affected by drift in the source voltage used.

In a more elegant version of the foregoing process, the voltage $F(X)$ may be computed as a function of an imput voltage or shaft rotation X which is varied (swept) periodically through its range of variation as a known periodic function of the time. It is then possible to measure the time at which the null detector responds by automatic means, and the relative time at which this occurs during the sweep cycle may be used to measure each root of the given equation.

The principle of the adjuster-type computer may also be applied to the solution of sets of equations. Given a set of machine equations

$$\left.\begin{array}{l} f_1(X_1, X_2, \ldots, X_n) = 0 \\ f_2(X_1, X_2, \ldots, X_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ f_n(X_1, X_2, \ldots, X_n) = 0 \end{array}\right\} \tag{2.71}$$

where the f's are reasonable single-valued functions of the X's, perhaps the most generally applicable method of solution is to generate a voltage

$$F(X_1, X_2, \ldots, X_n) = f_1^2 + f_2^2 + \cdots + f_n^2 \tag{2.72}$$

as a function of the input voltages or shaft rotations $X_1, X_2, \ldots, X_n$ by means of suitable computing elements, and to minimize this quantity by iterated adjustments of the X's.[1]

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[1] Murray, *op. cit.*

CHAPTER 3

THE APPLICATION OF D-C ANALOG COMPUTERS TO REPRESENTATIVE PRACTICAL PROBLEMS

3.1. PROBLEM PREPARATION

Introduction: Performance Studies. D-c analog computers are particularly well suited for applications involving the solution of sets of ordinary differential equations, and the majority of the sample problems presented in this chapter are of this general type. Applications involving the solution of ordinary algebraic or transcendental equations are discussed in Sec. 2.11.

Most of the problems discussed in this chapter involve d-c analog-computer studies of the performance of engineering systems. The various performance characteristics of dynamical, electrical, or electromechanical systems can usually be inferred from records of the system coordinates vs. the time. The use of an analog computer permits one to obtain such records quite rapidly as machine solutions of sets of differential equations (equations of motion or performance equations) which specify the design of the system under consideration. Wide ranges of variations in the design of an engineering system under study can be conveniently explored in terms of appropriate corresponding changes in the computer setup.

Parameter Studies: Design by Trial and Error. Methods involving the systematic substitution of different combinations of unknown design parameters are frequently used to find those "optimal" combinations of design parameters which will be the most useful for achieving a desired result. Trial-and-error methods of this type, made possible through high-speed computing methods, are frequently preferable to more conventional analytical engineering processes in the design of complex systems, such as nonlinear automatic control systems.

Parameter Studies: Analyses Based on Insufficient Data. Quite frequently, the values of parameters important for an analysis cannot be known with precision at the time the analysis is made. In such cases, bracketing of these parameters will permit the investigator to check the sensitivity of the results to changes in the parameters in question; such checks will help to evaluate the validity of results based on assumed approximate values of the parameters. The application of automatic computing machinery will, in fact, make it possible for the investigator to evaluate the effects of many combinations of the unknown parameters.

Preparation of Records. From the point of view of convenience in setting up the computer as well as for reference purposes, it is desirable to record all the information pertinent to the solution of a given problem as clearly and unambiguously as possible. For this purpose, it is useful to prepare two sets of forms, one summarizing the *problem specifications* and one constituting a *record of the computer solution.*

Problem-specifications Forms. The *problem-specifications forms*[1] will contain the following information:

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.
2. *Statement of the problem.* This paragraph will contain a concise statement of the computations to be performed and the specific results desired from the computations. Sketches and diagrams may be included.
3. *Mathematical relations.* All mathematical equations and inequalities relevant to the computations must be listed.
4. *Glossary of symbols.* All symbols used must be defined for rapid reference.
5. *Initial conditions and boundary conditions* (in the case of problems involving differential equations).
6. *Numerical values of constants.* The values of the constant coefficients as well as the units used must be specified.
7. *Estimated ranges of variables.* The information listed in this paragraph is to aid in an intelligent choice of scale factors (see Chap. 2).
8. *Parameters to be varied.* The values of the parameters to be changed or the ranges of parameter changes for use in parameter studies must be specified. The estimated total number of computer solutions to be thus obtained should also be given.
9. *Expected results of parameter variations* (estimates based on previous computations or practical experience). Such data may be helpful for checking computer solutions.
10. *Possible analytical or numerical checks noted by the customer*, including references to the literature.
11. *Use of results.* This (optional) paragraph may contain information as to the end use of the computations specified in the *problem statement;* such information may be useful in deciding on the repre-

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[1] This procedure was originated in the computer laboratory of the Reeves Instrument Corporation, New York City.

sentation of the results or in suggesting modifications to the problem statement.

12. *Accuracy requirements.*

A number of practical examples of .completed problem-specifications forms will be presented in the remaining sections of this chapter and may serve as models for other computations.

Records-of-computer-solution Forms. The computer laboratory should prepare "*records of computer solution*" forms for each problem as aids in setting up the computer and as records of the computation for later reference. These forms will contain the following information:

1. *Transformation equations* (see Chap. 2). This paragraph amounts to a record of the scale factors used. The units used for problem variables and machine variables must be specified. Changes in the scale factors made in the course of parameter studies must be carefully listed.
2. *Machine equations* (see Chap. 2). This paragraph will contain the relations actually established by the computing elements.
3. *Machine initial conditions* (initial conditions in terms of machine units and dial settings).
4. *Block diagrams.*
5. *Discussion.* Any *comments* regarding the solution of the problem or the operation of the computer should be recorded. In particular, it is important for future checks to list any failures of machine components during the computation. Any special setups or new operating methods should be noted for future reference.
6. *Enclosures* (computer records, graphs, tables of results, etc.).

A number of completed "records of the computation" forms for typical computations are included in the following sections and may serve as models for other computations. The reader should remember, in each case, that *the choice of computer setup and scale factors for a given problem is by no means necessarily unique.*

Evaluation of Machine Solutions. In many cases, the performance of an engineering system for a given combination of design parameters is obvious from an inspection of computer records showing the variation with time of one or several system coordinates. More frequently, no such direct interpretation of the computer records is possible, and the latter must be used indirectly to compute *figures of merit* of the system performance. Examples of such figures of merit are averages of maximum and minimum values of variables or their rates of change. In still other problems, it may also be necessary to analyze the system performance in terms of *cross plots* summarizing the results of several machine solutions (*example:* frequency-response curves, see Secs. 3.4, 3.7, and 3.8).

3.2. D-C ANALOG-COMPUTER STUDY OF A VIBRATION ABSORBER AND OF A SIMPLE AUTOMOBILE SUSPENSION

Performance of a Dynamic Vibration Absorber. As an illustrative example of a dynamical system, consider the dynamic vibration-absorber system of Fig. 3.1*a*. A mass m_1, restrained by a spring of stiffness k_1 and a dashpot whose damping coefficient is c_1, is made to oscillate in the y direction through the action of a periodic force

$$F(t) = a \sin \omega t \tag{3.1}$$

This situation is representative of many other situations involving the vibrations of machines containing more or less unbalanced components rotating at a circular frequency ω. In order to reduce the vibration of the main mass m_1, an absorber mass m_2 is coupled to m_1 through a spring of stiffness k_{12}. It may be shown[1] that the parameters m_2 and k_{12} of the vibration absorber can be chosen so that the main mass does not vibrate at all; in this case, the absorber mass m_2 will oscillate in such a way that the force it exerts on the main mass m_1 will just cancel the applied force F. These statements apply only if the force F is of the form (3.1).

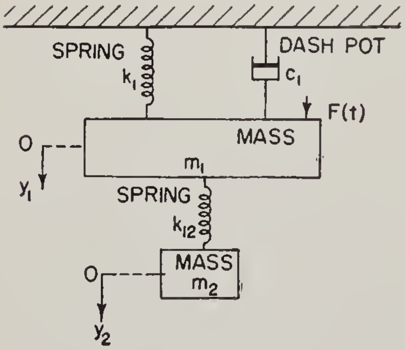

FIG. 3.1*a*. Dynamic vibration-absorber system.

Referring to Fig. 3.1*a*, let y_1 be the displacement in the y direction of the main mass m_1 and let y_2 be the displacement of the absorber mass m_2. The equation of motion of the main mass is

$$m_1 \frac{d^2y_1}{dt^2} = -k_1y_1 - c_1 \frac{dy_1}{dt} - k_{12}(y_1 - y_2) + a \sin \omega t,$$

where t is the time. The expression on the right is the sum of the forces acting on the main mass m_1, *viz.*, the restraining force of the main spring, the viscous damping force due to the dashpot, the force exerted by the coupling spring, and the driving force.

Similarly, the equation of motion of the absorber mass m_2 is found to be

$$m_2 \frac{d^2y_2}{dt^2} = k_{12}(y_1 - y_2)$$

The only force acting on m_2 is that due to the coupling spring.

A practical procedure for handling the study of such a dynamic vibration-absorber system in a d-c analog-computer laboratory is illustrated

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[1] Den Hartog, J. P., *Mechanical Vibrations*, 4th ed., McGraw-Hill, New York, 1956.

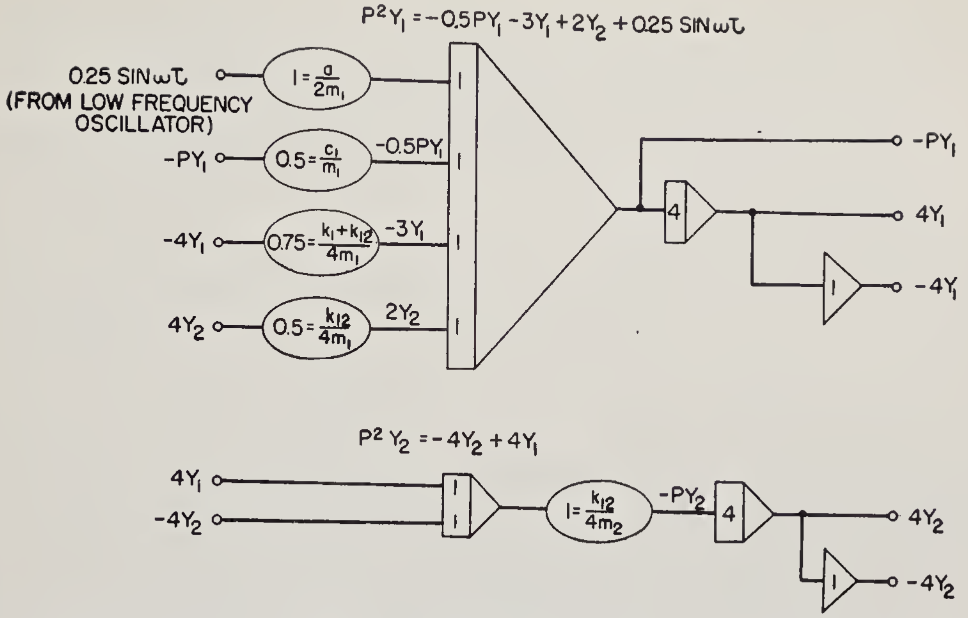

Fig. 3.1*b*. Block diagram of a d-c analog-computer setup for the study of a dynamic vibration-absorber system. The coefficients set up on each potentiometer are indicated; the numerical potentiometer settings refer to the sample problem treated in the text.

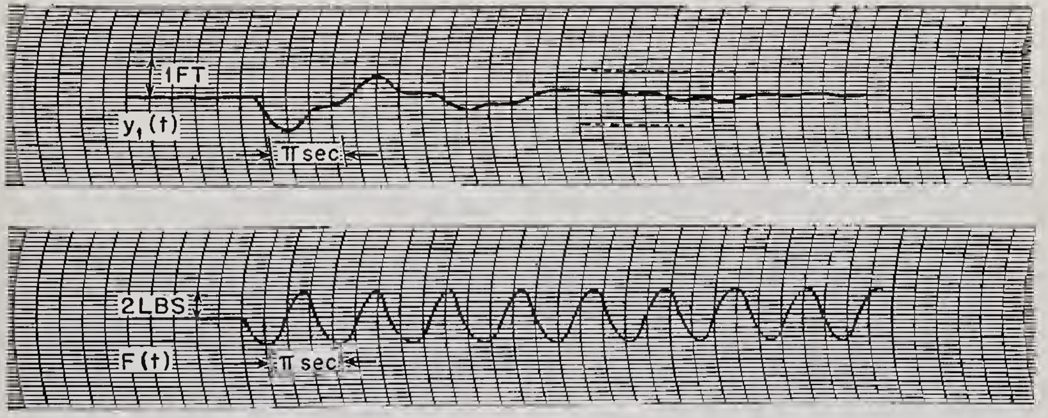

Fig. 3.1*c*. Computer record (Hagelbarger, Howe, and Howe, *op. cit.*), showing the effect of the dynamic vibration-absorber near resonance ($\omega = 2$ radians/sec). After an initial transient, the amplitude of y_1 is reduced to less than $\frac{1}{50}$ of the amplitude which would result without the vibration absorber (indicated in the broken lines). The motion of m_1 without the vibration absorber can be studied by setting $k_{12} = 0$ in the block diagram of Fig. 3.1*b*.

below in terms of a sample *problem statement* and a set of sample *computer setup records.*

Study of a Dynamic Vibration Absorber

A. Problem specifications

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.

2. *Statement of the problem.* Determine the effect of a dynamic vibration absorber on the motion of a mass (see Fig. 3.1*a*) driven by a force varying as a sinusoidal function of the time. In particular, show how the oscillation amplitude of the main mass is affected by changes in the frequency of the driving force.
3. *Mathematical relations*

$$(a)\quad m_1 \frac{d^2y_1}{dt^2} + c_1 \frac{dy_1}{dt} + k_1y_1 + k_{12}(y_1 - y_2) = a \sin \omega t$$

(MOTION of m_1)

$$(b)\quad m_2 \frac{d^2y_2}{dt^2} + k_{12}(y_2 - y_1) = 0$$

(MOTION of m_2)

4. *Glossary of symbols* (refer to Fig. 3.1*a*)

t	time
y_1	displacement of main mass
y_2	displacement of absorber mass
m_1	main mass
m_2	absorber mass
k_1	stiffness of main spring
k_{12}	stiffness of coupling spring
c_1	damping coefficient
a	amplitude of driving force
ω	circular frequency of driving force

5. *Initial conditions*

$$y_1 = y_2 = \frac{dy_1}{dt} = \frac{dy_2}{dt} = 0$$

6. *Numerical values of constants*

$m_1 = 1$ slug

$m_2 = 0.5$ slug

$k_1 = 1$ lb/ft

$k_{12} = 2$ lb/ft

$c_1 = 0.5$ lb-sec/ft

$a = 2$ lb

7. *Estimated ranges of variables*

$-2 \text{ ft} < y_1 < 2 \text{ ft} \qquad -2 \text{ ft} < y_2 < 2 \text{ ft} \qquad 0 \leq t < 200 \text{ sec}$

8. *Parameters to be varied.* Vary ω between 0.8 and 4 radians/sec in 10 steps.
9. *Expected results of parameter variations.* The amplitude of the steady-state periodic vibrations of the main mass will have a minimum for a certain circular frequency ω. In this "resonance" condition, the sinusoidal oscillations of the absorber mass will be 180 deg out of phase with respect to the driving force.

10. *Possible analytical or numerical checks noted by the customer.* This particular problem may be treated analytically.[1]
11. *Use of results.* Test problem.
12. *Accuracy required.* 0.5 to 1.5 per cent.

B. Record of computer solution

1. *Transformation equations.* (y_1, y_2, and t are measured in the ft/slug/sec system; all machine variables are measured in machine units. One machine unit equals 50 volts.)

$$y_1 = 8Y_1 \qquad y_2 = 8Y_2 \qquad t = \tau$$

NOTE: The relatively low scale factors for y_1 and y_2 may be necessary to keep the absolute values of PY_1 and PY_2 within the range of 1 machine unit; this choice of scale factors also makes it possible to represent voltages like $4Y_1$ and $4Y_2$ in the machine. The recorder scales are noted on the records.

2. *Machine equations*

$$(a)\ P^2Y_1 = -0.5PY_1 - 3Y_1 + 2Y_2 + 0.25 \sin \omega\tau$$
$$(b)\ P^2Y_2 = -4Y_2 + 4Y_1$$

3. *Machine initial conditions*

$$Y_1 = Y_2 = PY_1 = PY_2 = 0$$

4. *Block diagram.* See Fig. 3.1*b*. An electronic low-frequency oscillator or a signal generator of the type shown in Fig. 6.21, or circuit 4.6 of Table 4 may be used to generate the function sin $\omega\tau$. Loading corrections for all coefficient setting potentiometers must be taken into consideration.
5. *Discussion* (of special problems encountered during the computation, tentative conclusions from computer results, etc.) "Resonance" takes place near $\omega = 2$ radians/sec; this checks the theoretically computed value $\sqrt{k_{12}/m_2}$ for the resonance circular frequency.

A sample computer record for $\omega = 2$ radians/sec is shown in Fig. 3.1*c*.

An Automobile-suspension Problem.[2] Figure 3.2*a* illustrates a simple automobile-suspension problem. The automobile shown runs at a given speed over an obstacle which imparts the respective vertical dis-

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[1] See, for instance, Den Hartog, *op. cit.*

[2] The problem, but not the computer setup shown, is based on GEDA Analysis of a Standard Automobile Suspension System, *Report GER*-5262, Goodyear Aircraft Corporation, Akron, Ohio, Mar. 12, 1953.

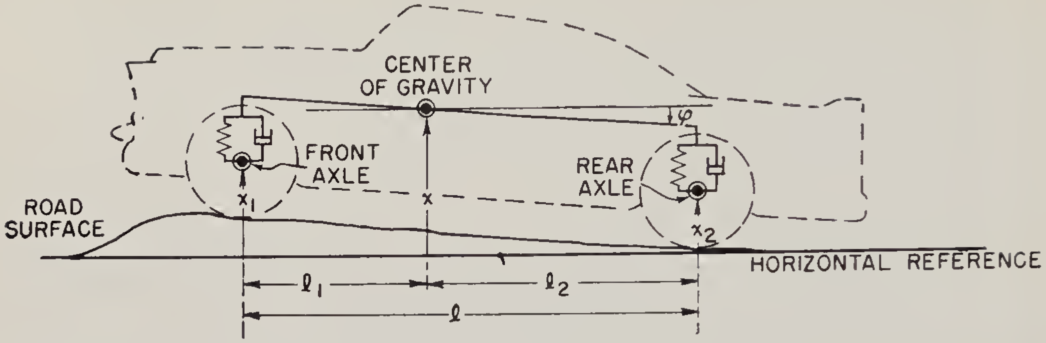

FIG. 3.2*a*. A simple automobile-suspension system, showing front and rear springs and shock absorbers. Small pitch angles φ are assumed. The displacements x_1, x_2, and x are measured vertically from the respective equilibrium positions of the axles and the center of gravity.

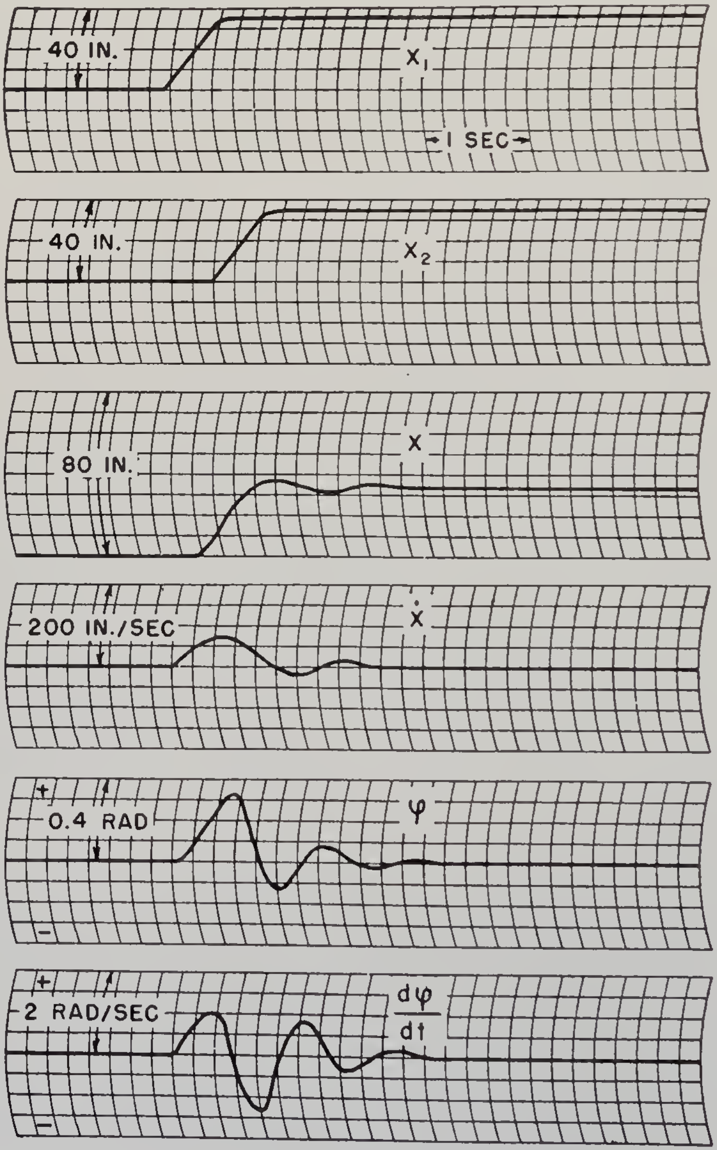

FIG. 3.2*b*. Computer record showing the response of the automobile running over a 15-deg incline at 15 mph (from Goodyear Aircraft Corporation *Report GER*-5262).

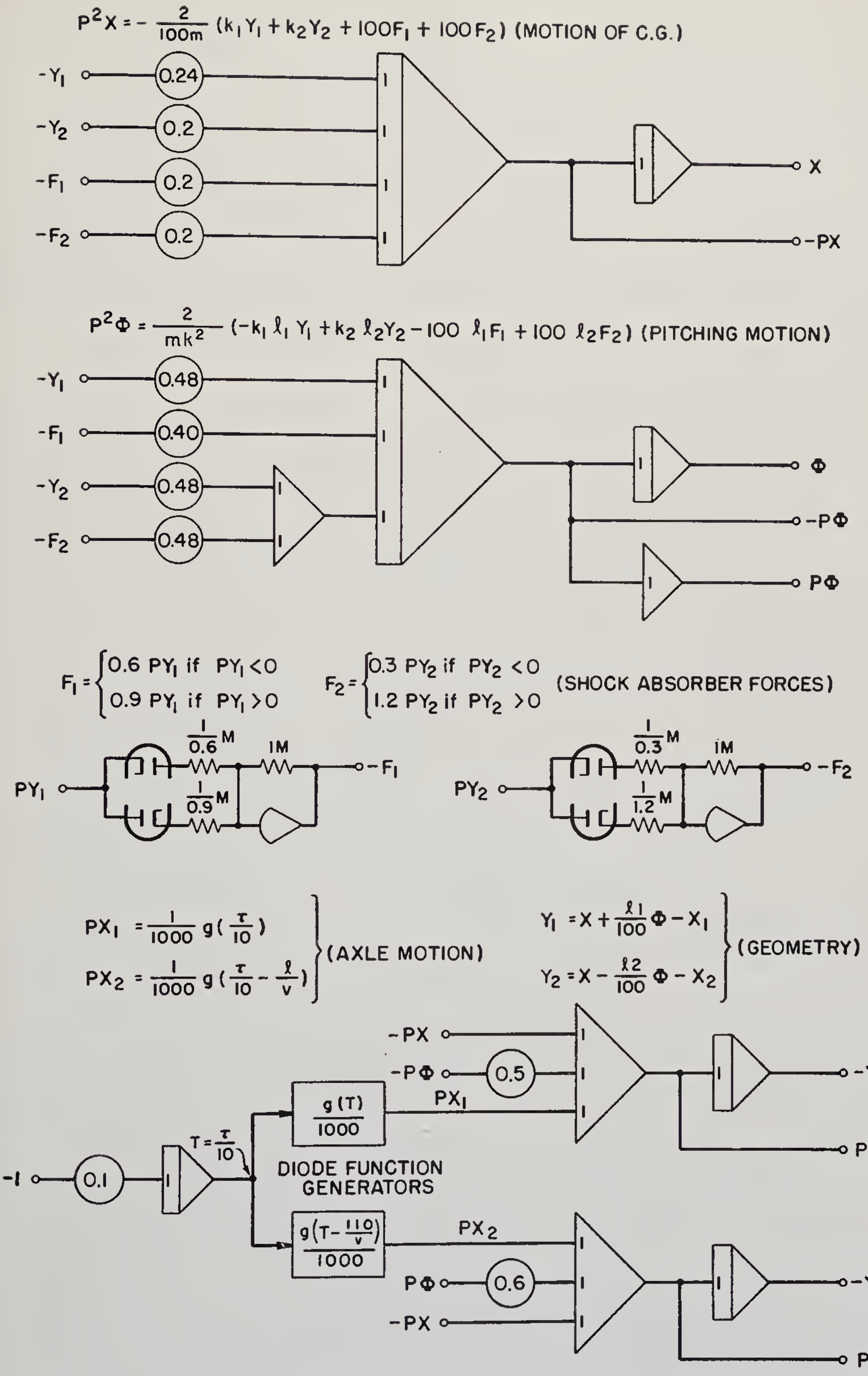
$p^2X = -\frac{2}{100m}(k_1Y_1 + k_2Y_2 + 100F_1 + 100F_2)$ (MOTION OF C.G.)
$-Y_1$
$-Y_2$
$-F_1$
$-F_2$
0.24
0.2
0.2
0.2
1
X
$-pX$
$p^2\Phi = \frac{2}{mk^2}(-k_1\ell_1Y_1 + k_2\ell_2Y_2 - 100\ell_1F_1 + 100\ell_2F_2)$ (PITCHING MOTION)
$-Y_1$
$-F_1$
$-Y_2$
$-F_2$
0.48
0.40
0.48
0.48
Φ
$-p\Phi$
$p\Phi$
$F_1 = \begin{cases} 0.6\,pY_1 \text{ if } pY_1 < 0 \\ 0.9\,pY_1 \text{ if } pY_1 > 0 \end{cases}$
$F_2 = \begin{cases} 0.3\,pY_2 \text{ if } pY_2 < 0 \\ 1.2\,pY_2 \text{ if } pY_2 > 0 \end{cases}$
(SHOCK ABSORBER FORCES)
pY_1
$\frac{1}{0.6}$M
$\frac{1}{0.9}$M
1M
$-F_1$
pY_2
$\frac{1}{0.3}$M
$\frac{1}{1.2}$M
1M
$-F_2$
$pX_1 = \frac{1}{1000}g(\frac{\tau}{10})$
$pX_2 = \frac{1}{1000}g(\frac{\tau}{10} - \frac{\ell}{v})$
(AXLE MOTION)
$Y_1 = X + \frac{\ell_1}{100}\Phi - X_1$
$Y_2 = X - \frac{\ell_2}{100}\Phi - X_2$
(GEOMETRY)
$-pX$
$-p\Phi$
0.5
pX_1
$-Y_1$
pY_1
-1
0.1
$T = \frac{\tau}{10}$
$\frac{g(T)}{1000}$
DIODE FUNCTION GENERATORS
$\frac{g(T - \frac{110}{v})}{1000}$
pX_2
$p\Phi$
0.6
$-pX$
$-Y_2$
pY_2

Fig. 3.2*c*. Block diagram of a d-c analog-computer setup representing the automobile-suspension system of Fig. 3.2*a* on a 10:1 (slow) time scale. Simple diode function generators (Sec. 6.7) introduce the nonlinear characteristics of double-acting shock absorbers (different damping coefficients in "jounce" and "rebound").

placements $x_1(t)$ and $x_2(t)$ to the front and rear axles; each pair of wheels is to move in unison. Each pair of springs or shock absorbers will then act in parallel, so that each effective spring stiffness or damping coefficient is twice that given for a single spring or shock absorber. Wheel motion as such is not treated in this simple example, but could easily be accounted for in terms of additional mass-spring-dashpot systems. A d-c analog-computer study of such an automobile suspension is outlined below; the glossary of symbols has been presented in the form of a table indicating the numerical values of constants and the estimated ranges of variables.

A. *Problem specifications*

1. *Statement of the problem.* Given the simple automobile-suspension system described above, determine the vertical displacement x of the automobile center of gravity and the pitch angle φ as functions of the time t for an obstacle defined by $x_1(t)$ and $x_2(t)$. The displacements x and φ are assumed to be small.
2. *Mathematical relations*

$$(a)\quad \frac{m}{2}\frac{d^2x}{dt^2} = -k_1y_1 - k_2y_2 - f_1\left(\frac{dy_1}{dt}\right) - f_2\left(\frac{dy_2}{dt}\right)$$

(MOTION OF CENTER OF GRAVITY)

$$(b)\quad \frac{m}{2}k^2\frac{d^2\varphi}{dt^2} = -k_1l_1y_1 + k_2l_2y_2 - l_1f_1\left(\frac{dy_1}{dt}\right) + l_2f_2\left(\frac{dy_2}{dt}\right)$$

(PITCHING MOTION)

$$(c)\quad f_1\left(\frac{dy_1}{dt}\right) = \begin{cases} 6\,\dfrac{dy_1}{dt} & \text{if } \dfrac{dy_1}{dt} < 0 \\ 9\,\dfrac{dy_1}{dt} & \text{if } \dfrac{dy_1}{dt} > 0 \end{cases}$$

$$f_2\left(\frac{dy_2}{dt}\right) = \begin{cases} 3\,\dfrac{dy_2}{dt} & \text{if } \dfrac{dy_2}{dt} < 0 \\ 12\,\dfrac{dy_2}{dt} & \text{if } \dfrac{dy_2}{dt} > 0 \end{cases}$$

(NONLINEAR DAMPING OF DOUBLE-ACTING SHOCK ABSORBERS)

$$(d)\quad y_1 = x + l_1\varphi - x_1 \qquad y_2 = x - l_2\varphi - x_2$$

(GEOMETRY)

$$(e)\quad \frac{dx_1}{dt} = g(t) \qquad \frac{dx_2}{dt} = g\left(t - \frac{l}{v}\right)$$

(AXLE MOTION)

3. *Glossary of symbols, numerical values of constants, and estimated ranges of variables* (see also Fig. 3.2*a*)

mass of car	$m = 10$ lb sec²/in.
radius of gyration (pitch)	$k = 50$ in.

spring stiffness, one front spring $k_1 = 120$ lb/in.
spring stiffness, one rear spring $k_2 = 100$ lb/in.
moment arm of front spring and shock absorber about c.g. $l_1 = 50$ in.
moment arm of rear spring and shock absorber about c.g. $l_2 = 60$ in.
wheelbase $l = 110$ in.
forward speed of car $v = 15$ mph

$$\text{vertical displacement of c.g.} \begin{cases} -100 \text{ in.} < x < 100 \text{ in.} \\ -200 \text{ in./sec} < \dfrac{dx}{dt} < 200 \text{ in./sec} \end{cases}$$

$$\text{pitch angle} \begin{cases} -1 \text{ radian} < \varphi < 1 \text{ radian} \\ -2 \text{ radians/sec} < \dfrac{d\varphi}{dt} < 2 \text{ radians/sec} \end{cases}$$

$$\begin{array}{l}\text{vertical displacement} \\ \quad \text{of front axle}\end{array} \begin{cases} -50 \text{ in.} < x_1 < 50 \text{ in.} \\ -500 \text{ in./sec} < \dfrac{dx_1}{dt} < 500 \text{ in./sec} \end{cases}$$

$$\begin{array}{l}\text{vertical displacement} \\ \quad \text{of rear axle}\end{array} \begin{cases} -50 \text{ in.} < x_2 < 50 \text{ in.} \\ -500 \text{ in./sec} < \dfrac{dx_2}{dt} < 500 \text{ in./sec} \end{cases}$$

4. *Initial conditions*

$$\text{For } t = 0,\ x = \varphi = y_1 = y_2 = \frac{dx}{dt} = \frac{d\varphi}{dt} = 0$$

5. *Accuracy required.* 1 to 2 per cent.

B. *Record of computer solution*

1. *Transformation equations*

$$x = 100X \quad \varphi = \Phi \quad x_1 = 100X_1 \quad x_2 = 100X_2 \quad y_1 = 100Y_1$$

$$y_2 = 100Y_2 \qquad f_1 = 10{,}000F_1 \qquad f_2 = 10{,}000F_2 \qquad t = \frac{\tau}{10} = T$$

$$p = 10P$$

2. *Machine equations*

$$(a)\ P^2X = -0.24Y_1 - 0.20Y_2 - 0.2F_1 - 0.2F_2$$

$$(b)\ P^2\Phi = -0.48Y_1 + 0.48Y_2 - 0.4F_1 + 0.48F_2$$

$$(c)\ F_1 = \begin{cases} 0.6PY_1 \text{ if } PY_1 < 0 \\ 0.9PY_1 \text{ if } PY_1 > 0 \end{cases} \qquad F_2 = \begin{cases} 0.3PY_2 \text{ if } PY_2 < 0 \\ 1.2PY_2 \text{ if } PY_2 > 0 \end{cases}$$

$$(d)\ Y_1 = X + 0.5\Phi - X_1 \qquad Y_2 = X - 0.6\Phi - X_2$$

$$(e)\ PX_1 = \frac{1}{1000}\, g\left(\frac{\tau}{10}\right) \qquad PX_2 = \frac{1}{1000}\, g\left(\frac{\tau}{10} - \frac{110}{v}\right)$$

3. *Machine initial conditions*

   For $\tau = 0$, $X = \Phi = Y_1 = Y_2 = PX = P\Phi = 0$

4. *Block diagram.* See Fig. 3.2*c*. Simple diode function generators (Sec. 6.7) are used to represent the nonlinear characteristics of the double-acting shock absorbers.

Figure 3.2*b* shows a computer record of the results obtained. *The techniques presented are useful for aircraft landing-gear design as well as for the analysis of automobile and railroad-car suspensions.*

3.3. REPRESENTATION OF DYNAMICAL SYSTEMS IN GENERAL. FORCES, STOPS, AND CONSTRAINTS

The d-c analog-computer representation of each of the dynamical systems discussed in Sec. 3.1 was introduced simply as a problem in the solution of differential equations. The block diagrams of Figs. 3.1 and 3.2 illustrate the following facts pertinent to practically all d-c analog-computer studies of dynamical systems.

1. A specific "block" of computing elements (see also Sec. 1.1) establishes a machine equation corresponding to the *equation of motion* for each "generalized coordinate" of the dynamical system.[1] The output voltages obtained from such a block will represent (generalized) *displacements* and/or *velocities.* As a rule, the initial values of these quantities are given.
2. The' input voltages "driving" each block will represent (generalized) *forces* or *accelerations.* In general, each such voltage is a function of one or more of the machine variables representing generalized displacements and/or velocities.
3. Coupling forces leading to logical interactions between two of the equations of motion will result in corresponding electrical interconnections between two blocks of computing elements.
4. It is usually possible to vary system parameters within reasonable limits and to study the resulting changes in system performance without radical changes in the computer setup.

The analogy existing between components of the actual system under consideration and "blocks" of computing elements in analog computers makes them eminently useful for engineering studies of dynamical systems; this is particularly true in cases involving nonlinear equations of motion.

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[1] The general formulation of such equations of motion (Lagrange's equations) is discussed, for instance, in R. B. Lindsay and H. Margenau, *Foundations of Physics,* Wiley, New York, 1936.

Representation of Typical Forces. In most practical applications, each equation of motion has the form

$$\frac{d^2x}{dt^2} = \frac{1}{m}(f_1 + f_2 + \cdots)$$

where m is a measure of inertia, and $f_1, f_2, \ldots$ are (generalized) forces or kinetic reactions given as functions of coordinates, velocities, and the time. Each such equation of motion will be represented by a corresponding machine equation

$$P^2X = F_1 + F_2 + \cdots \tag{3.2}$$

where $$F_1 = \frac{a_x}{\alpha_t^2 m} f_1 \qquad F_2 = \frac{a_x}{\alpha_t^2 m} f_2, \quad \cdots$$

are voltages generated as functions of the machine variables and/or of the time. For the sake of simplicity, the following discussion refers specifically to the linear motion of a particle of fixed mass m, so that the voltages $F_1, F_2, \ldots$ represent ordinary forces in the x direction. Analogous computer setups apply if x is one of several coordinates, or if x and m are respectively replaced by the angular displacement ϑ and the moment of inertia I of a rigid body rotating about a fixed axis.

Figure 3.3*a* shows a typical d-c analog-computer setup for the machine equation (3.2). One obtains $-PX$ and X by integrating the sum of the voltages appearing on the right side of Eq. (3.2) (see also Sec. 3.2). The remainder of this section will indicate suitable methods for generating functions $F_1(X,PX,\tau)$, $F_2(X,PX,\tau)$, . . . to represent typical physical forces.

Forces appearing as *functions of the time* were already encountered in Sec. 3.2. Methods for generating such functions are presented in Secs. 6.6 to 6.10, and in Table 4. Voltages representing forces can be suddenly applied and removed by means of switches (*function switches*) or relays (Sec. 6.7), as for tests of step-function response. The multivibrator circuits of Table 6 are useful for repeated switching operations. Randomly varying forces can be represented by voltages suitably derived from a noise generator (Secs. 3.9 and 6.6).

Figure 3.3*b* illustrates the generation of a voltage

$$F_1 = -KX = -\frac{k}{\alpha_t^2 m} X$$

representing a simple *elastic force* (spring force, see also Sec. 3.2) $f_1 = -kx$. The more general block diagram of Fig. 3.3*c* permits the representation of *nonlinear springs*, of *springs strained beyond their elastic limits*, of various *electrostatic forces* dependent on the displacement x, etc. *Simple diode function generators* (*Sec.* 6.7) *are particularly useful in such applications;*[1] thus the circuit of Fig. 3.3*d* simulates the action of a *single-acting spring,* and circuits 6.1*a*, *b*, and *c* of Table 6 represent spring systems having a *dead space* or *inert zone.*

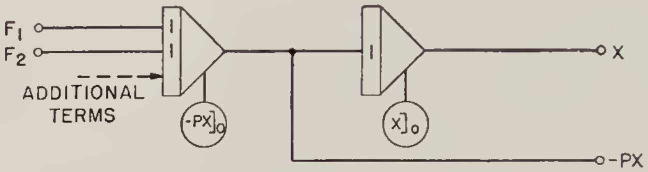

FIG. 3.3*a*. The machine equation $P^2X = F_1 + F_2 + \cdots$ describes the motion of a particle under the influence of a set of forces represented by voltages $F_1, F_2, \ldots$.

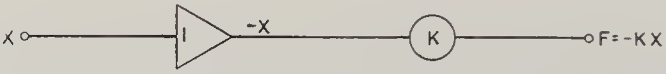

FIG. 3.3*b*. A voltage proportional to the displacement represents an elastic force, such as that exerted by a double-acting spring. For $K > 1$, a phase inverter of higher gain is used.

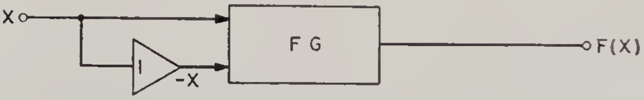

FIG. 3.3*c*. A more general type of force given as a function of position can be represented with the aid of a function generator.

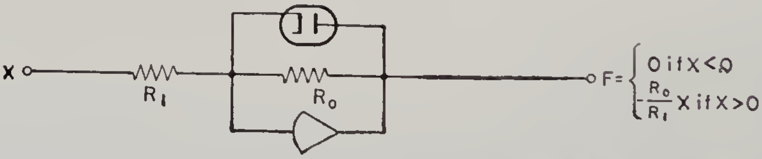

FIG. 3.3*d*. This simple diode limiter (see also Sec. 6.7) represents a single-acting spring, which exerts an elastic force for positive displacements only. The precision limiter circuit of Fig. 6.24*m* may be used for greater accuracy.

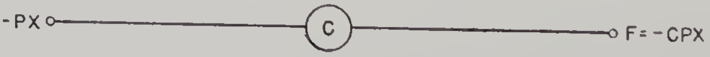

FIG. 3.3*e*. A voltage proportional to the velocity represents a viscous damping force.

Figure 3.3*e* shows the representation of a *linear damping force* due, for instance, to *viscous friction* (see also Sec. 3.2). Suitable function generators permit simulation of *nonlinear damping forces* by voltages of the form $-F(X)PX$ (Fig. 2.9) or $-F(PX)$ (Fig. 3.2*c*). Figure 3.3*f* illustrates the representation of *dry* (*coulomb*) *friction* and *starting friction* (see also circuits 6.4 and 6.6 of Table 6).

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[1] Morrill, C. D., and R. V. Baum, Diode Limiters Simulate Mechanical Phenomena, *Electronics*, **25**: 122, November, 1952.

Representation of Stops. Impact Forces. In Fig. 3.4*a*, the displacement x of the particle[1] shown is determined by the equation of motion

$$\frac{d^2x}{dt^2} = \frac{1}{m}(f_1 + f_2 + \cdots)$$

if

$$x \leq x_1 \tag{3.3}$$

In addition, the relation (3.3) is enforced by a more or less elastic *stop* or barrier. The voltage X corresponding to x must, then, be kept from exceeding a specified value X_1.

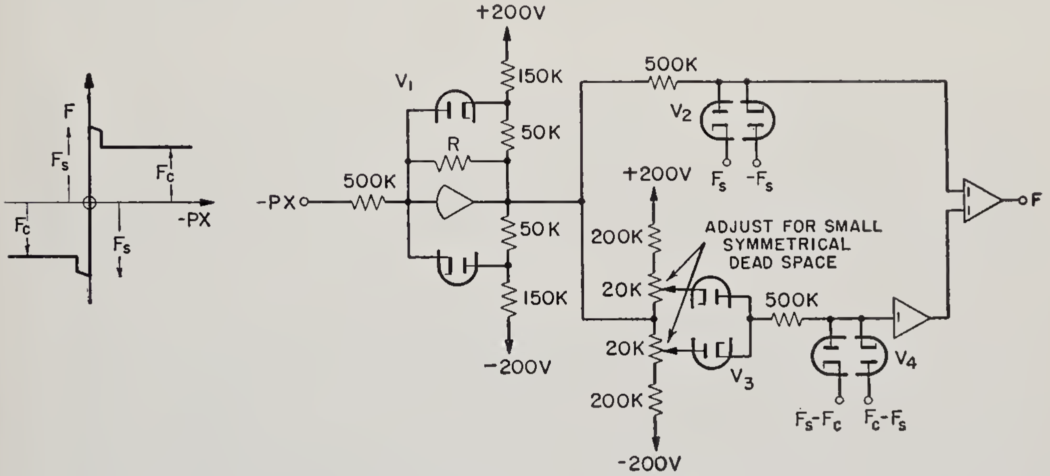

FIG. 3.3*f*. Representation of static friction (stiction) and dry (coulomb) friction. For $PX = 0$, the dead-space diodes V_3 (see also Sec. 6.7) do not conduct, and the circuit approximates the action of static friction by a voltage $-CPX$ limited in absolute value to F_S; if R exceeds 10 megohms, C will be greater than 20. Whenever a force exceeding that of static friction produces an appreciable value of $|PX|$, one diode of each pair shown will conduct. The output voltages of the two shunt limiters are then subtracted to yield $F = F_C$ or $F = -F_C$, corresponding to coulomb friction (Morrill and Baum, *op. cit.*).

The reader is warned that the stop is *not* correctly represented by a shunt limiter circuit (Fig. 6.23) connected so as to limit the output voltage X in Fig. 3.3*a*, since the artificially limited output voltage will not always equal the time integral of the voltage PX. *The correct analog-computer representation of a stop or barrier actually introduces the impact force experienced by the particle as it penetrates the barrier slightly.*

An *elastic stop* will exert a strong elastic force proportional to the barrier penetration. Figure 3.4*b* shows how a feedback-limited high-gain amplifier (see also Sec. 6.7) may be used to generate a corresponding

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[1] The following discussion applies, again, to other types of (generalized) coordinates, such as rotations about a fixed axis.

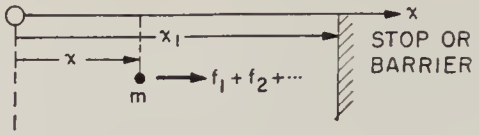

FIG. 3.4*a*. Linear motion of a particle restrained by a fixed or movable stop.

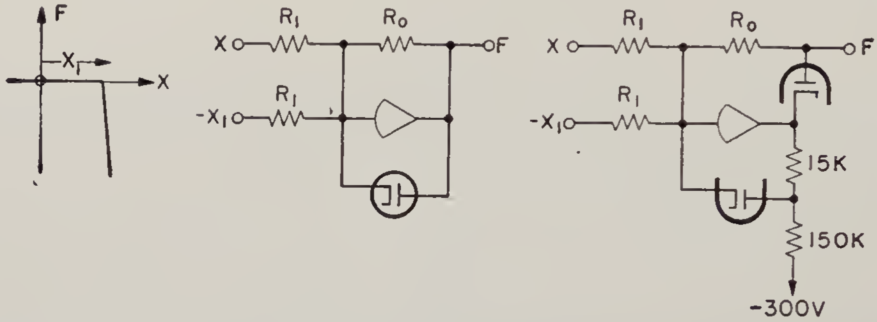

FIG. 3.4*b*. Representation of an elastic stop by means of high-gain limiter circuits (see also Figs. 6.23 and 6.24). In the first circuit, the output voltage F is zero while the diode conducts for $X \leq X_1$; for $X > X_1$, the output voltage is $-\frac{R_o}{R_1}X = -K_S X$. In the second circuit, the upper diode is cut off for $X \leq X_1$; the lower diode prevents overloads.

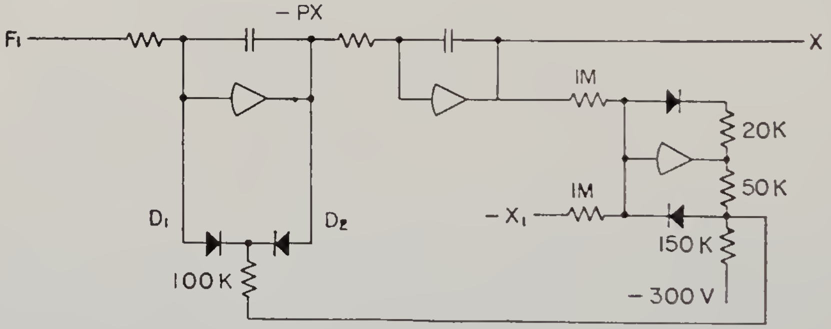

FIG. 3.4*c*. Representation of an inelastic stop. Diodes D_1, D_2 comprise a unidirectional switch which is closed by a comparator circuit (Fig. 6.27*a*) to apply a strong viscous force whenever $X > X_1$ and $PX > 0$. Note that the simulated particle will not stick to the stop (G. A. Korn, *Trans. IRE*-PGEC, March, 1960). A voltage proportional to $X_1 - X$ (elastic force) could be added to the switch input for representation of semielastic stops.

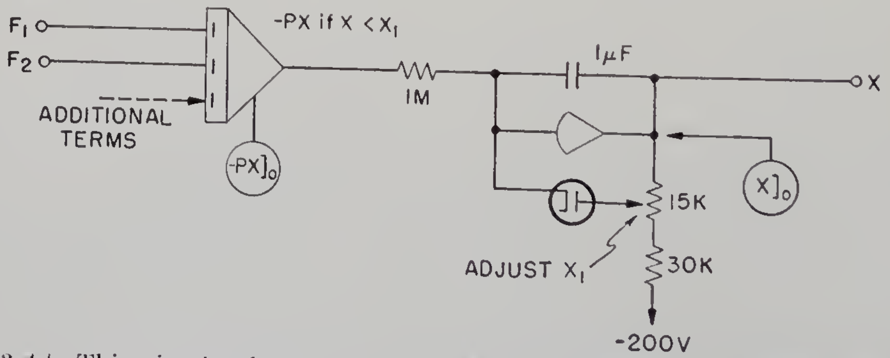

FIG. 3-4*d*. This simple shortcut representation represents an inelastic stop only roughly, since the output of the first integrator is not returned to zero on impact.

voltage

$$F = \begin{cases} 0 & \text{if } X \leq X_1 \\ -K_S(X - X_1) & \text{if } X > X_1 \end{cases}$$

which is added to the other force terms F_1, F_2, . . . indicated in the machine equation of motion (3.2). The value of K_S may exceed several thousand.

An *inelastic stop* may be described in terms of a strong viscous force proportional to the rate of penetration. The corresponding voltage would be

$$F = \begin{cases} 0 & \text{if } X \leq X_1 \\ -C_S P(X - X_1) & \text{if } X > X_1 \end{cases}$$

Figure 3.4*c* shows how one can avoid the differentiation by representing the equation of motion in the form

$$-PX = -\frac{1}{P}(F_1 + F_2 + \cdots + F)$$

The addition of a voltage proportional to $X_1 - X$ permits the representation of a *semielastic stop* which exerts both elastic and dissipative forces. Figure 3.4*d* shows a short-cut representation of inelastic stops; this is only approximately correct, since *the output voltage of the first integrator is equal to $-PX$ only before impact.*

If the particle is limited to the interval between *two stops* ($x_1 \geq x \geq x_2$), the circuits of Fig. 3.4 may be duplicated (with appropriately reversed diodes) for the second stop, or one may use one of the dead-space circuits 6.1*a*, *b*, or *c* of Table 6 (see also Fig. 3.5*b*). The circuits of Fig. 3.4 permit many other variations and generalizations.[1] In particular, the stops themselves may be moving bodies, so that x_1 becomes a variable. Since the impact force on the stop will be equal and opposite to the force represented by F in Fig. 3.4, the computer can easily deal with motions involving impacts of two or more rigid bodies. In the case of *oblique impacts* on smooth barriers, one must consider components of the impact force in the coordinate directions (see also Sec. 3.11). It is, for instance, possible to simulate the rebound of a billiard ball from a barrier.

Example. **Backlash in Spring-loaded Couplings (Linkages, Cams) and Gears.** Figure 3.5*a* illustrates a type of motion which is encountered in many dynamical systems involving spring-loaded linkages, cams, or gears. It is desired to investigate the linear motion of two bodies coupled through a pair of elastic stops and a loading spring intended to modify the backlash. Given the respective masses m and m_1 of the two bodies, their equations of motion are

$$\frac{d^2x}{dt^2} = \frac{1}{m}(f_1 + f) \qquad \frac{d^2x_1}{dt^2} = \frac{1}{m_1}(g_1 - f)$$

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[1] See also Morrill and Baum, *op. cit.;* and C. A. Meneley and C. D. Morrill, Application of Electronic Differential Analyzers to Engineering Problems, *Proc. IRE,* **41**: 1487, 1953.

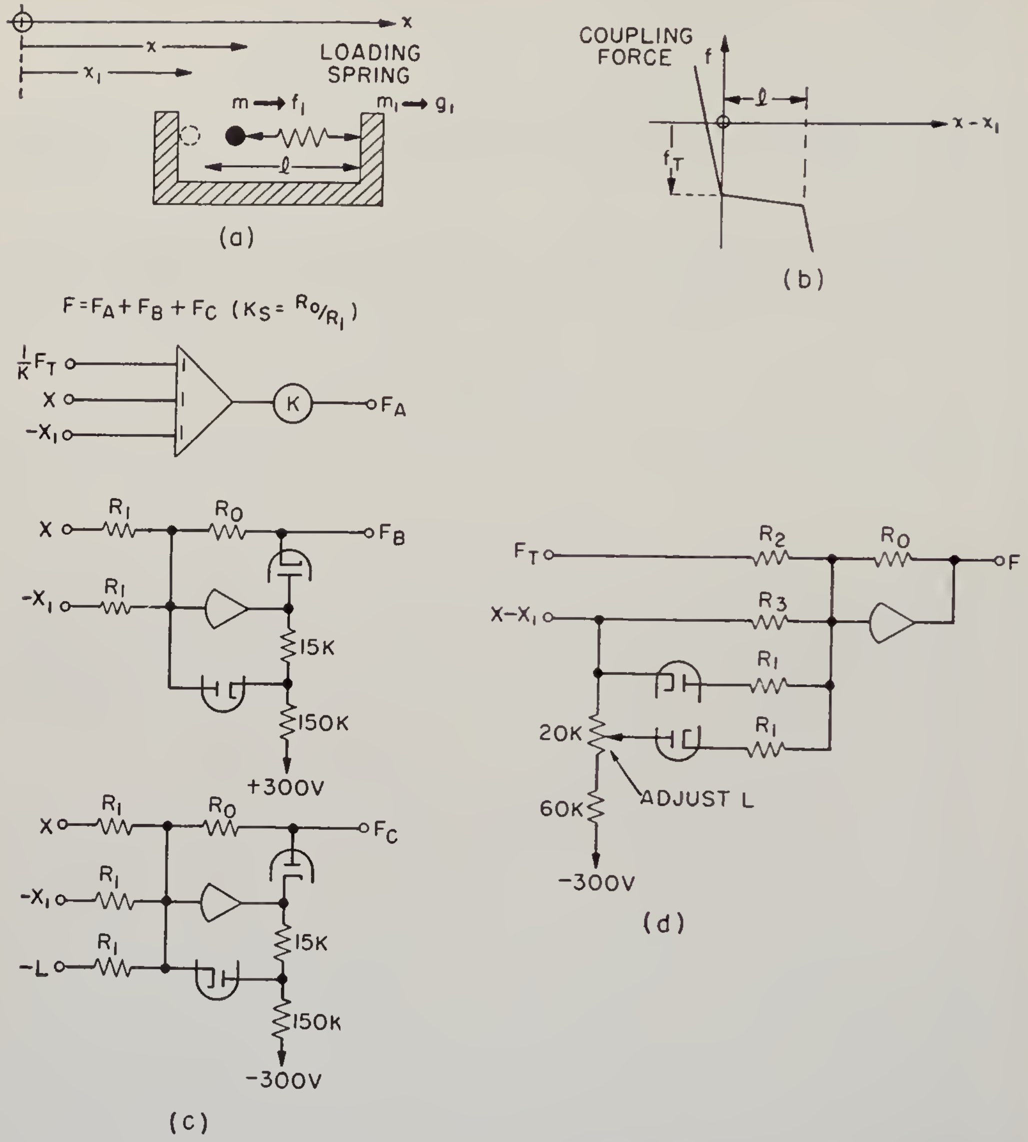

FIG. 3.5. Representation of the coupling force associated with spring-loaded linkages, cams, gears, etc., subject to backlash (the special case $K = F_T = 0$ corresponds to ordinary backlash, without a loading spring). These computer setups apply, in particular, to the representation of backlash in spring-loaded or ordinary gears (see text).

f_1 and g_1 are external forces in the x direction, and f is the *coupling force* given by

$$\left.\begin{aligned}
&f = f_A + f_B + f_C \\
&f_A = -f_T - k(x - x_1) \qquad \text{(FORCE DUE TO LOADING SPRING)} \\
&f_B = \begin{cases} 0 & \text{if } x - x_1 \geq 0 \\ -k_S(x - x_1) & \text{if } x - x_1 < 0 \end{cases} \qquad \text{(FORCE DUE TO LEFT STOP)} \\
&f_C = \begin{cases} 0 & \text{if } x - x_1 \leq l \\ -k_S(x - x_1 - l) & \text{if } x - x_1 > l \end{cases} \qquad \text{(FORCE DUE TO RIGHT STOP)}
\end{aligned}\right\} \tag{3.4}$$

where f_T is the take-up force of the loading spring, k is its stiffness, and k_S is the stiffness of the stops. Figure 3.5*b* shows how the coupling force f varies with the relative displacement $x - x_1$.

Given the transformation equations

$$X = a_x x \qquad X_1 = a_x x_1 \qquad \tau = \alpha_t t \tag{3.5}$$

let

$$F_1 = \frac{a_x}{\alpha_t^2 m} f_1 \qquad G_1 = \frac{a_x}{\alpha_t^2 m_1} g_1$$

$$F_T = \frac{a_x}{\alpha_t^2 m} f_T \qquad K = \frac{k}{\alpha_t^2 m} \qquad K_S = \frac{k_S}{\alpha_t^2 m} \qquad L = a_x l$$

The machine equations become

$$P^2 X = F_1 + F \qquad P^2 X_1 = G_1 - \frac{m}{m_1} F \tag{3.6}$$

where the voltage F, which represents the coupling force (3.4) on the mass m, is given by

$$\left.\begin{aligned} F &= F_A + F_B + F_C \\ F_A &= -F_T - K(X - X_1) \\ F_B &= \begin{cases} 0 & \text{if } X - X_1 \geq 0 \\ -K_S(X - X_1) & \text{if } X - X_1 < 0 \end{cases} \\ F_C &= \begin{cases} 0 & \text{if } X - X_1 \leq L \\ -K_S(X - X_1 - L) & \text{if } X - X_1 > L \end{cases} \end{aligned}\right\} \tag{3.7}$$

Figure 3.5*c* indicates how the three terms F_A, F_B, and F_C may be generated as functions of X and X_1. The voltage F_A represents the elastic force of the loading spring and is generated as in Fig. 3.3*b*. F_B and F_C represent the forces due to the elastic stops and are generated in the manner of Fig. 3.4*b*. Figure 3.5*d* shows a simpler but less accurate circuit for generating the voltage F.

The situation described by Fig. 3.5 reduces to the case of *simple backlash* (no loading spring) if $f_T = k = F_T = K = 0$.

The computer setups of Fig. 3.5 apply, in particular, to the representation of *backlash in gears* with or without spring loading. Let $x = \vartheta$ and $\frac{1}{n} x_1 = \vartheta_1$ denote the respective angular displacements of two shafts coupled by gears with or without spring loading; $1/n$ is the gear ratio. The equations of motion are

$$\frac{d^2x}{dt^2} = \frac{1}{m}(f_1 + f) \qquad \frac{1}{n}\frac{d^2x_1}{dt^2} = \frac{1}{m_1}(g_1 - nf)$$

where m and m_1 are now the respective moments of inertia associated with the first and second shaft, and f_1 and g_1 are external torques. f is the coupling torque on the first shaft and is given by Eq. (3.4) if f_T is interpreted as the take-up torque, k and k_S as rotational stiffnesses, and l as the backlash angle, all referred to the first shaft. Given the transformation equations (3.5), the machine equations become

$$P^2 X = F_1 + F \qquad P^2 X_1 = nG_1 - n^2 \frac{m}{m_1} F \tag{3.8}$$

where F_1, G_1, and F are defined as before. The voltage F obtained from the circuits of Fig. 3.5 now represents the coupling torque (torque transmitted by the gears) on the first shaft.

One may represent *backlash with inelastic stops* or *semielastic stops* by applying dead-space voltages (obtained as in Fig. 3.5*c* or *d*) in the manner of Fig. 3.4*c*. The backlash circuit 6.2 of Table 6 is an example of such a computer setup. This circuit is frequently used to represent the effect of backlash on the displacement x of a load, and the reaction of the load on the displacement x_1 of the driving member is neglected. Such an approximation is justified in the case of a geared-down light load not subject to appreciable disturbing torques, such as a potentiometer (Sec. 3.4). In the general case, the effect of the coupling force or torque on both driving and driven member must be represented in the computer setup as specified by the equations of motion (3.6) or (3.8).

Representation of Constraints: A Simple Example.[1] The dynamical system shown in Fig. 3.6*a* comprises a weight of mass m coupled to a gear of radius r and moment of inertia I through a toothed rack. The system is described by the vertical displacement x of the weight and the angular displacement ϑ of the gear about its fixed axis. If there is no backlash, x and ϑ are not independent coordinates but must satisfy the *accessory condition* or *constraint*

$$x + r\vartheta = 0 \tag{3.9}$$

The constraint gives rise to a force f on the weight and to a corresponding torque rf on the gear, so that the equations of motion are

$$\frac{d^2x}{dt^2} = \frac{1}{m}f - g \qquad \frac{d^2\vartheta}{dt^2} = \frac{1}{I}(rf - q) \tag{3.10}$$

where g is the acceleration of gravity, and q is an external torque.

The force f must vary precisely so as to enforce the constraint (3.9). In the simple example shown, it is easy to eliminate f and ϑ from Eqs. (3.9) and (3.10); the resulting relation

$$\left(m + \frac{I}{r^2}\right)\frac{d^2x}{dt^2} = -mg + \frac{q}{r} \tag{3.11}$$

describes the motion in terms of x alone. The correct constraint force (force exerted by the gear teeth) is

$$f = m\,\frac{gI + rq}{mr^2 + I}$$

The simple differential equation (3.11) is easily solved on a d-c analog

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[1] To the best of the writers' knowledge, D. Greenwood (lectures presented at the University of Southern California, Los Angeles, 1954) originated the methods described here by representing (linear) constraints in terms of elastic forces. The writers are grateful to Dr. Greenwood for introducing them to this interesting subject.

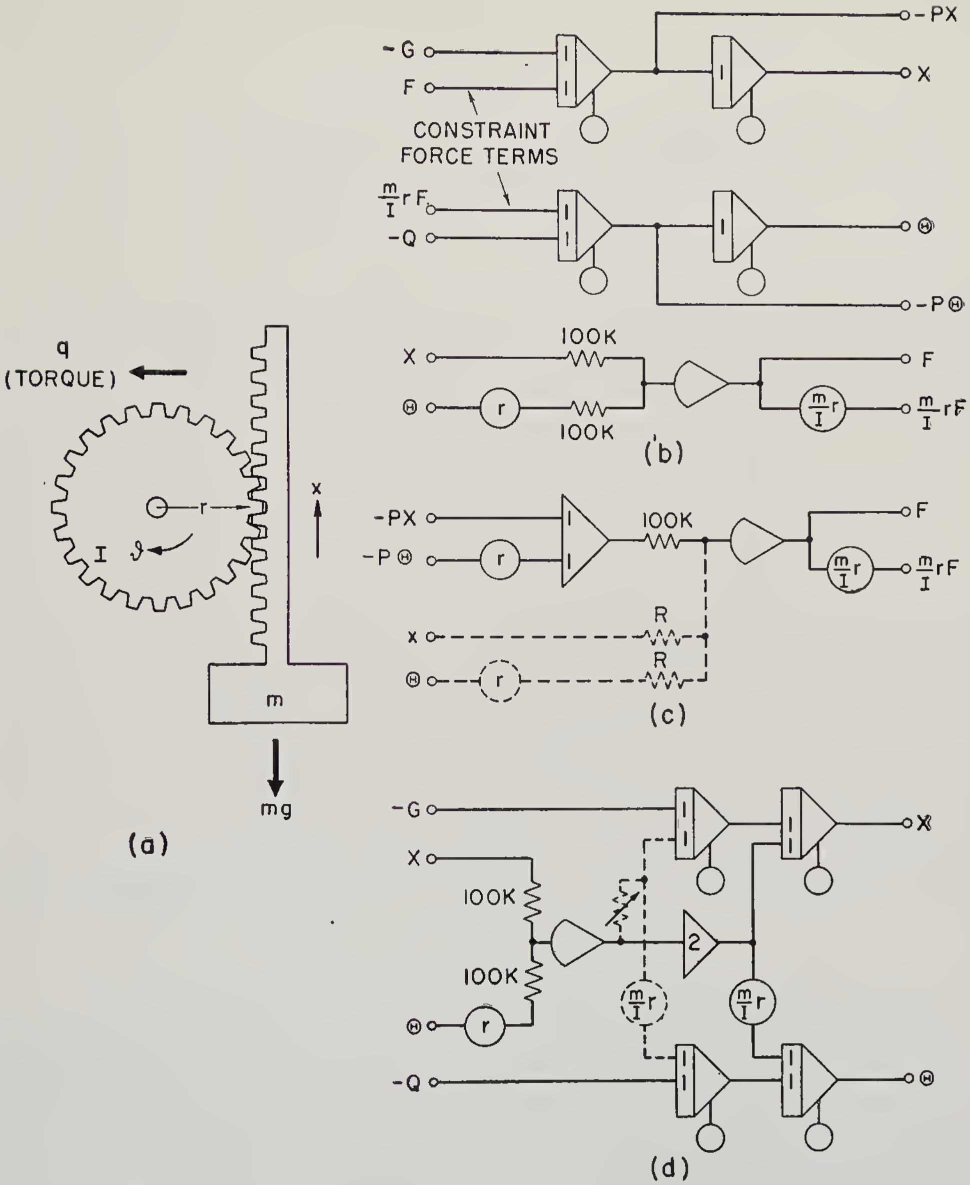

FIG. 3.6. Representation of linear constraints in terms of elastic and/or viscous constraint forces. (In the example shown, $r \leq 1$, $mr/I \leq 1$.) A small integrating capacitor (10 to 200 $\mu\mu$f) connected across the high-gain amplifier may benefit both stability and accuracy. The high-gain amplifier may have to be feedback-limited to avoid momentary overloads (comparator circuit 6.6 of Table 6).

computer. It is, however, instructive to deal with the equations of motion (3.10) in their original form, because (1) it is not always possible to reduce the number of variables explicitly, and (2) it may be useful to study the effect of a constraint imposed on a system *already set up on the computer.* If x, ϑ, and t are represented by corresponding machine variables

$$X = a_x x \qquad \Theta = a_x \vartheta \qquad \tau = \alpha_t t$$

the machine equations corresponding to Eqs. (3.9) and (3.10) become

$$X + r\Theta = 0$$

and

$$P^2X = F - G \qquad P^2\Theta = \frac{m}{I} rF - Q$$

with

$$G = \frac{a_x}{\alpha_t^2} g, \qquad Q = \frac{a_x}{\alpha_t^2 I} q$$

The machine variable F represents the constraint force f and must be made to vary so that $X + r\Theta = 0$; the initial values of X and Θ must, of course, satisfy the constraint.

In the computer setup of Fig. 3.6*b*

$$F = -\frac{|A|}{2}(X + r\Theta)$$

so that $f = \frac{\alpha_t^2 m}{a_x} F$ is approximated by an *elastic force* resisting violations of the constraint (3.9) with a high stiffness represented by the large (negative) gain A of a high-gain d-c amplifier. If the resulting feedback arrangement is stable, the constraint can be enforced within better than 0.2 volt; errors correspond to small elastic deformations of the teeth in Fig. 3.6*a*.

In the computer setups indicated by the solid lines of Figs. 3.6*c* and *d*,

$$F = -|A|P(X + r\Theta)$$

i.e., f is approximated by a *viscous force* with a very large damping coefficient. The dash lines show how the viscous force may be supplemented by an elastic force. The exact type of constraint force used makes no difference, as long as the "error" $X + r\Theta$ is kept small; the use of viscous forces will tend to produce more stable computer setups.

Representation of Constraints: General Case. In the general case, the constraints need not be linear. Each constraint

$$\varphi(x_1, x_2, \ldots, x_n) = 0 \tag{3.12}$$

imposed on the generalized coordinates $x_1, x_2, \ldots, x_n$ of a dynamical system gives rise to n generalized forces (constraint-force components)

$$\lambda(t)\frac{\partial\varphi}{\partial x_1}, \lambda(t)\frac{\partial\varphi}{\partial x_2}, \ldots, \lambda(t)\frac{\partial\varphi}{\partial x_n}$$

which act on the respective generalized coordinates $x_1, x_2, \ldots, x_n$ so as to enforce the constraint.[1] The constraint force component $\lambda(t)\,\frac{\partial\varphi}{\partial x_k}$ corresponding to each constraint must be added to the other force terms appearing in the equation of motion for x_k. For a large class of systems subject to a constraint (3.12), the resulting equations of motion may be written in the form

$$\frac{d^2x_k}{dt^2} = \frac{1}{m_k}\left[\lambda(t)\,\frac{\partial\varphi}{\partial x_k} + f_{1k} + f_{2k} + \cdots\right] \qquad (k = 1, 2, \ldots, n) \qquad (3.13)$$

where m_k is a measure of inertia, and $f_{1k}, f_{2k}, \ldots$ are (generalized) forces or kinetic reactions given as functions of coordinates and velocities.

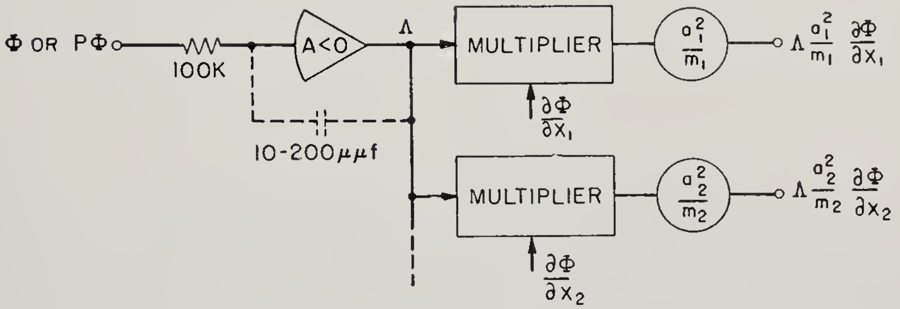
Φ OR PΦ
100K
A<0
Λ
MULTIPLIER
$\frac{a_1^2}{m_1}$
$\Lambda \frac{a_1^2}{m_1} \frac{\partial\Phi}{\partial X_1}$
$\frac{\partial\Phi}{\partial X_1}$
10-200μμf
MULTIPLIER
$\frac{a_2^2}{m_2}$
$\Lambda \frac{a_2^2}{m_2} \frac{\partial\Phi}{\partial X_2}$
$\frac{\partial\Phi}{\partial X_2}$

FIG. 3.7. Representation of constraints.

If the variables $x_1, x_2, \ldots, x_n$ and t are represented by the corresponding machine variables

$$X_1 = a_1x_1, X_2 = a_2x_2, \ldots, X_n = a_nx_n \qquad \tau = \alpha_t t$$

the machine equations corresponding to Eqs. (3.12) and (3.13) are, respectively,

$$\varphi\left(\frac{X_1}{a_1}, \frac{X_2}{a_2}, \ldots, \frac{X_n}{a_n}\right) \equiv \Phi(X_1, X_2, \ldots, X_n) = 0 \qquad (3.14)$$

and

$$P^2X_k = \Lambda\,\frac{a_k^2}{m_k}\,\frac{\partial\Phi}{\partial X_k} + F_{1k} + F_{2k} + \cdots \qquad (k = 1, 2, \ldots, n) \qquad (3.15)$$

The first term in Eq. (3.15) represents the constraint force component; *Λ is a machine variable which must be made to vary so that the constraint* (3.14) *is enforced.* Figure 3.7 suggests a computer setup in which Λ is obtained as the output voltage of a high-gain amplifier whose input voltage is

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[1] It is assumed that $\varphi(x_1, x_2, \ldots, x_n)$ is continuous and has single-valued, piecewise-continuous derivatives for all $x_1, x_2, \ldots, x_n$ satisfying Eq. (3.12). n equations of motion and m constraints should just suffice to determine n generalized coordinates and m different "Lagrange multipliers" $\lambda(t)$ as functions of the time t. The initial values of $x_1, x_2, \ldots, x_n$ must satisfy all constraints. See also Lindsay and Margenau, *op. cit.*, for a more general form of Lagrange's equations (3.13).

$$\Phi \equiv \Phi(X_1,X_2, \ . \ . \ . \ ,X_n) \qquad \text{or} \qquad P\Phi \equiv \sum_{k=1}^{n} \frac{\partial \Phi}{\partial X_k} PX_k$$

If the resulting feedback arrangement is stable, Λ approximates the ideal value $\frac{1}{\alpha_t^2}\lambda(t)$, and the computer represents the constraint forces either as elastic forces minimizing the "error" Φ, or as viscous forces minimizing the "error rate" $P\Phi$, and thus the power dissipated by the constraint forces. The analog-computer representation of constraints implies small violations of the constraints, corresponding to small deformations of constraining structures, gears, levers, ropes, etc. In a sense, the analog-computer representation is, thus, more "natural" than the analytical representation.

3.4. D-C ANALOG-COMPUTER STUDIES OF AUTOMATIC CONTROL SYSTEMS: SERVOMECHANISMS AND PROCESS CONTROL

Perhaps the most extensive and fruitful applications of d-c analog computers have been in the field of automatic control engineering. D-c analog-computing elements lend themselves naturally to the representation of feedback loops analogous to those used in control systems, and the analog nature of the computer input and output data permits one to introduce components of actual systems into the feedback loops for *partial system tests.* Investigations of different components and systems can thus be made under very realistic conditions (see also Sec. 3.8).

The following section is intended as an introduction to the analysis and synthesis of automatic control systems by means of d-c analog computers. For an exhaustive treatment of the mathematical theory of servomechanisms and other feedback systems, the reader is referred to the textbooks[1] on this subject.

Performance Equations of a Simple Servomechanism. Figure 3.8*a* shows a simple servomechanism designed to position a load so as to fol-

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[1] Brown, G. S., and D. P. Campbell, *Principles of Servomechanisms*, Wiley, New York, 1948; Chestnut, H., and R. W. Mayer, *Servomechanisms and Regulating System Design*, Vols. I and II, Wiley, New York, 1951 and 1955; Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948; James, H. M., N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms*, MIT Radiation Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947; Lauer, H., R. Lesnick, and L. E. Matson, *Servomechanism Fundamentals*, McGraw-Hill, New York, 1947; Thaler, G. J., and R. G. Brown, *Servomechanism Analysis*, McGraw-Hill, New York, 1954; Truxal, J., *Automatic Feedback Control System Synthesis*, McGraw-Hill, New York, 1955.

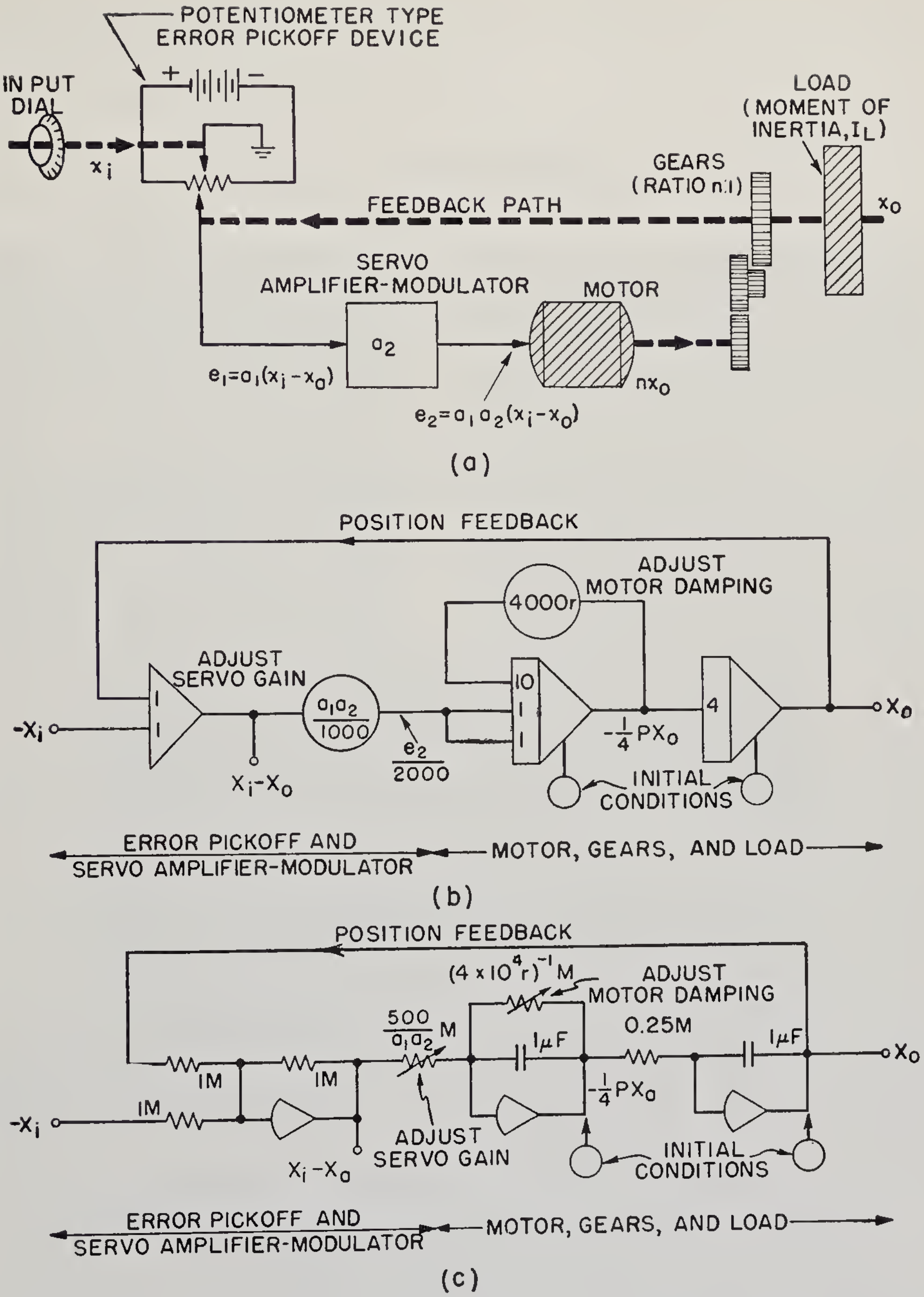

Fig. 3.8. A simple servomechanism designed to make the angular displacement x_o of a load follow the angular displacement x_i of an input dial, and two d-c analog-computer setups simulating this servomechanism on a 1:1 time scale. Note how easily changes in the design parameters of amplifier, motor, and load can be introduced into the computer setup.

low the motion of a control dial. A potentiometer-type pickoff device measures the *output error*

$$\epsilon = x_i - x_o \tag{3.16}$$

and produces a d-c *error voltage*

$$e_1 = a_1\epsilon = a_1(x_i - x_o)$$

which modulates a 60-cps a-c carrier. The modulated a-c is amplified and controls the torque of a two-phase servomotor so as to reduce the error (3.16). The amplitude of the motor control-field voltage is

$$e_2 = a_2e_1 = a_1a_2\epsilon = a_1a_2(x_i - x_o)$$

where a_2 is the amplifier-modulator gain. The (linearized) equation of motion of the system consisting of motor, gear train, and load is

$$(I_L + n^2I_M)\frac{d^2x_o}{dt^2} + n^2r\frac{dx_o}{dt} = na_3e_2 = na_1a_2a_3(x_i - x_o) \tag{3.17}$$

where I_L is the moment of inertia of the load and I_M is the moment of inertia of the motor armature including that of the shaft and gear pinion; the effect of the inertia of the other gears is considered negligible. n is the gear ratio between motor and load, r is the motor damping coefficient, and a_3 is a parameter relating the motor stall torque to the input voltage. The values of r and a_3 may be obtained from the manufacturer's data on the speed-torque characteristics of the particular motor used.[1] The effects of any time delays involved in "building up" the servo torque in the motor mechanism are considered as negligible in this analysis [see also Eq. (3.20)].

D-c Analog-computer Setup for the Performance Equation of the Simple Servomechanism. A d-c analog-computer setup establishing a machine equation corresponding to the servo performance equation (3.17) will now be designed for the following combination of parameter values:

$$I_L = 0.05 \text{ slug-ft}^2 \qquad I_M = 2 \times 10^{-5} \text{ slug-ft}^2$$
$$n = 100 \qquad a_1 = 20 \text{ volts/radian} \qquad a_2 = 25$$
$$a_3 = 2 \times 10^{-5} \text{ ft-lb/volt}$$

It is desired to study the effect on the servo performance of varying the motor damping coefficient r between the values

$$r \approx 0 \qquad \text{and} \qquad r = 2 \times 10^{-4} \text{ ft-lb-sec}$$

Physically, one may change the value of r, for instance, by changing the resistance of the motor armature winding.

If x_o, x_i, and ϵ are assumed to vary between the limits plus and minus

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[1] See, for instance, Lauer, Lesnick, and Matson, *op. cit.*

2 radians, the problem variables x_o, x_i, and t can be represented by the corresponding machine variables

$$X_o = \tfrac{1}{2}x_o \qquad X_i = \tfrac{1}{2}x_i \qquad \tau = t$$

where the voltages X_o and X_i are measured in machine units of, say, 100 volts. The scale factors chosen are, of course, subject to later revision. The resulting d-c analog-computer solution will be on a 1:1 time scale. The machine equation corresponding to Eq. (3.17) becomes

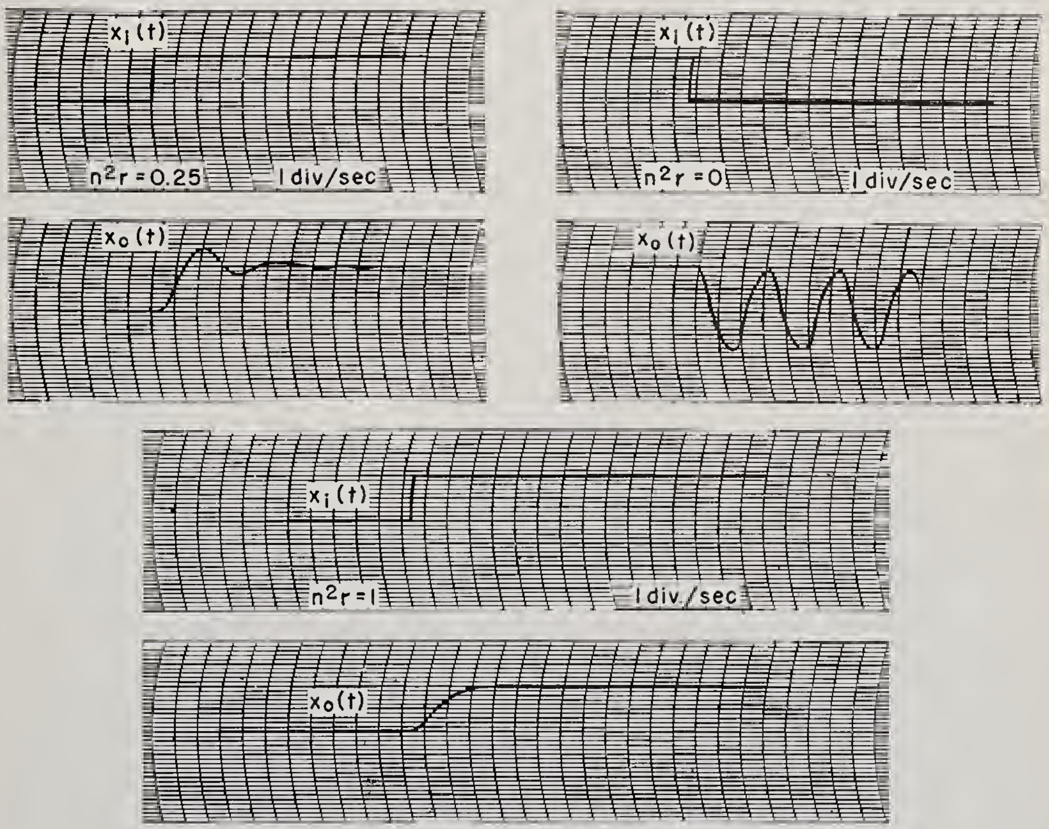

FIG. 3.9. D-c analog-computer records (Hagelbarger, Howe, and Howe, *op. cit.*), showing the response of the servomechanism of Fig. 3.8 to a step-function input displacement for three values of the motor-damping coefficient r.

$$P^2X_o + 4 \times 10^4 rPX_o = 8\,\frac{a_1a_2}{1000}\,(X_i - X_o) = 4(X_i - X_o)$$

Figure 3.8*b* shows the corresponding block diagram. The computer setup is essentially the same as that for the damped harmonic oscillator discussed in Sec. 2.7. Figure 3.8*c* shows a similar computer setup suitable for analog computers with plug-in feedback networks. Adjustable resistors are used instead of coefficient-setting potentiometers.

D-c Analog-computer Study of the Simple Servomechanism: Transient Response. The computer records of Fig. 3.9 show the response of the simple servomechanism to a step displacement of the input shaft for three values of the damping coefficient r. Parameter studies (Sec. 3.1) of this type can be made to test the transient response of a servomechanism for any number of combinations of the design parameters; the computer records may be used to pick a combination of design parameters which will result in small errors $x_i - x_o$ without any tendency toward uncontrolled oscillations or hunting. A number of quantitative figures of merit for servomechanism performance will be discussed below.

A Linear Servomechanism with Rate Feedback and Equalizing Network. Figures 3.10*a* and 3.10*b* illustrate a somewhat more complex servomechanism incorporating rate feedback from a d-c tachometer as well as a special equalization network which produces both derivative and integral control. The linearized performance equation is

$$[(I_L + n^2 I_M)p^2 + n^2 rp]x_o = na_2a_3 \left[a_1 \frac{\alpha p + 1}{\beta p + 1} (x_i - x_o) - nk_1 p x_o \right] \tag{3.18}$$

where k_1 is the tachometer sensitivity, and a_1, α, and β are parameters describing the design of pickoff and equalizing network. The other symbols are the same as those used in Fig. 3.8.

The computer block diagram of Fig. 3.10*c* has been drawn intentionally so as to emphasize the analogy between components of the actual servomechanism and corresponding blocks of computing elements; the control signal can be traced through the computer setup just as it can be traced through the servomechanism itself. *Note that the addition of the equalizing network and the rate feedback to the servomechanism of Fig.* 3.8 *has been represented by very simple modifications of the computer setup of Fig.* 3.8*c*. Different combinations of the equalizer parameters α and β are easily translated into corresponding combinations of capacitances C_1 and C_o.

Analysis and Synthesis of Servomechanisms: Evaluation of Computed Transient Response Records. In parameter studies of automatic control systems, such as the simple study of optimum damping illustrated by the computer records of Fig. 3.9, it becomes necessary to formulate qualitative criteria or quantitative figures of merit which will permit definite choices of "optimum" combinations of the design parameters on the basis of the corresponding computer records. Most observers will agree that the servo output displacement x_o in the third computer record of Fig. 3.9 seems to follow the input displacement x_i more faithfully than that in the second computer record; but a choice between the values of the damping constant r leading to the first and third computer records of Fig. 3.9 is more difficult. Since no one number could possibly give a complete description of the system performance, the choice of a quantitative

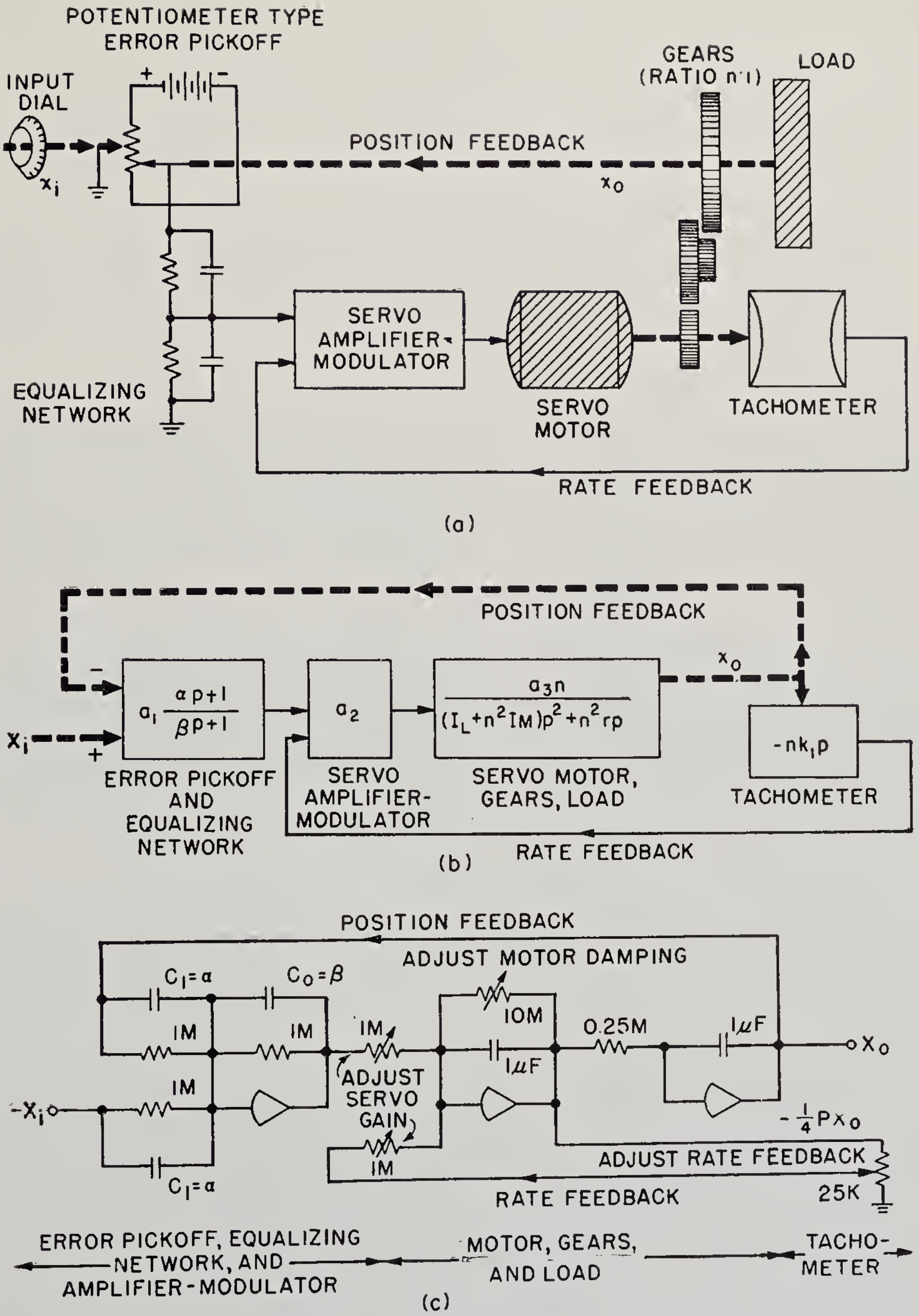

FIG. 3.10. Semischematic representation (Fig. 3.10*a*), functional block diagram (Fig. 3.10*b*), and analog-computer setup (Fig. 3.10*c*) for the linear servomechanism of Fig. 3.8 improved by the addition of an equalizing network and rate feedback. Note the analogy between components of the actual servomechanism and "blocks" of computing elements.

figure of merit will depend on the nature of the particular performance characteristic most important in each case.

A number of possible figures of merit for judging the transient performance of servomechanisms are listed below. All of them are obtained more or less directly from d-c analog-computer records of the *error* (3.16) for various input signals $x_i(t)$.

1. A somewhat crude figure of merit is the reciprocal of the time taken by the absolute value of the error ϵ to "recede" below a certain specified percentage of the input voltage after a step input voltage has been applied.

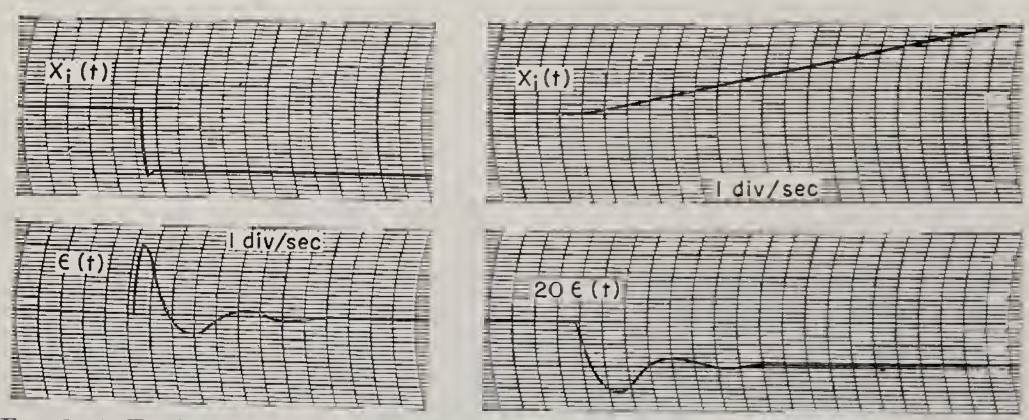

FIG. 3.11. Typical error records for a step input signal and for a ramp input signal (Hagelbarger, Howe, and Howe, *op. cit.*).

2. The *displacement figure of merit* M_0 is defined as the ratio of the output displacement x_o to the error ϵ when all derivatives of the output displacement x_o are zero. The value of M_0 is easily obtained from a computer record of $\epsilon(t)$ for a step input signal x_i (see Fig. 3.11*a*).
3. The *rate figure of merit* M_1 is defined as the ratio of the output rate px_o to the error ϵ when the output acceleration and all higher derivatives of x_o are zero. This quantity may be obtained from the computer records of $\epsilon(t)$ and px_o for a *ramp function input signal*

$$x_i = (\text{constant})\ t$$

(see Fig. 3.11*b*).
4. The *acceleration figure of merit*[1] M_2 is defined as the ratio of the acceleration p^2x_o to the error ϵ when px_o equals zero. The value of M_2 may be obtained from computer records of $\epsilon(t)$ and p^2x_o for a step input signal x_i.

If the servomechanism under consideration is linear, the displacement and rate figures of merit are simply related to the first two coefficients in a power-series expansion of the transfer function

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[1] Lauer, Lesnick, and Matson, *op. cit.*

$$f(p) = \frac{\epsilon}{x_i}$$

in terms of p.[1] If the servomechanism is not linear, all figures of merit may vary appreciably with the magnitude of the input signals.

A very useful measure[2] of servo performance is the *mean square error*

$$\overline{\epsilon^2} = \frac{1}{t_1} \int_0^{t_1} \epsilon^2(t)\, dt \tag{3.19}$$

obtained by integration of $\epsilon^2(t)$ over a fixed time t_1 corresponding to the computer run. If this measure is used, the "optimum" combination of design parameters will be the one to minimize the expression (3.19) for a given input signal, such as a unit step function. In more sophisticated studies, the measure might be a suitably weighted average of the quantity (3.19) over a set of different typical input signals. The mean-square-error criterion permits consideration of "random" effects, such as noise, stray input signals, and disturbing torques represented by appropriate voltages (Sec. 3.9).

Analysis and Synthesis of Linear Servomechanisms: Transfer Function or Frequency Analysis Methods. Powerful modern methods of servomechanism analysis[3] and synthesis are based on studies of the performance of servomechanisms components for steady-state sinusoidal oscillations of their input and output signals. More specifically, the ratio of the output and input amplitudes as well as the phase difference between the sinusoidal input and output oscillations are plotted as functions of the circular frequency ω of the oscillations, and the properties of the component or group of components in question are described in terms of these *frequency-response characteristics.*

The transient response of a linear servomechanism can be related to its frequency-response characteristics by means of Fourier analysis. It is true that the transient response of a servomechanism can be obtained directly by means of a d-c analog computer. The frequency-response method of describing servo performance is, nevertheless, particularly useful for the synthesis of linear servomechanisms because of the simple relations existing between the individual frequency-response characteristics of "transducers," such as amplifiers, networks, motors, etc., and the frequency-response characteristics of combinations of such devices connected in cascade. The amplitude ratio of the combination, for each value of the circular frequency, will be the product of the individual amplitude ratios, whereas the phase shifts are simply added. These relations help the designer to choose, for instance, the type of equalizing networks he must insert into a feedback loop in order to obtain a desired frequency response. The frequency characteristics of a servomechanism frequently not only show up undesirable conditions but also indicate what to do about them. No similarly simple relationships exist between the transient responses of individual and cascaded transducers.

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[1] Hagelbarger, D. W., C. E. Howe, and R. M. Howe, Investigation of the Utility of an Electronic Analog Computer in Engineering Problems, *UMM*-28, Engineering Research Institute, University of Michigan, Ann Arbor, Apr. 1, 1949.

[2] James, Nichols, and Phillips, *op. cit.*

[3] Refer to the textbooks quoted on p. 92.

D-c analog computers can be at least as useful for computing the frequency-response characteristics of servomechanism components and complete servomechanisms as they are for computing transient responses. After the performance equations of the devices in question have been set up in the usual manner, it is only necessary to provide a source of sinusoidal input signals and a means for measuring the relative amplitudes and phases of the input and output voltages. Although ordinary computer records of the input and output signals will serve this latter purpose, it is preferable to use a convenient signal generator capable of generating sinusoidal voltages of different frequencies and of measuring the amplitude and phase of an output signal by comparison with a reference signal. A signal source of this type is shown in Fig. 6.21.

An interesting application of such d-c analog-computer methods is the automatic computation of loop transfer functions[1] for use in phase-decibel plots and other powerful graphical methods of control-system analysis and synthesis.

Frequency-analysis methods apply, strictly speaking, only to devices whose performance equations are linear; but the frequency-response characteristics of many nonlinear devices are at least qualitatively interesting. Sometimes a linear or nonlinear transducer whose performance equation is not known can be represented in the computer by a block of operational amplifiers or networks having similar frequency-response characteristics for most input amplitudes.

Representation of Nonlinear Transfer Characteristics. Although important properties of many control systems can be studied in terms of linearized relations between system variables, it is frequently absolutely necessary to explore the effects of *nonlinear* relationships existing between the input and output signals of control-system components. The use of analog computers for such investigations is particularly important because it is often difficult or impossible to apply analytical methods. Many nonlinear effects (*e.g.*, limit stops, static and coulomb friction) are easily represented by simple diode function generators in the manner of Sec. 3.3.

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[1] James, Nichols, and Phillips, *op. cit.*

More Elaborate Representation of a Servomechanism.[1] The computer setup of Fig. 3.12 is a more accurate representation of the servomechanism shown in Fig. 3.10. Several simple "blocks" of computing elements have been combined to account for complicated nonlinear effects.

The modulated-carrier waveform has been introduced explicitly. The motor torque t_M is related to the instantaneous control-field voltage v_C and the motor speed px_1 by the nonlinear differential equation[2]

$$t_M = \frac{b_1}{b_2p + 1} v_C v_R - b_3 \left[\frac{1}{b_2p + 1} v_C \right]^2 px_1 - b_4 v_R^2 px_1 - b_5(px_1)^3 \quad (3.20)$$

where v_R is the sinusoidal reference-field voltage; the parameters b_1, b_2, b_3, b_4, and b_5 must be matched to the voltage-torque-speed characteristics of the motor used.

Special function generators (see also Sec. 6.7 and Table 6) simulate potentiometer granularity, amplifier saturation, static and coulomb friction, and tachometer dead space. The simple representation of gear backlash used neglects the reaction of the geared-down load on the motor; a more accurate representation of backlash, gear loading, and gear compliance is described in Sec. 3.3.

Computer setups like that of Fig. 3.12 can yield important design information, such as the servo frequency response for different input amplitudes and the effects of various manufacturing tolerances on servo jitter.

Representation of Relays, Clutches, and Other Nonlinear Transducers. Figures 3.13*a* to 3.13*e* illustrate the d-c analog-computer representation of relays and solenoid-actuated clutches.[3] Computer setups of this type

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[1] Based on GEDA Simulation of a Carrier-type Instrument Servomechanism, *Report GER*-5779, Goodyear Aircraft Corporation, Akron, Ohio, Apr. 21, 1954. See also Meneley and Morrill, *op. cit.*

[2] If, as is frequently the case, one does not wish to introduce the carrier waveform explicitly, the nonlinear voltage-torque-speed relation may be approximated by

$$t_M = \frac{c_1}{c_2p + 1} e_2 - c_3 \left[\frac{1}{c_2p + 1} e_2 \right]^2 px_1 - c_4 px_1 - c_5(px_1)^3$$

where e_2 is the amplitude of v_C, as in Eq. (3.17). The last term is often omitted. See also L. O. Brown, Transfer Function for a Two-phase Induction Motor, *Trans. AIEE*, preprint T-1-347, **70,** 1951; and R. J. W. Koopman, Characteristics of Two-phase Servomotors, *Elec. Eng.*, **68:** 775, 1949.

[3] The Simulation of a Relay Servomechanism, *Report GER*-2734, Goodyear Aircraft Corporation, Akron, Ohio, July 16, 1951.

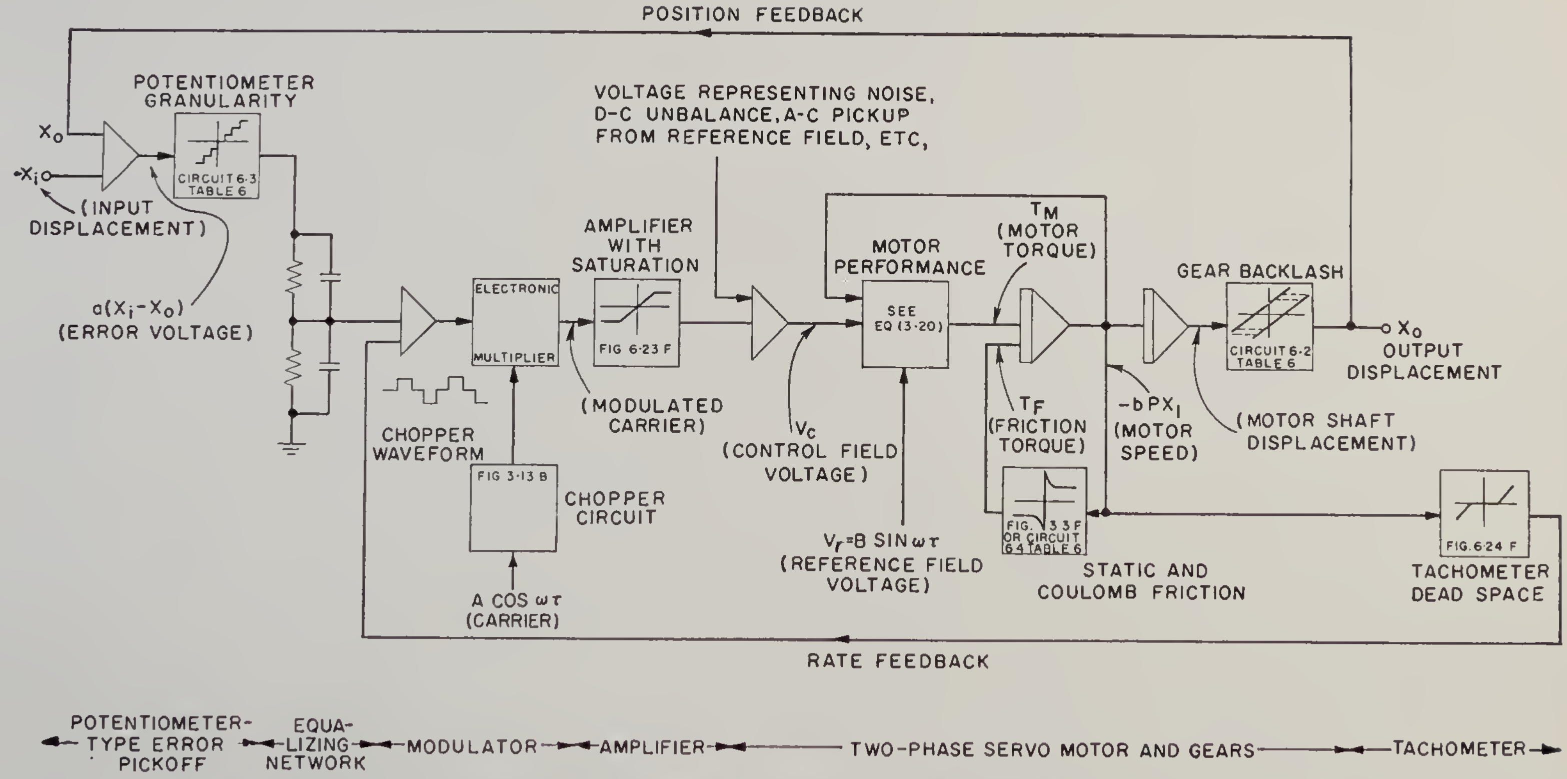
POSITION FEEDBACK
POTENTIOMETER GRANULARITY
X_o
X_i
(INPUT DISPLACEMENT)
CIRCUIT 6-3 TABLE 6
$a(X_i - X_o)$ (ERROR VOLTAGE)
VOLTAGE REPRESENTING NOISE, D-C UNBALANCE, A-C PICKUP FROM REFERENCE FIELD, ETC,
ELECTRONIC MULTIPLIER
AMPLIFIER WITH SATURATION
FIG 6-23 F
(MODULATED CARRIER)
CHOPPER WAVEFORM
FIG 3-13 B
CHOPPER CIRCUIT
A COS ωτ (CARRIER)
V_c (CONTROL FIELD VOLTAGE)
MOTOR PERFORMANCE
SEE EQ (3-20)
V_r=B SIN ωτ (REFERENCE FIELD VOLTAGE)
T_M (MOTOR TORQUE)
T_F (FRICTION TORQUE)
FIG. 3 3 F OR CIRCUIT 6 4 TABLE 6
STATIC AND COULOMB FRICTION
$-bPX_1$ (MOTOR SPEED)
GEAR BACKLASH
CIRCUIT 6-2 TABLE 6
(MOTOR SHAFT DISPLACEMENT)
X_o OUTPUT DISPLACEMENT
FIG. 6-24 F
TACHOMETER DEAD SPACE
RATE FEEDBACK
POTENTIOMETER-TYPE ERROR PICKOFF
EQUALIZING NETWORK
MODULATOR
AMPLIFIER
TWO-PHASE SERVO MOTOR AND GEARS
TACHOMETER

FIG. 3.12. A more elaborate d-c analog-computer setup representing the servomechanism of Fig. 3.10 shows how simple "blocks" of computing elements may be combined to account for complicated nonlinear effects. Simulation on a 1:1 time scale and partial system testing is possible if sufficiently fast computing elements are available (based on Goodyear Aircraft Corporation *Report GER*-5779).

have served not only for engineering design studies but also as valuable aids to the theoretical exploration of nonlinear control systems.[1]

Figure 3.13*f* indicates a possible representation of a solenoid valve whose displacement y is determined by an input voltage e, and a hydraulic actuator whose piston displacement x_o is, in turn, controlled by the valve displacement. The performance equations of the hydraulic control mechanism have been approximated in the form[2]

$$y = \frac{a_1}{(bp + 1)(mp^2 + rp + k)} e \qquad x_o = \frac{a_2}{p} y \tag{3.21}$$

where a_1 and a_2 are gain coefficients, b is the time delay due to the solenoid, and m, r, and k are, respectively, the mass, damping coefficient, and spring constant associated with the valve. The rate px_o of the output motion is limited by the flow capacity of the hydraulic system, and the piston displacement x_o is limited by inelastic stops (see also Sec. 3.3).

The backlash circuit 6.2 of Table 6 represents a transducer whose transfer characteristics depend on the "direction" of the input rate of change. The addition of appropriate function generators to this circuit permits simulation of *hysteresis loops* such as those typical of magnetic circuits. Similar arrangements make it possible to simulate the performance cycles of heat engines, pumps, storage batteries, etc.

Applications of D-c Analog Computers to Process Control.[3] Figures 3.14*a* and 3.14*b* (based on *EASE Application Bulletin* 3, see footnote) illustrate a typical process-control problem. A temperature-sensitive transmitter pneumatically positions a valve which controls the heat supply of a chemical process so as to minimize deviations of the process tem-

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[1] Rogers, T. A., and W. C. Hurty, Relay Servomechanisms: The Shunt-motor Servo with Inertia Load, *Trans. ASME*, Paper 50-S-13, p. 1163, November, 1950; Walters, L. G., A Study of the Series-motor Relay Servomechanism, Ph.D. thesis, University of California, Los Angeles, 1951; GEDA Simulation Study of a Relay Servomechanism, *Report GER*-4799, Goodyear Aircraft Corporation, Akron, Ohio, May 9, 1952.

[2] McCann, G. D., C. H. Wilts, and B. N. Locanthi, *Trans. AIEE*, **68**, Paper 49–165, 1949. For a more complete discussion of hydraulic servomechanisms, see M. R. Hannah, Frequency Response Measurements of a Hydraulic Power Unit, Paper 47-A-70, ASME Annual Meeting, Dec. 1–5, 1947.

[3] Philbrick, G. A., Designing Industrial Controllers by Analog, *Electronics*, June, 1945; Temperature Control a Simple Problem on Computer, *EASE Application Bulletin* 3, Berkeley Division, Beckman Instruments, Inc., Richmond, Calif.; Simulation of a Process Controller, *Report GER*-4689, Goodyear Aircraft Corporation, Akron, Ohio, Mar. 11, 1952; Hovius, R. L., C. D. Morrill, and N. P. Tomlinson, Industrial Uses of Analog Computers, *Instruments and Automation*, **28**: 594, April, 1955. See also P. R. Bell and H. A. Straus, Electronic Pile Simulator, *Rev. Sci. Instr.*, **21**: 760, 1950; and Determination of Turbine Speed Controller Settings for Shell Development Company, Emeryville, California, *Application Bulletin* 1, Beckman Instruments, Inc. (now Berkeley Division, Richmond, Calif.).

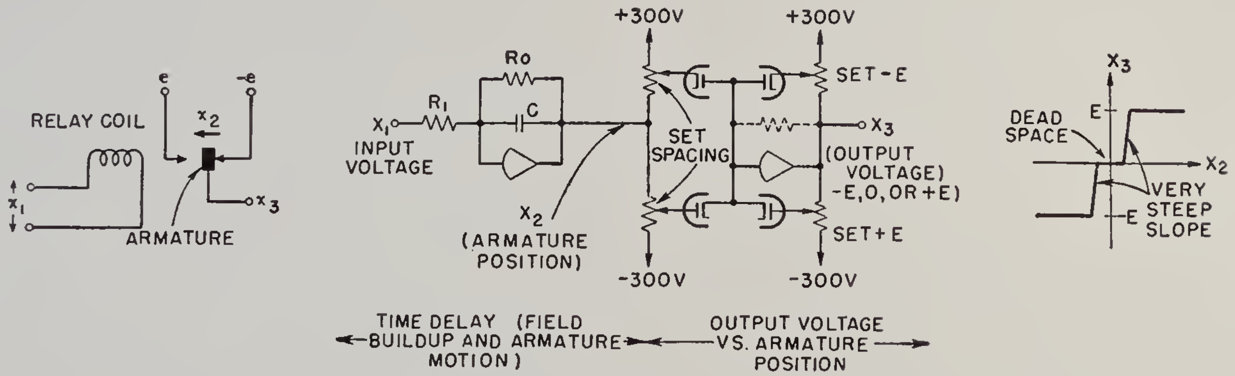
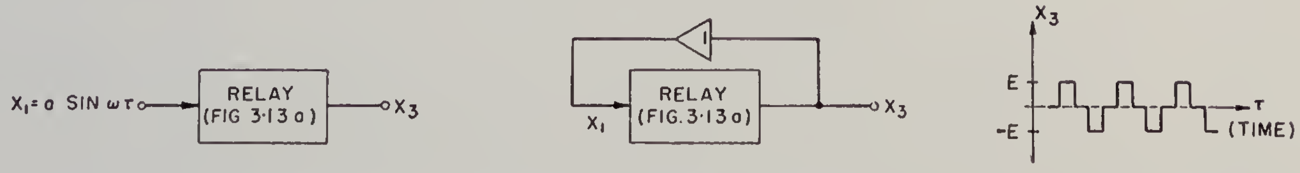


Fig. 3.13*a*. Representation of a polarized double-throw relay. The delays involved in field-current buildup and armature motion have been approximated by the transfer function $\frac{X_1}{X_2} = -\frac{R_o}{R_1}\frac{1}{R_oCP + 1}$. This approximation is not unreasonable for well-designed relay circuits and sufficient armature damping. A more accurate representation involves simulation of the spring-restrained armature with damping and stops in the manner of Sec. 3.3. Such a representation will permit detailed investigations of contact bouncing, effects of friction, etc.



(*b*) (*c*) (*d*)

Fig. 3.13*b*, *c*, *d*. Generation of relay-chopper waveform: externally driven relay (*b*), self-excited relay (buzzer) (*c*), and waveform (*d*).

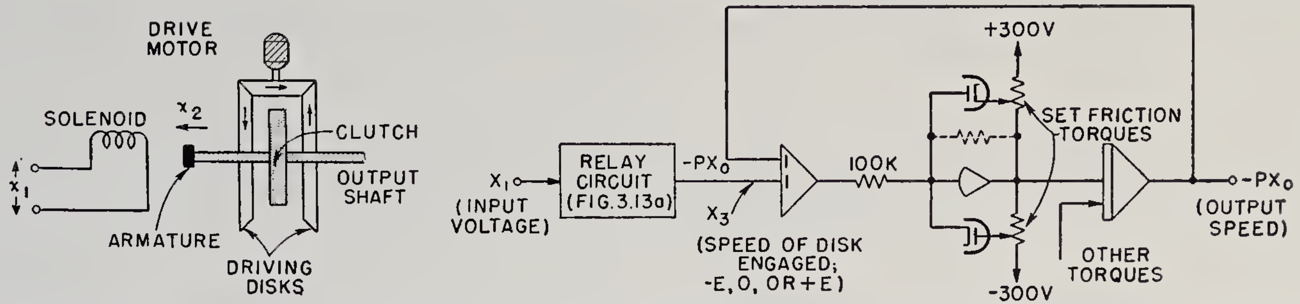


FIG. 3.13*e*. Representation of a solenoid-operated dry-disk clutch. The operating delays have been approximated as in Fig. 3.13*a*. The dry-friction torque (see also Sec. 3.3) is developed from the difference between the output speed and the speed of the driving disk engaged.

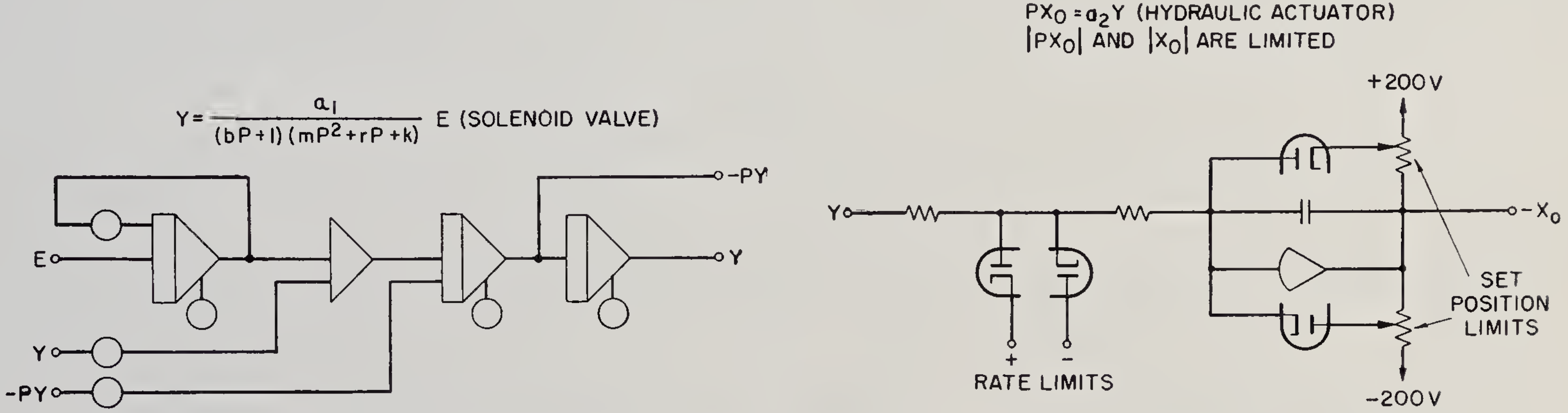

$Y = \frac{a_1}{(bP+1)(mP^2+rP+k)} E$ (SOLENOID VALVE)
$PX_0 = a_2 Y$ (HYDRAULIC ACTUATOR)
$|PX_0|$ AND $|X_0|$ ARE LIMITED
E
Y
-PY
-PY
Y
Y
+200V
-200V
$-X_0$
SET POSITION LIMITS
+
−
RATE LIMITS

FIG. 3.13*f*. Block diagram of a d-c analog-computer setup approximately simulating a hydraulic control mechanism with rate and position limiting. The P^2 term in the transfer function of the control valve is sometimes omitted.

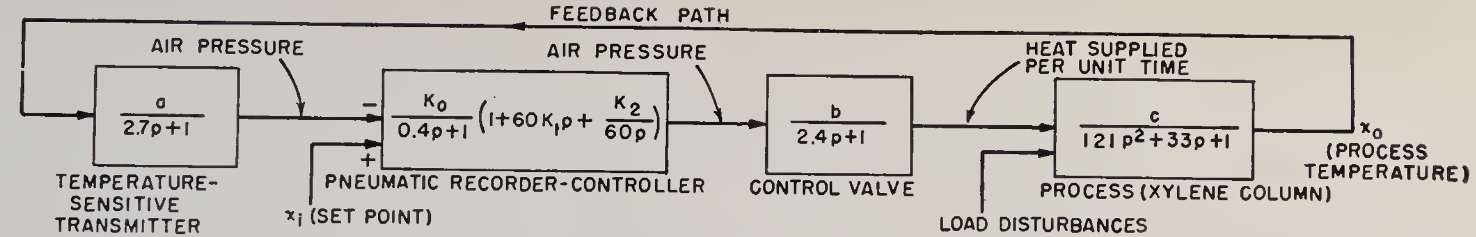


FIG. 3.14*a*. Functional block diagram of a process-control system (temperature control for a xylene column). The parameters *a*, *b*, and *c* depend on the units employed, with $abc = 8.32$. The controller parameters K_o, K_1, and K_2 are respectively known as the *proportional gain*, the *rate time* (minutes), and the *reset* rate (repeats per minute). $K_o = 1.8$, $K_1 = 0.12$ min, and $K_2 = 7$ repeats per minute were found to be favorable controller settings.

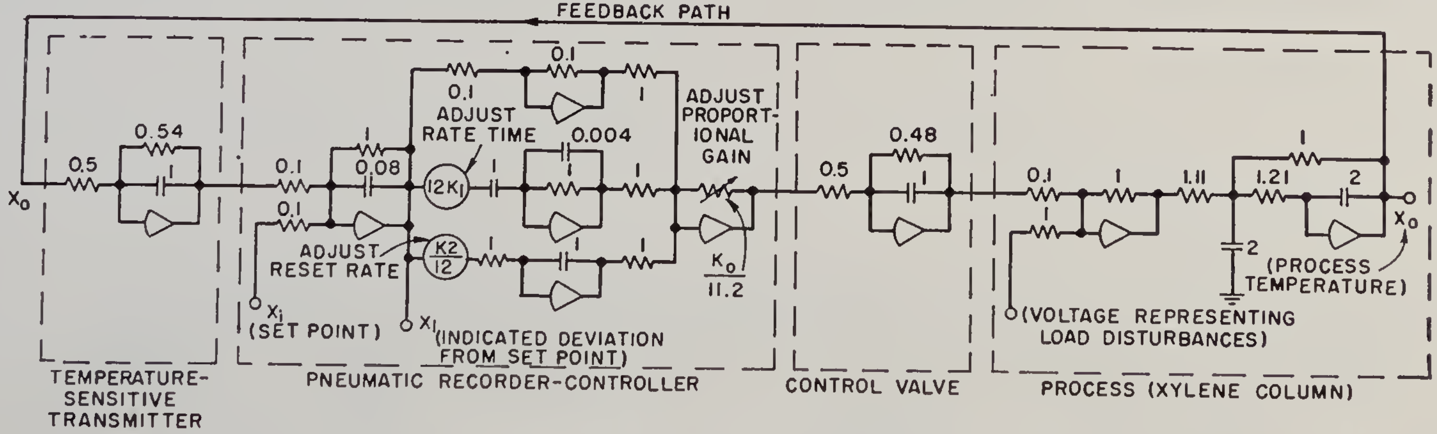


FIG. 3.14*b*. A d-c analog-computer setup representing the automatic control system of Fig. 3.14*a* on a fast time scale ($P = 5p$). The voltage X_1 corresponds to the temperature deviation indicated by a pneumatic recorder associated with the controller mechanism. Note how individual system components are represented by "blocks" of direct-analog operational amplifiers (see also Secs. 1.1 and 1.5). Nonlinearities may be introduced in the manner of Fig. 3.12.

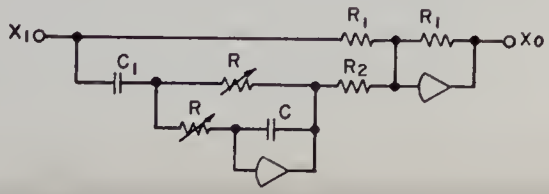


FIG. 3.14*c*. Example of an operational amplifier circuit whose transfer function $1 - \frac{R_1}{R_2}\frac{RC_1P}{R^2C_1CP^2 + 2RCP + 1}$ approximates the time-delay operator $e^{-\tau_L P}$ if, in accordance with Eq. (3.23), $R = \tau_L/4C$, $C_1 = 4C/3$, $R_2 = R_1/3$. Similar circuits may be cascaded to yield more accurate approximations of the form (3.22).

perature from a *set point.* The pneumatic control mechanism records the difference between set-point temperature and process temperature and operates on this difference (error) so as to introduce *proportional control, rate control,* and *reset rate* (integral control). The resulting automatic control system constitutes a *feedback loop* similar to that of a servomechanism; the dynamic characteristics of the servo load are here replaced by the physical characteristics of the process to be controlled.

In Fig. 3.14*a*, the performance of the temperature-detecting element, of the control valve, and of the process itself have been described by linear differential equations expressed in transfer-function form. It is desired to determine combinations of proportional control, rate control, and reset rate which will yield optimum recovery of the process temperature after a step-function disturbance; optimum recovery may be defined by inspection or by some error criterion, as for servomechanisms. The computer setup of Fig. 3.14*b* represents the process-control system of Fig. 3.14*a* on a 1:5 (fast) time scale. Direct-analog operational amplifiers (Sec. 1.5) implement the various transfer functions. Nonlinear transfer characteristics, such as dead space, backlash, or control limits are easily introduced in the manner of Fig. 3.12 (see also Table 6). The reader may find it instructive to add various effects of this sort to the computer setup of Fig. 3.14*b*.

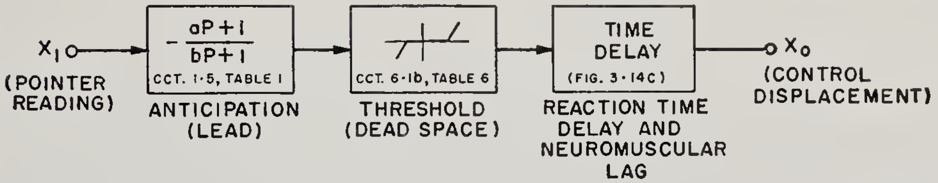


FIG. 3.14*d*. A possible computer representation of a human operator adjusting a continuous control so as to reduce an error indicated by a pointer reading (from Goodyear Aircraft Corporation *Report GER*-6608).

Representation of Time Delays.[1] *Time delays* in control systems are exemplified by the dead-period lag (transport lag) introduced by a pipe line between the reaction chamber in Fig. 3.14*a* and the temperature-sensitive transmitter. At the time t the latter will, then, sense the temperature $x(t - t_L)$ if the process temperature is $x(t)$; t_L is the time delay due to the pipe line.

A d-c analog computer can introduce time delays by means of a two-pen servo recorder (Sec. 7.3) which records a function $X(\tau)$ with one pen and uses the second pen carriage to read the recorded curve after a suitable time delay τ_L. Magnetic recording devices can be used in an analogous manner.

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[1] Morrill, C. D., A Sub-audio Time Delay Circuit, *Trans. PGEC, IRE,* **3**: 45, June, 1954; Cunningham, W. J., Time-delay Networks for an Analog Computer, *Trans. PGEC, IRE,* **3**: 16, December, 1954; Single, C. H., and G. S. Stubbs, Transport Delay Simulation Circuits, *Report WAPD-T*-38 *and supplement,* Westinghouse Electric Corporation, Atomic Power Division.

More frequently, one represents *constant* time delays in the form of an operation on $X(\tau)$:

$$X(\tau - \tau_L) = (1 - \tau_L P + \tfrac{1}{2}\tau_L^2 P^2 \mp \ldots)X(\tau) = e^{-\tau_L P}X(\tau)$$

The *time-delay operator* $e^{-\tau_L P}$ may then be replaced by the somewhat primitive approximation $1/(1 + \tau_L P)$, or by one of the successively better approximations

$$\prod_{i=1}^{m}\left(1 - \frac{2a_i\tau_L P}{1 + a_i\tau_L P + b_i\tau_L^2 P^2}\right) \qquad (m = 1, 2, \ldots) \qquad (3.22)$$

with suitably chosen coefficients a_i and b_i. It is often sufficient to use a single factor ($m = 1$), with

$$a_1 = \tfrac{1}{2} \qquad b_1 = \tfrac{1}{12} \qquad (m = 1) \qquad (3.23)$$

For $m = 2$, the coefficients a_1, a_2, b_1, b_2 satisfy the relations

$$a_1 + a_2 = \tfrac{1}{2} \qquad a_1a_2 + b_1 + b_2 = \tfrac{3}{28} \qquad a_1b_2 + a_2b_1 = \tfrac{1}{84}$$
$$b_1b_2 = \tfrac{1}{1680} \qquad (m = 2) \qquad (3.24)$$

so that

$$e^{-\tau_L P} \approx \frac{1680 - 840\tau_L P + 180\tau_L^2 P^2 - 20\tau_L^3 P^3 + \tau_L^4 P^4}{1680 + 840\tau_L P + 180\tau_L^2 P^2 + 20\tau_L^3 P^3 + \tau_L^4 P^4} \qquad (3.25)$$

One may estimate the errors involved in such approximations by computing the gain and phase shift for sinusoidal inputs ($P = j\omega$).

Operators of the form (3.22) can be implemented, factor by factor, with direct-analog operational amplifiers (Fig. 3.14*c*; see also Tables 2 and 3), or the classical differential-analyzer technique may be employed. In the latter case, each integrator can be associated with a potentiometer whose setting is proportional to $1/\tau_L$, so that the combined transfer function is $-\dfrac{\text{const}}{\tau_L P}$; the potentiometers may be ganged on one shaft to permit very convenient adjustment of the delay time τ_L.[1]

Control Systems Comprising Human Operators. In real-time studies of control systems comprising human operators, one may introduce actual human operators in simulated work positions. *Such techniques permit studies of human-engineering factors, such as the design of displays, con-*

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[1] Rudolph, J. A., GEDA Circuits to Obtain a Variable Time Delay, *GEDA Application Report* 2, Goodyear Aircraft Corporation, Akron, Ohio, June 15, 1955.

trol levers, communication links, and crew task assignments. Sometimes a human operator can be represented by a block of computing elements during at least part of a study for the sake of simplicity, or to permit changes in time scale. Figure 3.14*d* shows a possible computer representation of a human operator adjusting a single continuous control so as to reduce an error indicated by a pointer reading.[1] No parameter values are given, since they will depend radically on the task, on the type of input signals under consideration, on the design of the operating position, and on the individual operator. Changes in operator training, fatigue, etc., can be represented by suitably determined parameter changes.

Use of Integrators as Storage Devices. Sampled-data Control Systems. A parallel-feedback integrator disconnected from its input resistor or resistors constitutes an analog *storage device* which "holds" its last-computed output voltage (HOLD condition, Sec. 7.1). Integrators placed in the HOLD condition by means of relays or diode switches (Sec. 6.9) can represent information storage devices. An interesting application of this technique is the analog-computer representation of sampled-data control systems.[2] Here, periodically switched storage integrators are used to "hold" values of sampled variables (representing, for instance, servo output or error) during each sampling period; the stored sample is then replaced by a new sample value (Fig. 7.30).

D-c Analog Computers as Control-system Components.[3] D-c analog representation of automatic control systems has proved to be a powerful aid in the design of prototype models and pilot plants and in the determination of starting procedures and of optimum controller settings after changes in raw materials or other conditions. But the utility of d-c analog techniques for automatic control applications is not restricted to computations of this type. *Special-purpose d-c analog computers, which may often be conveniently assembled from the standard components of commercially available machines, can themselves serve as control-system elements in many applications suited to their characteristics.* A single commercial multipurpose computer might be adapted to furnish components controlling several phases of an industrial process. The great flexibility of d-c analog-computing elements may permit not only refinements of existing control techniques but also entirely new applications. Thus it has been suggested that analog computers be used for continuous recomputation of optimum set points as functions of the composition or quality of raw materials entering a process.[1] It is conceivable that such techniques will permit the use of less pure or cheaper raw materials.

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[1] Mead, R. J., and N. Diamantides, Application of GEDA to Human Dynamics Studies, *Report GER*-6608, Goodyear Aircraft Corporation, Akron, Ohio, Mar. 4, 1955; see also Final Report, Investigation of Vestibular and Body Reactions to the Dynamic Response of a Human Operator, *Report GER*-5452, Goodyear Aircraft Corporation, Akron, Ohio, Nov. 25, 1953.

[2] Wadel, L. B., Analysis of Combined Sampled and Continuous-data Systems on an Electronic Analog Computer, lecture presented at the IRE National Convention, New York, 1955.

[3] Bibbero, R. J., Process Regulation with Analog Control, *Automatic Control*, Feb., 1955.

In a closely related application, a centrally located d-c analog computer achieves integrated remote control of an entire power distribution system. The Goodyear Economic Dispatch Computer[2] reduces the cost of power distribution by controlling the throttles of 35 turbogenerator units in nine widely scattered power plants of the Ohio Edison Company[3] in accordance with current data on generator and station availability, area protection, system load, line losses, varying fuel costs, and other conditions. The machine is easily modified or expanded to account for changes in the power-distribution system and will, in fact, serve as a model for the evaluation of different system layouts and scheduling methods.

3.5. D-C ANALOG-COMPUTER SETUP FOR A TRAJECTORY COMPUTATION: THE TRAJECTORY OF A SLOW SPHERICAL PROJECTILE WITH AIR RESISTANCE

As a simple example of a *trajectory computation* in two dimensions, consider the computation of the trajectory of an old-fashioned spherical cannonball with air resistance. This example will serve as an introduction to more complex ballistic computations involving more complicated projectiles, such as rockets.

The cannonball is fired from a cannon with a given muzzle velocity v_0 and at a given elevation angle ϑ_0. It is desired to find the trajectory of the cannonball with respect to a rectangular cartesian coordinate system whose x and y axes are horizontal and vertical, respectively, and whose origin is located at the cannon muzzle (trajectory origin).

The only forces acting on the cannonball are the force of gravity and the aerodynamic drag. The latter force is assumed to be proportional to the square of the instantaneous velocity of the projectile. Since the aerodynamic force acting on a body in flight is usually specified in terms of its components in the direction of the velocity vector (drag) and at right angles to this direction (lift and side force), it is often convenient to write the equations of motion of the projectile with respect to "wind axes" along and perpendicular to the velocity vector. Let ϑ be the angle between the horizontal and the positive instantaneous velocity direction. Referring to Fig. 3.15*a* and to the glossary of symbols (see page 113) the equation of motion in the positive velocity direction is

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[1] *Ibid.*

[2] Memorandum by C. D. Morrill and J. A. Blake, *A Computer for Economic Scheduling and Control of Power Systems;* and Goodyear Economic Dispatch Computer, *Report GER*-6969, Goodyear Aircraft Corporation, Akron, Ohio, Aug. 30, 1955.

[3] Massillon, Ohio.

$$m\frac{dv}{dt} = -mg \sin \vartheta - rv^2$$

The equation of motion (upward) at right angles to the velocity direction may be expressed in the form

$$mv\frac{d\vartheta}{dt} = -mg \cos \vartheta$$

Note that the acceleration in the direction perpendicular to the instantaneous velocity is equal to $v(d\vartheta/dt)$.

These two differential equations permit the computation of the projectile velocity v and the angle ϑ as functions of the time t. The instantaneous cartesian coordinates x and y of the cannonball are then obtained from v and ϑ through the relations

$$\frac{dx}{dt} = v \cos \vartheta \qquad \frac{dy}{dt} = v \sin \vartheta \tag{3.26}$$

A practical procedure for handling this trajectory problem in an analog-computer laboratory is illustrated below in terms of a sample *problem statement* and a set of sample *computer-setup records.*

Trajectory Computation for a Slow Spherical Cannonball

A. Problem specifications

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.
2. *Statement of the problem.* It is desired to obtain the parametric representation

$$x = x(t) \qquad y = y(t)$$

   of the trajectory of a slow spherical cannonball whose equations of motion are given below.
3. *Mathematical relations*

(*a*) $m\dfrac{dv}{dt} = -mg \sin \vartheta - rv^2$ (MOTION IN THE DIRECTION OF THE INSTANTANEOUS VELOCITY)

(*b*) $mv\dfrac{d\vartheta}{dt} = -mg \cos \vartheta$ (MOTION AT RIGHT ANGLES TO THE INSTANTANEOUS VELOCITY)

$$\left.\begin{aligned} &(c)\ \frac{dx}{dt} = v \cos \vartheta \\ &(d)\ \frac{dy}{dt} = v \sin \vartheta \end{aligned}\right\} \quad \text{(TRANSFORMATION TO EARTH AXES)}$$

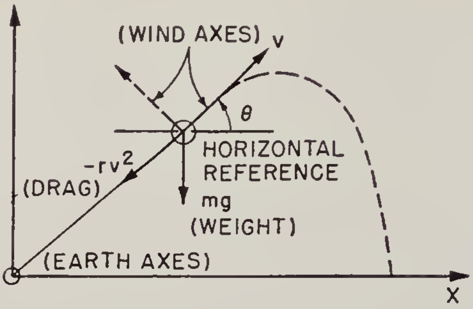

FIG. 3.15*a*. Computation of the trajectory of a slow spherical cannonball.

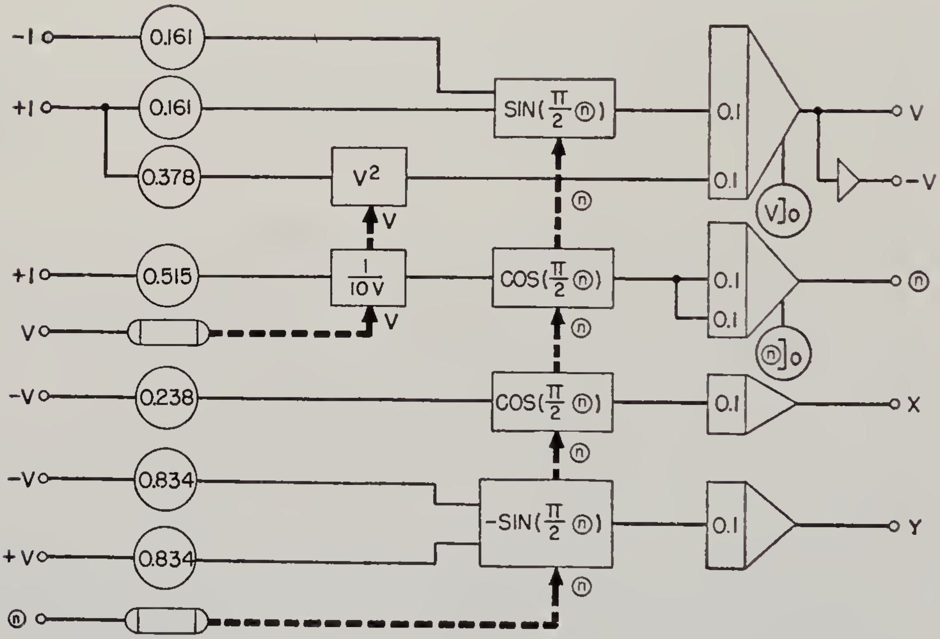
-1
+1
+1
V
-V
-V
+V
Θ
0.161
0.161
0.378
0.515
0.238
0.834
0.834
V^2
V
$\frac{1}{10V}$
V
$\text{SIN}(\frac{\pi}{2}\Theta)$
$\text{COS}(\frac{\pi}{2}\Theta)$
$\text{COS}(\frac{\pi}{2}\Theta)$
$-\text{SIN}(\frac{\pi}{2}\Theta)$
Θ
Θ
Θ
Θ
0.1
0.1
0.1
0.1
0.1
0.1
$V]_0$
$\Theta]_0$
V
-V
Θ
X
Y

FIG. 3.15*b*. Block diagram of a possible computer setup for the trajectory problem. Potentiometer-loading corrections (Sec. 6.9) must be used if good accuracy is desired. Solid lines indicate electrical connections, and broken lines indicate mechanical connections (servo shafts). The machine equations are

$$10PV = -0.161 \sin\left(\frac{\pi}{2}\Theta\right) - 0.378V^2$$

$$10P\Theta = -1.03\left(\frac{1}{10V}\right)\cos\left(\frac{\pi}{2}\Theta\right)$$

$$10PX = 0.238V \cos\left(\frac{\pi}{2}\Theta\right)$$

$$10PY = 0.834V \sin\left(\frac{\pi}{2}\Theta\right)$$

4. *Glossary of symbols*

m	mass of cannonball
t	time
$v(t)$	velocity of cannonball
$-rv^2$	aerodynamic drag
g	acceleration of gravity
$\vartheta(t)$	angle between velocity vector and horizontal
$x(t)$ $y(t)$	rectangular cartesian coordinates of cannonball in the horizontal and vertical directions, respectively
v_0	value of $v(t)$ for $t = 0$
ϑ_0	value of $\vartheta(t)$ for $t = 0$

5. *Initial conditions.* For $t = 0$, $x = y = 0$

$$\left.\frac{dx}{dt}\right]_0 = v_0 \cos \vartheta_0$$

$$\left.\frac{dy}{dt}\right]_0 = v_0 \sin \vartheta_0$$

See below for the values of v_0 and ϑ_0 to be used.

6. *Numerical values of constants*

$g = 32.2$ ft/sec^2
$m = 0.621$ slug (20 lb)
$r = 4.7 \times 10^{-5} \dfrac{\text{lb}}{\text{ft}^2}$ sec^2

7. *Estimated ranges of variables*

$0 < t < 25$ sec
$0 < x < 21{,}000$ ft
$0 < y < 6{,}000$ ft (negative values of y are not of interest)
$100 < v < 1{,}000$ ft/sec
$-45° < \vartheta \leq 21°$

8. *Parameters to be varied*

Take $v_0 = 1{,}000$, 950, and 1,050 ft/sec
Take $\vartheta_0 = 20°$, 19°, and 21°
(Nine solutions required)

9. *Expected results of parameter variations.* Increases in the value of v_0 are expected to increase the values of x and y for each value of the time t.

10. *Possible analytical or numerical checks noted by the customer.* If there is no air resistance ($r = 0$), then

$$x = v_0 t \cos \vartheta_0 \qquad y = v_0 t \sin \vartheta_0 - \tfrac{1}{2} g t^2$$

11. *Use of results.* To prepare ballistic tables for an eighteenth-century cannon. In particular, the sensitivity of the trajectory to small variations in the initial conditions is of interest.
12. *Accuracy required.* Better than 3 per cent.

B. Record of computer solution

1. *Transformation equations.* (x, y, v, and t are measured in the ft/slug/sec system; ϑ is measured in radians; all machine variables are measured in machine units. One machine unit equals 100 volts, or a servo shaft rotation corresponding to 100 volts input to the servomechanism.)

$$x = 21{,}000X \qquad y = 6{,}000Y$$

$$v = 1{,}000V \qquad \vartheta = \frac{\pi}{2}\Theta$$

$$t = \tfrac{1}{2}\tau \qquad \frac{d}{dt} = 2P$$

Note: The choice of the scale factor for ϑ depends on the types of function potentiometers used to generate voltages proportional to sin ϑ and cos ϑ. The choice made here permits the use of commercial sine and cosine potentiometers whose wiper positions correspond to $\vartheta = \pm \pi/2$ when the input voltage Θ to the servomechanism equals plus or minus 1 machine unit, respectively.

The 2:1 (slow) time scale was chosen tentatively to permit greater accuracy in the computer servomechanisms.

The recorder scales are noted on the records.

2. *Machine equations*

$$(a)\quad 10PV = -0.161 \sin\left(\frac{\pi}{2}\Theta\right) - 0.378V^2$$

$$(b)\quad 10P\Theta = -0.103\,\frac{1}{V}\cos\left(\frac{\pi}{2}\Theta\right)$$

Introducing $1/V = 10(1/10V)$ to permit the use of a function potentiometer (see Sec. 2.2), (*b*) becomes

$$(b)\quad 10P\Theta = -1.03\left(\frac{1}{10V}\right)\cos\left(\frac{\pi}{2}\Theta\right)$$

$$(c)\quad 10PX = 0.238V\cos\left(\frac{\pi}{2}\Theta\right)$$

$$(d)\quad 10PY = 0.834V\sin\left(\frac{\pi}{2}\Theta\right)$$

3. *Machine initial conditions*

$$X_0 = 0 \qquad Y_0 = 0 \qquad V_0 = 0.001v_0 \qquad \Theta_0 = 0.637\vartheta_0$$

4. *Block diagram* (see Fig. 3.15*b*). Integrators employing 10 μf integrating capacitors are indicated to facilitate operation on a reduced time scale. Loading corrections for potentiometers should be observed.
5. *Discussion* (of special problems encountered during the computation, tentative conclusions from computer results, etc.)
   None.
6. *Enclosure.* Dated and initialed computer records (x vs. t, y vs. t, and y vs. x).

More General Trajectory Computations. The foregoing example will serve to illustrate a method for solving ballistic trajectory problems on d-c analog computers. If the simple cannonball is replaced by a rocket or other self-propelled body, the effects of mass changes and thrust forces will have to be considered.[1] In the case of more complicated nonspherical projectiles, the effects of the aerodynamic lift as well as of various control forces must be taken into consideration. In addition, the "moment equation" describing the rotation of the projectile about the pitch axis must be solved, and the interaction between this rotation and the translatory motion must be determined. An elementary discussion of some of these questions for a special case is presented in Sec. 3.6. Many three-dimensional as well as two-dimensional trajectory problems can be solved successfully by means of d-c analog computers subject to the accuracy limitations of the latter.

3.6. INTRODUCTION TO D-C ANALOG-COMPUTER METHODS FOR SOLVING AIRCRAFT FLIGHT EQUATIONS

Description of the Motion of an Aircraft: Motion in the Pitch Plane. The motion of an aircraft in the atmosphere, like the motion of the projectile discussed in Sec. 3.5, may be conveniently specified by reference to *wind axes* along and perpendicular to the direction of the instantaneous velocity vector and centered in the center of gravity of the aircraft; the velocity direction is, of course, opposite to that of the relative wind in a stationary air mass.

As an important special case, it will be interesting to consider the motion of an aircraft which is not turning, slipping, or skidding. The motion will, then, take place in the vertical plane, pictured in Fig. 3.16.

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[1] See J. B. Rosser, R. R. Newton, and G. L. Gross, *Mathematical Theory of Rocket Flight*, McGraw-Hill, New York, 1947, for a detailed discussion of the equations of motion of rockets.

The *trajectory* of the aircraft center of gravity in this "pitch plane" may be described, in the manner shown in Sec. 3.5, by specifying the magnitude $v(t)$ of the instantaneous velocity and the angle $\vartheta(t)$ between the velocity direction and the horizontal as functions of the time t. The rectangular cartesian coordinates $x(t)$ and $y(t)$ of the trajectory are, again, related to $v(t)$ and $\vartheta(t)$ by

$$\frac{dx}{dt} = v \cos \vartheta \qquad \frac{dy}{dt} = v \sin \vartheta \tag{3.27}$$

In addition to the motion of the aircraft center of gravity, the pitch-attitude angle $\varphi(t)$, defined as the angle between some preferred longitudinal axis fixed in the aircraft and the horizontal reference, must be specified as a function of the time t.

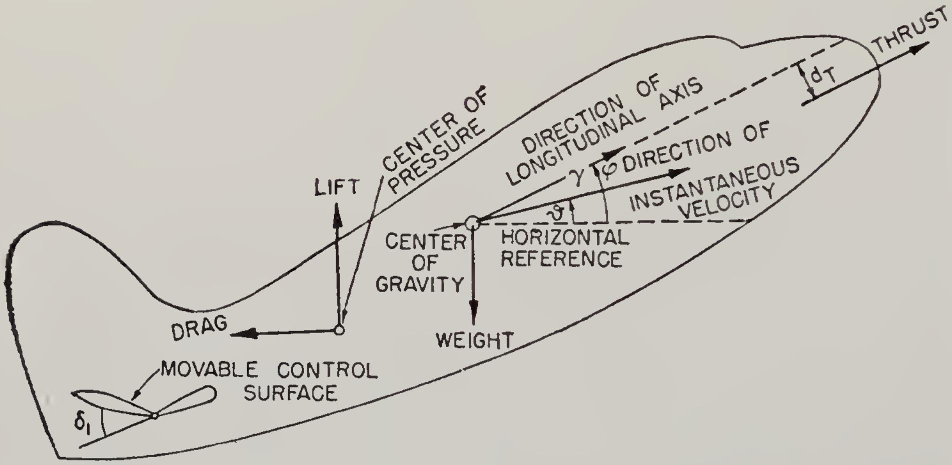

FIG. 3.16. Description of the two-dimensional motion of an aircraft in the pitch plane. Note that neither the location of the center of pressure nor of the center of gravity is necessarily fixed with respect to the airframe.

The angle $\gamma(t)$ between the longitudinal axis and the velocity direction, or

$$\gamma(t) = \varphi(t) - \vartheta(t) \tag{3.28}$$

is known as the pitch *angle of attack*.

The motion of the aircraft in the pitch plane can be *controlled* by changing the deflection angles of control surfaces such as elevators, spoilers, elevons, or movable wings and by changing the magnitude, direction, or point of attack of the thrust propelling the aircraft. The various control parameters will be denoted by $\delta_1, \delta_2, \ldots$.

Flight Equations for Motion in the Pitch Plane. The system of forces acting on the aircraft can usually be reduced to the four forces indicated in Fig. 3.16. These may be listed as follows:

1. The *thrust* T due to the power plant or power plants. In most conventional aircraft, the thrust direction and point of attack are not

used for control purposes. The longitudinal axis may then be chosen to coincide with the thrust axis. The latter does not necessarily pass through the center of gravity of the aircraft.

2. The *weight* mg of the aircraft, where m is the aircraft mass and g is the acceleration of gravity.
3. The *drag* D is the component of the total aerodynamic force in the negative velocity direction. For each particular aircraft configuration, the drag will appear as a function

$$D = D(v,\gamma,\rho) \tag{3.29}$$

of the velocity v, the angle of attack γ, and the air density ρ, and possibly also of some of the control parameters $\delta_1, \delta_2, \ldots$.

4. The *lift* L is the component of the total aerodynamic force at plus 90 deg with respect to the velocity direction. For each particular aircraft configuration, the lift will appear as a function

$$L = L\left(v,\gamma,\frac{d\varphi}{dt}, \delta_1,\delta_2, \ldots ,\rho\right) \tag{3.30}$$

of the velocity v, the angle of attack γ, the pitching rate $d\varphi/dt$, the control surface deflections $\delta_1, \delta_2, \ldots$, and the air density ρ.

In the case of certain jet aircraft, and especially of rockets, the aircraft cannot be regarded as a rigid body but contains a large amount of moving fuel and exhaust gases which may subject it to various kinetic reaction forces in addition to the jet thrust;[1] these forces are not considered in the analysis presented below.

The equation of motion of the aircraft in the positive velocity direction is, by reference to Fig. 3.16,

$$m\frac{dv}{dt} = T\cos\gamma - mg\sin\vartheta - D(v,\gamma,\rho) \tag{3.31}$$

The equation of motion at right angles to the instantaneous velocity direction may be written in the form

$$mv\frac{d\vartheta}{dt} = -mg\cos\vartheta + L\left(v,\gamma,\frac{d\varphi}{dt}, \delta_1,\delta_2, \ldots ,\rho\right) \tag{3.32}$$

The equation of motion describing the changes of the pitch angle φ takes the form

$$I\frac{d^2\varphi}{dt^2} = d_T T + M\left(v,\gamma,\frac{d\varphi}{dt}, \delta_1,\delta_2, \ldots ,\rho\right) \tag{3.33}$$

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[1] See, for instance, Rosser, Newton, and Gross, *op. cit.*

where M is the *pitch moment* due to the aerodynamic forces and d_T is the distance between the thrust axis and the center of gravity. The moment M may be computed from known values of the drag D and the lift L if the location of the point of attack of the total aerodynamic force (center of pressure, see Fig. 3.16) is known, but it is usually simpler to obtain M as a function of v, γ, $d\varphi/dt$, δ_1, δ_2, . . . , and ρ directly from wind-tunnel data.

The three flight equations (3.31), (3.32), and (3.33), in conjunction with the definition (3.28) of the angle of attack γ, determine the motion of the aircraft in the pitch plane and permit one to compute $v(t)$, $\vartheta(t)$, $\varphi(t)$, and $\gamma(t)$ as functions of the time t if proper initial conditions are given.

D-c Analog-computer Solution of the Flight Equations. The flight equations are usually suitable for d-c analog-computer solution. The equations of motion of the projectile discussed in Sec. 3.5 are special cases of the flight equations (3.31) and (3.32); the d-c analog-computer setup discussed in Sec. 3.5 is thus an example of the d-c analog-computer solution of a flight problem.

At least for low values of the velocity v, it is often possible to express the functions D, L, and M to a good approximation in the form

$$\left.\begin{aligned} D(v,\gamma,\rho) &= c_1\rho D_1(v)D_2(\gamma) \\ L\left(v,\gamma,\frac{d\varphi}{dt},\delta_1,\delta_2,\ \ldots\ ,\rho\right) &= c_2\rho L_1(v)\left[L_2(\gamma) + L_3\left(\frac{d\varphi}{dt}\right)\right. \\ &\qquad\left. + L_4(\delta_1) + \cdots\right] \\ M\left(v,\gamma,\frac{d\varphi}{dt},\delta_1,\delta_2,\ \ldots\ ,\rho\right) &= c_3\rho M_1(v)\left[M_2(\gamma) + M_3\left(\frac{d\varphi}{dt}\right)\right. \\ &\qquad\left. + M_4(\delta_1) + \cdots\right] \end{aligned}\right\} \quad (3.34)$$

where c_1, c_2, and c_3 are constants. The expressions (3.34) involve only functions of single variables and are particularly easily computed by means of servo-driven function potentiometers which are capable of simultaneous multiplication and function generation. Functions of two or more variables can also be generated directly (Sec. 6.13). D-c analog-computer solutions of the flight equations based on these techniques can be successfully applied to the computation of aircraft trajectories even in the case of more or less complicated maneuvers such as stalls, landings, and rocket-assisted take-offs.

Aircraft Motion in Three Dimensions. If the aircraft motion under consideration is not confined to the pitch plane, a total of six generalized coordinates must be specified as functions of the time through a set of six flight equations.[1] The solution of the general problem by means of a d-c analog computer is not more difficult in principle than the determination of the motion in the pitch plane, but the number of computing elements needed in the former case will be materially larger. The transformations relating the variation of the three-dimensional velocity vector to the cartesian coordinates of the three-dimensional trajectory will also involve more complex trigonometric expressions than the corresponding transformation equations (3.27) for motion in the pitch plane.[2]

Flight Simulators. An analog computer capable of solving aircraft flight equations on a 1:1 time scale is frequently referred to as a *flight simulator.* If input voltages corresponding to the control surface deflections are supplied to the machine either manually or by means of appropriate control mechanisms, the machine variables describing the attitude and trajectory of the aircraft must vary as the corresponding aircraft coordinates would in actual flight. These voltages may be recorded for later study of the simulated aircraft performance and may also actuate real or simulated flight instruments.

The operation of a flight simulator on the ground is vastly cheaper as well as safer than operation of actual aircraft. For this reason, analog computers are used in increasing numbers as flight simulators for the following purposes:

1. *Training.* Flight simulators are invaluable as safe, rapid, and reliable synthetic training devices. The use of flight simulators will, for instance, permit realistic instruction in instrument flying, aerial gunnery, and power-plant control for single- and multiengine aircraft. Instruction can be given under controlled conditions which can be easily evaluated and repeated by the instructor.

Such synthetic training devices are also useful for evaluating flight procedures, crew cooperation, and job assignments among air crews.

2. *Aircraft design and redesign.* A flight simulator is a flexible computer which can simulate aircraft parameter changes in hundreds of "test flights" under controlled laboratory conditions.

Such flight simulation permits one to test the behavior of new aircraft configurations in standard maneuvers as well as in potentially dangerous stalls, spins, etc. If desired, the maneuvers in question can also be performed on a slow time scale.

Again, the cost of testing a modern multiengine aircraft can amount to thousands of dollars per hour, so that very substantial savings can be realized by performing as many preliminary tests as possible on a flight simulator.

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[1] See, for instance, R. von Mises, *Theory of Flight,* McGraw-Hill, New York, 1945.

[2] Refer to *Application Bulletin* 2, Computer Greatly Simplifies Yaw Damper Design, Berkeley Division, Beckman Instruments, Inc., Richmond, Calif., for a simple d-c analog-computer study of aircraft roll and yaw.

3. *Partial system testing.* The output voltages of a flight simulator may be used to change the attitude of a servo-driven *flight table* or to control the "altitude" in a pressure chamber. Flight instruments such as gyros and altimeters may thus be tested under conditions closely simulating those of actual flight. Flight simulators are particularly useful for the design of automatic pilots (see Sec. 3.8).

Because of the limitations imposed on the computing time by d-c analog computing elements (see Chap. 4), the latter may not be so suitable for use in flight simulators intended for training purposes as electro-mechanical analog computers. The relatively high accuracy, low cost, and flexibility of d-c analog computers make them, however, eminently useful as flight simulators used for design and partial system testing.

3.7. ANALYSIS OF SMALL AIRCRAFT MOTIONS IN THE PITCH PLANE: D-C ANALOG-COMPUTER SOLUTION OF THE LINEARIZED FLIGHT EQUATIONS

The solution of the set of flight equations (3.28), (3.31), (3.32), and (3.33) for aircraft motion in the pitch plane can be simplified very considerably if it is necessary to determine only *small changes* $\bar{v}$, $\bar{\vartheta}$, and $\bar{\varphi}$ of the variables v, ϑ, and φ from their constant values for some condition of steady-state or equilibrium flight. Steady-state or equilibrium motion in the pitch plane corresponds to a condition of steady horizontal flight, climb, or dive given by a set of values of the three variables v, ϑ, and φ; it is desired to investigate the results of disturbing aerodynamic forces and/or control surface deflections.

Linearized Flight Equations for Motion in the Pitch Plane. Steady-state or equilibrium motion in the pitch plane may be defined by the relations

$$\frac{dv}{dt} = \frac{d\vartheta}{dt} = \frac{d\varphi}{dt} = 0 \qquad (3.35)$$

It will be assumed that the elevator deflection δ_1 is the only control parameter and that the air density ρ is constant. Let $\bar{v}$, $\bar{\vartheta}$, $\bar{\varphi}$, $\bar{\gamma}$, and $\bar{\delta}_1$ be the variations of v, ϑ, φ, γ, and δ_1 from their respective equilibrium values v_1, ϑ_1, φ_1, γ_1, and δ_{11}, so that

$$\left.\begin{array}{ccc} v = v_1 + \bar{v} & \vartheta = \vartheta_1 + \bar{\vartheta} & \varphi = \varphi_1 + \bar{\varphi} \\ \gamma = \gamma_1 + \bar{\gamma} & \delta_1 = \delta_{11} + \bar{\delta}_1 & \end{array}\right\} \qquad (3.36)$$

Since $\bar{v}$, $\bar{\vartheta}$, $\bar{\varphi}$, and $\bar{\gamma}$ are assumed to be small, one may write

$$\left.\begin{aligned}
&\sin\vartheta = \sin\vartheta_1 + (\cos\vartheta_1)\bar{\vartheta} \qquad \cos\vartheta = \cos\vartheta_1 - (\sin\vartheta_1)\bar{\vartheta}\\
&D(v,\gamma,\rho) = D(v_1,\gamma_1,\rho) + D_v\bar{v} + D_\gamma\bar{\gamma}\\
&L\left(v,\gamma,\delta_1,\frac{d\varphi}{dt},\rho\right) = L(v_1,\gamma_1,\delta_{11},0,\rho) + L_v\bar{v} + L_\gamma\bar{\gamma} + L_{\delta_1}\bar{\delta}_1 + L_{\dot{\varphi}}\frac{d\bar{\varphi}}{dt}\\
&M\left(v,\gamma,\delta_1,\frac{d\varphi}{dt},\rho\right) = M(v_1,\gamma_1,\delta_{11},0,\rho) + M_v\bar{v} + M_\gamma\bar{\gamma} + M_{\delta_1}\bar{\delta}_1 + M_{\dot{\varphi}}\frac{d\bar{\varphi}}{dt}
\end{aligned}\right\} \qquad (3.37)$$

where the derivatives of D, L, and M with respect to v, γ, δ_1, and $d\varphi/dt$ are all computed for the equilibrium values of v, γ, δ_1, and $d\varphi/dt$.

By introducing the relations (3.35), (3.36), and (3.37) into the flight equations (3.28), (3.31), (3.32), and (3.33) it may be shown that the variations $\bar{v}$, $\bar{\vartheta}$, $\bar{\varphi}$, $\bar{\gamma}$, and $\bar{\delta}_1$ must satisfy the *linearized flight equations*

$$\left.\begin{array}{lll}
(a) & m\dfrac{d\bar{v}}{dt} = -D_v\bar{v} - D_\gamma\bar{\gamma} - (mg\cos\vartheta_1)\bar{\vartheta} & \\
 & & \text{(MOTION IN VELOCITY DIRECTION)}\\
(b) & mv_1\dfrac{d\bar{\vartheta}}{dt} = L_v\bar{v} + L_\gamma\bar{\gamma} + L_{\delta_1}\bar{\delta}_1 + L_{\dot{\varphi}}\dfrac{d\bar{\varphi}}{dt} + (mg\sin\vartheta_1)\bar{\vartheta} & \text{(LIFT)}\\
(c) & I\dfrac{d^2\bar{\varphi}}{dt^2} = M_v\bar{v} + M_\gamma\bar{\gamma} + M_{\delta_1}\bar{\delta}_1 + M_{\dot{\varphi}}\dfrac{d\bar{\varphi}}{dt} & \\
 & & \text{(PITCHING MOTION)}\\
(d) & \bar{\gamma} = \bar{\varphi} - \bar{\vartheta} & \text{(ANGLE OF ATTACK DEFINITION)}
\end{array}\right\} \qquad (3.38)$$

for small motions in the pitch plane. Although the analysis of the linear equations (3.38) is by far simpler than that of the corresponding general flight equations, such analysis has been very fruitful, especially in connection with the following applications:

1. *Studies of aircraft stability.* The equilibrium motion of an aircraft is said to be *stable* if the aircraft tends to return to that equilibrium condition after a small disturbance due to aerodynamic disturbing forces or control deflections. In the case of stable motion, small changes in speed and attitude will tend to remain small; such a situation is frequently desirable.

2. *Studies of aircraft response to controls.* The linearized flight equations are frequently used for analyzing the response of different aircraft configurations to motions of the control surfaces. This may be accomplished by studying the transient behavior of the aircraft after a control-surface deflection. Another particularly useful method for analyzing the sensitivity of the aircraft attitude to control motions applied at various rates is based on the study of steady-state sinusoidal variations of the flight variables. It is possible, for instance, to study the *amplitude and phase response* of the attitude-angle changes φ and flight-path angle ϑ to

applied sinusoidal changes in the elevator deflection at different frequencies. The amplitude response will indicate the sensitivity of the aircraft motion to control motions applied at different rates, whereas the phase response will show how quickly the aircraft can follow the control motion. Aerodynamic *transfer functions* like $\bar{\varphi}/\bar{\delta}_1$ or $\bar{\vartheta}/\bar{\delta}_1$ may be derived from the linearized flight equations (3.38) in much the same manner as voltage transfer functions are derived from electrical circuit equations.[1]

Figures 3.18*b* and *c* show typical plots of the computed amplitude and phase response of the attitude angle φ of a B-25J aircraft[2] to small sinusoidal changes in the elevator deflection δ_1. The results of such computations can be applied to the study of more realistic maneuvers by means of Fourier analysis.

Finally, the study of small aircraft motions is important in connection with the design of automatic pilots (see Sec. 3.8).

D-c Analog-computer Solution of the Linearized Flight Equations. The solution of aircraft linearized flight equations has been one of the most fruitful applications of d-c analog computers. This is true not only because of the savings in time and labor effected but also because of the ease of eventually introducing the effects of nonlinearities such as stops limiting the control-surface deflections (see Sec. 3.3) and of performing partial system tests.

A d-c analog computer set up to solve the linearized flight equations may be used to study the transient response of the aircraft with respect to small disturbances as well as the amplitude and phase response to sinusoidal disturbances of different frequencies. The amplitude and phase response may be measured exactly as shown in Sec. 3.4 for the case of servomechanisms simulated by the computer. D-c analog computers have become invaluable tools for the study of aircraft stability and control in an ever-increasing number of instances.

A typical d-c analog-computer setup for such purposes is described below in terms of the problem specifications and computer-solution-record forms for a typical case.

A. Problem specifications

1. *Introduction.* This paragraph will contain problem designations, authorization references, etc.
2. *Statement of the problem.* Set up a d-c analog computer to solve the linearized flight equations of an aircraft for small motions in the pitch plane. It is desired to compute variations in the speed,

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[1] Seamans, R. C., B. G. Bromberg, and L. E. Payne, Application of the Performance Operator to Aircraft Automatic Control, *MIT DIC Report*, 1947 (Instrumentation Laboratory).

[2] Hagelbarger, Howe, and Howe, *op. cit.*

pitch-attitude angle, and flight-path angle of the aircraft as functions of the time for given initial conditions if the elevator-deflection angle is given as a function of the time.

3. *Mathematical relations*

$$(a)\quad m\frac{d\bar v}{dt} = -D_v\bar v - D_\gamma\bar\gamma - (mg\cos\vartheta_1)\bar\vartheta \qquad \text{(MOTION IN VELOCITY DIRECTION)}$$

$$(b)\quad mv_1\frac{d\bar\vartheta}{dt} = L_v\bar v + L_\gamma\bar\gamma + L_{\delta_1}\bar\delta_1 + L_{\dot\varphi}\frac{d\bar\varphi}{dt} + (mg\sin\vartheta_1)\bar\vartheta \qquad \text{(LIFT)}$$

$$(c)\quad I\frac{d^2\bar\varphi}{dt^2} = M_v\bar v + M_\gamma\bar\gamma + M_{\delta_1}\bar\delta_1 + M_{\dot\varphi}\frac{d\bar\varphi}{dt} \qquad \text{(PITCHING MOTION)}$$

$$(d)\quad \bar\gamma = \bar\varphi - \bar\vartheta \qquad \text{(ANGLE OF ATTACK DEFINITION)}$$

These equations were derived under the following assumptions which flight tests have shown reasonably valid for aircraft of the type under consideration: (*a*) variables deviate from the equilibrium conditions by small increments only; (*b*) all coefficients remain constant throughout the maneuvers considered; (*c*) the coupling between the motion in the pitch plane (longitudinal motion) and the roll and yaw (lateral motion) is negligible; (*d*) the thrust remains constant throughout the maneuver.

4. *Glossary of symbols*

t	time
v_1	aircraft velocity v in equilibrium condition
ϑ_1	flight-path angle ϑ in equilibrium condition
φ_1	pitch-attitude angle φ in equilibrium condition
$\bar v$	change in velocity
$\bar\vartheta$	change in flight-path angle
$\bar\varphi$	change in pitch-attitude angle
$\bar\gamma = \bar\varphi - \bar\vartheta$	angle-of-attack change
$\bar\delta_1$	change in elevator deflection (upward)
g	acceleration of gravity
m	aircraft mass
I	pitch moment of inertia
$D(v,\gamma)$	drag
$L\left(v,\gamma,\delta_1,\frac{d\varphi}{dt}\right)$	lift
$M\left(v,\gamma,\delta_1,\frac{d\varphi}{dt}\right)$	pitch aerodynamic moment

The constant coefficients D_v, L_v, M_γ, M_δ, etc., are the partial derivatives of D, L, and M, respectively, with respect to the subscript variables; all these derivatives are obtained for the equilibrium conditions, with

$$\delta_1 = \frac{dv}{dt} = \frac{d\vartheta}{dt} = \frac{d\varphi}{dt} = \frac{d\gamma}{dt} = 0$$

5. *Initial conditions.* As specified for each particular problem within the ranges given below.
6. *Numerical values of constants*[1]

$\frac{D_v}{m} = +0.02$ ft/sec^2

$\frac{D_\gamma}{m} = +16.6$ ft/sec^2

$g = 32.2$ ft/sec^2

$\vartheta_1 = 0$

$v_1 = 270$ ft/sec

$\frac{L_v}{m} = +0.237$ ft/sec^2

$\frac{L_\gamma}{m} = +238$ ft/sec^2

$\frac{L_{\delta_1}}{m} = -26.6$ ft/sec^2

$\frac{L_{\dot\varphi}}{m} = +1.68$ ft/sec

$\frac{M_v}{I} = 0$

$\frac{M_\gamma}{I} = -11.9 \frac{1}{\text{sec}^2}$

$\frac{M_{\delta_1}}{I} = +10.3 \frac{1}{\text{sec}^2}$

$\frac{M_{\dot\varphi}}{I} = -0.679 \frac{1}{\text{sec}}$

7. *Estimated ranges of variables*

$0 \leq t < 250$ sec

$-25° < \bar\vartheta < 25°$

$-25° < \bar\varphi < 25°$

$-25° < \bar\gamma < 25°$

$-25° < \bar\delta_1 < 25°$

-50 ft/sec $< \bar v < 50$ ft/sec

$25° = 0.436$ radian

8. *Parameters to be varied.* The effects of certain changes in aircraft configurations may be investigated by changes in the constants above.
9. *Expected results of parameter variations.* None noted here.[2]
10. *Possible analytical and numerical checks noted by the customer*

    a. A constant upward change $\bar\delta_1$ in the elevator displacement should be followed by an increase $\bar\varphi$ in the attitude angle. The resulting increase in the angle of attack γ will create additional lift, which in turn will increase the flight-path angle ϑ.

    b. The steady-state response of the aircraft to elevator deflections varying as sinusoidal functions of the time t may be determined analytically by replacing each operator d/dt in the linearized equations of motion by $j\omega$, where ω is the circular frequency of the sinusoidal oscillations.

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[1] From *Operating Manual for the Goodyear Electronic Analog Computer*, Goodyear Aircraft Corporation, Akron, Ohio.

[2] See, for instance, Von Mises, *op. cit.*

11. *Use of results*

    a. To investigate small deviations from equilibrium flight conditions in the pitch plane.
    b. To investigate effects of changes in the aircraft configuration on aircraft stability in the pitch plane (longitudinal stability).
    c. Use of the flight simulator for testing both simulated and real automatic pilots (see Sec. 3.8).

12. *Accuracy required.* 0.5 to 1.5 per cent.

B. Record of computer solution

1. *Transformation equations.* All problem variables are measured in the ft/slug/sec system of units; all angles are measured in radians. All machine variables are measured in machine units. One machine unit equals 100 volts.

$$t = \tau \text{ (1:1 time scale)}$$
$$\bar{\vartheta} = 0.5\bar{\Theta} \qquad \bar{\varphi} = 0.5\bar{\Phi} \qquad \bar{\gamma} = 0.5\bar{\Gamma} \qquad \bar{\delta}_1 = 0.5\bar{\Delta}_1$$
$$\bar{v} = 50\bar{V}$$

2. *Machine equations*

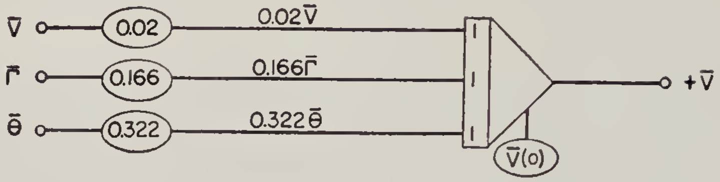
$$(a)\ P\bar{V} = -0.02\bar{V} - 0.166\bar{\Gamma} - 0.322\bar{\Theta} \quad \text{(MOTION IN VELOCITY DIRECTION)}$$
$$(b)\ P\bar{\Theta} = 0.0879\bar{V} + 0.882\bar{\Gamma} - 0.0986\bar{\Delta}_1 + 0.00622P\bar{\Phi} \quad \text{(LIFT)}$$
$$(c)\ P^2\bar{\Phi} = -11.9\bar{\Gamma} + 10.3\bar{\Delta}_1 - 0.679P\bar{\Phi} \quad \text{(PITCHING MOTION)}$$
$$(d)\ \bar{\Gamma} = \bar{\Phi} - \bar{\Theta} \quad \text{(ANGLE OF ATTACK DEFINITION)}$$

3. *Machine initial conditions.* As specified for each particular problem.
4. *Block diagram* (see Fig. 3.17). The following types of function generators may be needed for generating $-\bar{\Delta}_1$ (simulated elevator deflection) as a function of the machine time τ.

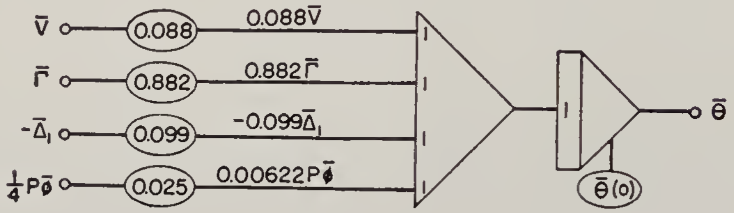
    a. Potentiometer for manual operation.
    b. Step-function generator (voltage source and switch).
    c. Sine or cosine generator capable of continuous operation (Table 4 or Fig. 6.21).
    d. Computer setups to generate (constant) $\cdot\ \tau$ and (constant) $\cdot\ \tau^2$ (Table 4).
    e. Other function generators.

5. *Discussion.* None.

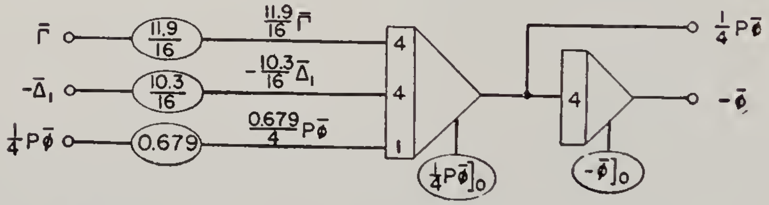
(a) $P\bar{V} = -(0.02\bar{V} + 0.166\bar{\Gamma} + 0.322\bar{\theta})$
(MOTION IN VELOCITY DIRECTION)



(b) $P\bar{\theta} = 0.0879\bar{V} + 0.882\bar{\Gamma} - 0.0986\bar{\Delta}_1 + 0.00622P\bar{\phi}$
(LIFT)



(c) $P^2\bar{\phi} = -11.9\bar{\Gamma} + 10.3\bar{\Delta}_1 - 0.679P\bar{\phi}$ (PITCHING MOTION)



(d) $\bar{\Gamma} = (\bar{\phi} - \bar{\theta})$ (ANGLE OF ATTACK DEFINITION)

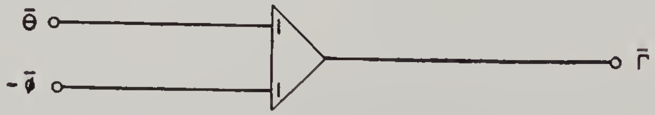


FIG. 3.17. Block diagram of a flight-simulator setup for the study of small aircraft motions in the pitch plane. Note how each equation of motion is represented by a block of computing elements. The simulated "aircraft" may be "steered" by varying the input voltage $-\bar{\Delta}_1$, which corresponds to a change in the elevator deflection.

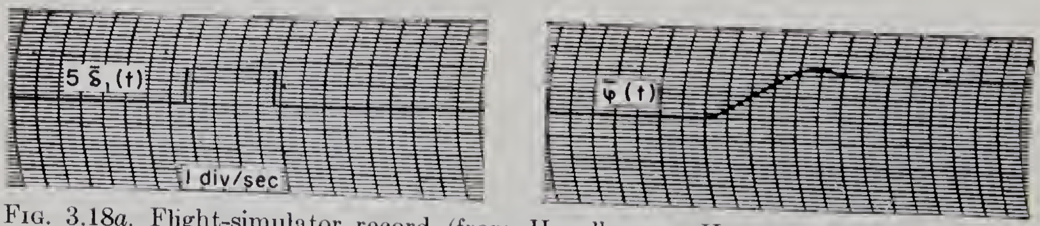
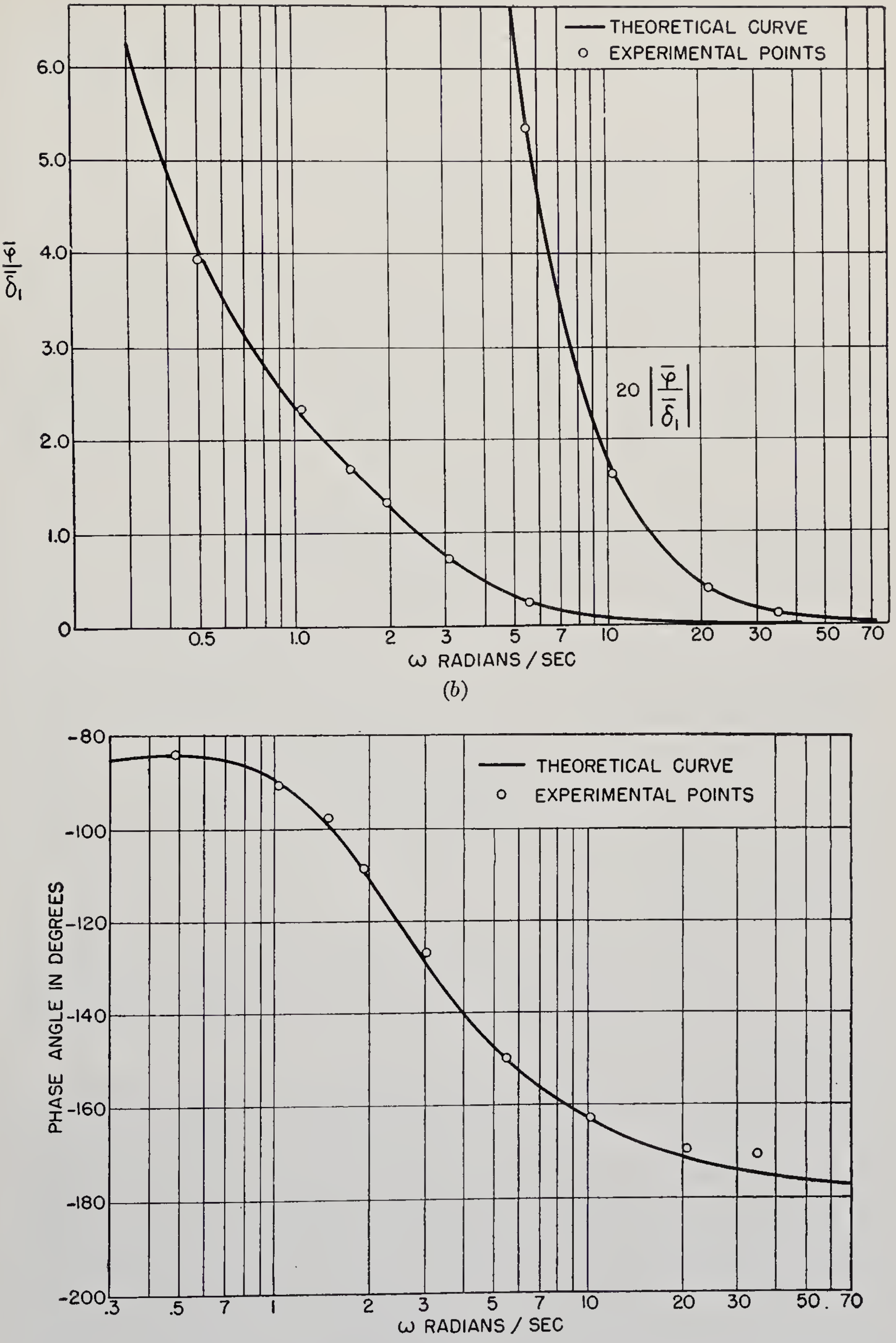


FIG. 3.18a. Flight-simulator record (from Hagelbarger, Howe, and Howe, *op. cit.*) of the attitude change of an aircraft in response to an elevator deflection. The solutions of Fig. 3.18a to *c* are not for the aircraft configuration discussed in the text.



(*b*)

(*c*)

FIG. 3.18*b* and *c*. Experimental and calculated amplitude and phase response of the aircraft attitude change $\bar{\varphi}$ to sinusoidal elevator deflections (from Hagelbarger, Howe, and Howe, *op. cit.*). The solutions of Fig. 3.18*a* to *c* are not for the aircraft configuration discussed in the text.

6. *Enclosures* (see also Fig. 3.18). Dated and initialed computer records ($\bar{\delta}_1$, $\bar{v}$, $\bar{\gamma}$, $\bar{\varphi}$, $\bar{\vartheta}$, vs. t); plotted amplitude and phase characteristics of the "transfer function" $\bar{\varphi}/\bar{\delta}_1$ for sinusoidal elevator motion.

3.8. APPLICATION OF D-C ANALOG COMPUTERS TO THE DESIGN AND TESTING OF AUTOMATIC PILOTS

A number of techniques for employing d-c analog computers as flight simulators for the design of automatic pilots will now be outlined in more detail. An automatic pilot is a system of servomechanisms designed to adjust the control-surface deflections and possibly the throttle settings of an aircraft continuously or intermittently. The automatic pilot will operate in such a manner that all or some of the coordinates describing the aircraft attitude and position and/or their derivatives correspond as closely as possible to certain desired "reference" values which may be fixed or vary as functions of the time. Automatic pilots used for *attitude control* will position control surfaces so as to maintain a desired attitude in roll, pitch, and yaw or so as to produce desired rates of turn or rates of climb or descent. Automatic pilots may also achieve *altitude control* and *speed control* as well as control of the *angle of attack*. Automatic control devices intended primarily for *trajectory control* are not usually referred to as automatic pilots, but they may be studied in much the same manner. Devices in this latter category include equipment used for blind landings as well as long-range-navigation, homing, and guidance apparatus. They will usually comprise a computer which translates position data from some type of navigational device into orders for an automatic pilot which regulates control-surface deflections and/or throttle settings.

Automatic-pilot Operation. The nature of automatic-pilot operation as well as the great importance of analog computers in the development of automatic pilots can perhaps be understood best after an inspection of the block-diagram representation of an automatic-pilot system shown in Fig. 3.19*a*.

In this block diagram, the aircraft is represented as a "transducer" whose input data are instantaneous values of control-surface deflections and throttle settings. The output data are the instantaneous values of the coordinates specifying the aircraft attitude and position. The automatic pilot, in turn, measures the attitude and position "errors," or differences, between the actual and desired values of the attitude and position coordinates by means of gyros, accelerometers, pressure-sensitive devices, etc. These data are reduced to input signals for the servomechanisms which position the control surfaces and throttles. The latter are made to control the aircraft attitude and position so as to minimize the attitude and position errors.

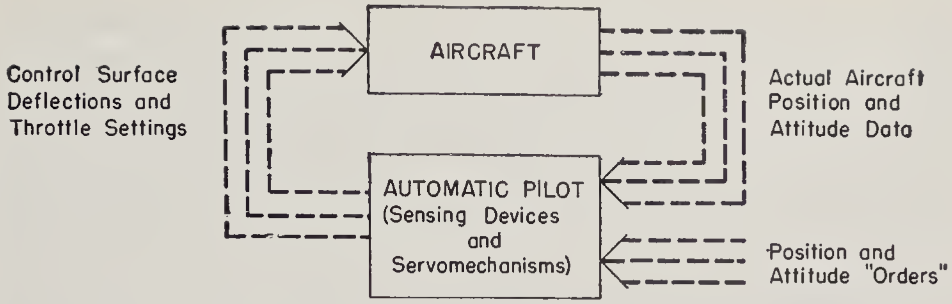

Control Surface Deflections and Throttle Settings
AIRCRAFT
Actual Aircraft Position and Attitude Data
AUTOMATIC PILOT (Sensing Devices and Servomechanisms)
Position and Attitude "Orders"

FIG. 3.19*a*. Block-diagram representation of autopilot operation.

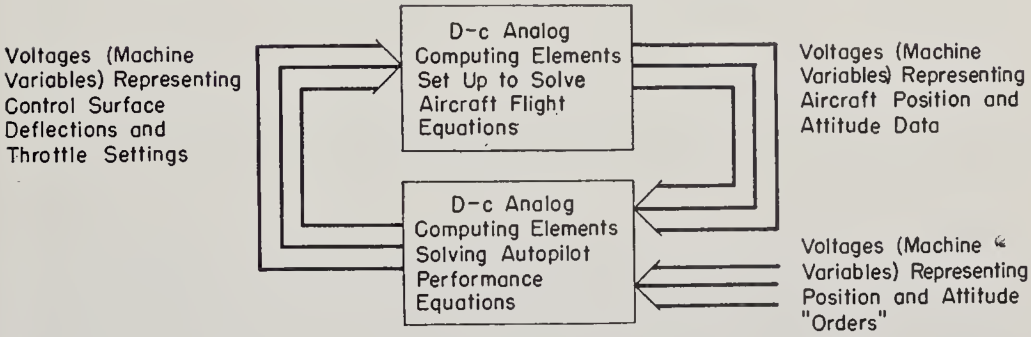

Voltages (Machine Variables) Representing Control Surface Deflections and Throttle Settings
D-c Analog Computing Elements Set Up to Solve Aircraft Flight Equations
Voltages (Machine Variables) Representing Aircraft Position and Attitude Data
D-c Analog Computing Elements Solving Autopilot Performance Equations
Voltages (Machine Variables) Representing Position and Attitude "Orders"

FIG. 3.19*b*. General type of d-c analog-computer setup for simulating the operation of an aircraft-autopilot system.

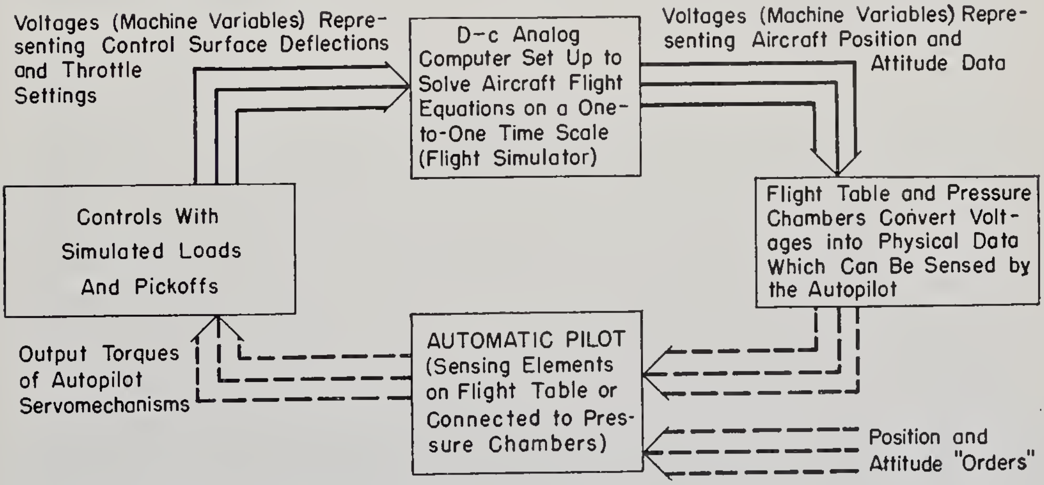

Voltages (Machine Variables) Representing Control Surface Deflections and Throttle Settings
D-c Analog Computer Set Up to Solve Aircraft Flight Equations on a One-to-One Time Scale (Flight Simulator)
Voltages (Machine Variables) Representing Aircraft Position and Attitude Data
Controls With Simulated Loads And Pickoffs
Flight Table and Pressure Chambers Convert Voltages into Physical Data Which Can Be Sensed by the Autopilot
Output Torques of Autopilot Servomechanisms
AUTOMATIC PILOT (Sensing Elements on Flight Table or Connected to Pressure Chambers)
Position and Attitude "Orders"

FIG. 3.19*c*. Block diagram showing how a d-c analog computer may be used as a flight simulator for partial system testing.

In Fig. 3.19*a*, the automatic pilot, like the aircraft, is represented as a "transducer"; its "input signals" are the aircraft attitude and position data as well as the preset orders, and its output data are control-surface deflections and throttle settings. Together the aircraft and the automatic pilot constitute a *feedback loop*.

A properly designed automatic pilot for a certain aircraft must control that aircraft in accordance with its "orders" to within a specified error range and without any tendency toward instability or uncontrolled oscillations.

Since the automatic pilot comprises a number of servomechanisms, it will itself contain a number of internal feedback loops used to obtain the correct control-surface deflections and throttle settings as functions of the input data. The entire system consisting of the automatic pilot and the aircraft is, then, a rather complex multiloop feedback system whose analysis by ordinary means may become quite complicated, especially if the performance of system components cannot be described in terms of linear differential equations.

Simulation of the Aircraft–Automatic-pilot System. Figure 3.19*b* illustrates the use of a d-c analog computer as a flight simulator for automatic-pilot design. The performance of the aircraft is represented by a block of computing elements set up for solution of the flight equations. This block of computing elements accepts voltages proportional to the control-surface deflections and throttle settings; output voltages proportional to the corresponding attitude and position coordinates of the aircraft are obtained through continuous integration of the flight equations. The automatic pilot may, in turn, be represented by a block of computing elements which accept attitude and position data as input voltages and continuously compute output voltages proportional to the control-surface deflections and throttle settings produced by the automatic pilot. *The entire feedback loop of Fig.* 3.19*a is thus "simulated" by the "analogous" computer setup of Fig.* 3.19*b*.

Figure 3.19*c* shows schematically how a *partial system test* of an automatic pilot may be performed through the use of a d-c analog computer set up as a flight simulator operating on a 1:1 time scale. The output voltages of the flight simulator are used to position a *flight table* and, if desired, to vary the pressures in small pressure chambers in accordance with the simulated aircraft motion. The sensing elements of an actual automatic pilot under test are used to detect these changes and will cause the servomechanisms of the automatic pilot to position control surfaces and throttles loaded artificially by springs, motors, or solenoids.

Pickoff potentiometers permit one to obtain voltages proportional to the control-surface deflections and throttle settings; these voltages, in turn, become the input voltages of the flight simulator, so that the entire control loop is, again, simulated; a setup of this type permits very realistic testing of automatic-pilot performance.

A Practical Design Procedure. The design of a projected automatic pilot for a given aircraft may proceed in a series of steps constituting successive approximations to the realistic simulation of actual flight conditions. Such a stepwise procedure will permit the designer to introduce himself gradually to the difficulties involved in ascertaining the performance of a complicated automatic control system.

1. The flight equations of the aircraft under consideration are set up on the d-c analog computer, and the resulting flight-simulator setup is checked.

2. Frequently the most effective approach to control-system design is through an initial study of a greatly simplified system. Thus each servomechanism of a projected automatic pilot may be initially represented by a perfect proportional follow-up device having a certain constant gain. This amounts to the assumption of a perfect servomechansim which can translate any order into a corresponding proportional control-surface deflection or change in throttle setting.

3. As the simulated servomechanism gains in such a computer setup are raised in order to obtain sufficiently low attitude and position errors ("tight" control), "hunting" or instability will usually result. The next step in the design of each automatic-pilot servomechanism may be to introduce equalizing networks (rate feedback, integral feedback, or even nonlinear types of feedback) into the control loop while still assuming "perfect" servomechanisms having unlimited amplitude and frequency response.

4. After a preliminary choice of the equalization devices has been made in this manner, it becomes necessary to make a choice of actual error-sensing devices and servomotors which most closely approach the desired ideal characteristics within the working frequency range of the automatic pilot while still satisfying given space, weight, and power requirements.

It will usually be found that especially the high-frequency response of actual servomotors does not correspond to the idealized conditions considered in step 3.

The equations of motion of the actual automatic-pilot components are next included in the d-c analog-computer setup. In other words, each servomechanism of the automatic pilot is represented by d-c analog computing elements in the manner discussed in Sec. 3.4 in as great detail as is found necessary. The design parameters of the various servomechanisms and equalization networks may then be varied until the best possible response is approximated. The values of the parameters so obtained will serve as basic design data for the automatic pilot.

5. After a model of the actual automatic pilot has been assembled according to these considerations, it may be connected together with the flight simulator for a *partial system test* during which further adjustments can be made.

6. The equipment is then packaged for an actual flight test, is given the usual drop and vibration tests, and is checked out in flight. Although the flight test must always be the final criterion of the suitability of an automatic-pilot design, a large amount of actual flight time may be saved by the procedure outlined above.

The d-c analog-computer setup for the automatic-pilot development outlined above may be operated on an expanded time scale if this is desirable; only the flight simulator used for partial system testing must necessarily be operated on a 1:1 time scale. It may, however, often be useful to utilize the same flight simulator for both partial system testing and simulation of the automatic pilot, since by convenient alternation of the two methods, it is possible to make rapid checks on proposed changes in the automatic-pilot design.

It will usually be advantageous to utilize both the frequency-analysis and transient-analysis methods discussed in Sec. 3.4 *throughout the automatic-pilot development procedure outlined above.* The use of the d-c analog computer also makes it possible to introduce nonlinear characteristics of both aircraft and servomechanisms, such as torque and speed limitations of motors, stops limiting control-surface deflections, and backlash in gear trains (Secs. 3.3 and 3.4).

Flight-simulator study of Automatic-pilot Performance for Small Motions in the Pitch Plane. The use of a d-c analog computer for the design and testing of a simple automatic pilot intended to control aircraft motions in the pitch plane (see Sec. 3.7) is an interesting application of the principles outlined above. The proposed automatic pilot is to "sense" the pitching motion of a certain aircraft by means of free and rate gyros and to control the elevator deflections so as to keep the pitch-attitude angle as closely as possible equal to a reference attitude set into the automatic pilot. If the reference attitude angle is constant, the automatic pilot will attempt to maintain a constant aircraft attitude φ equal to the preset reference attitude. If the reference attitude is "programmed" as a function of time, the automatic pilot will attempt to make the aircraft attitude follow the reference.

Block diagrams of d-c analog-computer setups used in connection with the development of such an automatic pilot are shown in Figs. 3.20*a* to 3.20*e*. Since conventional automatic pilots are, in general, not used during violent maneuvers, it will be sufficient to study small motions of the aircraft about an equilibrium condition. For the sake of simplicity, only motions in the pitch plane are considered, as in Sec. 3.7. The equivalent three-dimensional problem is, again, no more difficult in principle, but the resulting computations are more complex.

The design parameters of any automatic pilot must depend strongly on the design of the aircraft to be controlled, since the response of different aircraft to similar control-surface deflections may be entirely different. Assuming that the aircraft under consideration is of a type similar to that discussed in Sec. 3.7, a flight simulator of the type shown in Fig. 3.20*a* may be used to compute the attitude changes of the aircraft in response to applied changes in the elevator deflection. The machine equations and the computer setup of the flight simulator shown are those derived in Sec. 3.7; the effects of reasonable changes in aircraft configuration (changes in the areas of wing, stabilizer, or elevator surfaces; length of the fuselage; etc.) or equilibrium speed may be introduced by changing coefficient settings (Sec. 3.7). The flight simulator will accept an input voltage proportional to the elevator-deflection changes $\bar{\delta}_1$ and will continuously compute output voltages proportional to the pitch attitude-angle change $\bar{\varphi}$ and the pitch rate $d\bar{\varphi}/dt$ on a 1:1 time scale. A real or simulated automatic pilot may then be made to "sense" the simulated pitching motion and can, in turn, be made to "control" the simulated elevator deflections, so that the performance of the entire control loop can be studied.

First Approximation to the Performance of an Actual Automatic Pilot. Figure 3.20*b* shows a d-c analog-computer setup simulating the performance of a hypothetical automatic pilot which makes the elevator-surface deflection $\bar{\delta}_1$ equal and opposite to the weighted sum

$$a_1\left(k_1\frac{d\bar{\varphi}}{dt} + \bar{\varphi} - \bar{\varphi}_r\right) = a_1[(k_1p + 1)\bar{\varphi} - \bar{\varphi}_r]$$

Such an automatic pilot would have to incorporate a "perfect" servomechanism capable of positioning the control surface *instantaneously* in accordance with the input signals supplied from the free gyro, which "senses" the attitude error $(\bar{\varphi}_r - \bar{\varphi})$, and the rate gyro, which senses the pitch rate $d\bar{\varphi}/dt$. The computer setup of Fig. 3.20*b* may be said to constitute a first approximation to the performance of an actual automatic pilot.

Second Approximation to the Performance of an Actual Automatic Pilot. In an actual automatic pilot, the control-surface deflection cannot follow the input signals instantaneously, mainly because of the inertia, spring loads, and friction associated with the mechanical components of the system. As a second approximation to the performance of an actual automatic pilot, the transfer function of the automatic-pilot servomechanism may be approximated by the simple expression

$$\frac{\text{constant}}{bp + 1} \tag{3.39}$$

so that

$$\bar{\delta}_1 = -\frac{a_1}{bp + 1}[k_1p\bar{\varphi} + (\bar{\varphi} - \bar{\varphi}_r)] \tag{3.40}$$

The somewhat crude approximation (3.39) is frequently useful for investigating the approximate operation of automatic control mechanisms. Using the transformation equations

$$t = \tau \qquad \bar{\varphi} = 0.5\bar{\Phi} \qquad \bar{\varphi}_r = 0.5\bar{\Phi}_r \qquad \bar{\delta}_1 = 0.5\bar{\Delta}_1 \tag{3.41}$$

corresponding to those used in Sec. 3.7, the machine equation corresponding to (3.40) becomes

$$bP\bar{\Delta}_1 = -[a_1(\bar{\Phi} - \bar{\Phi}_r) + a_1k_1P\bar{\Phi} + \bar{\Delta}_1] \tag{3.42}$$

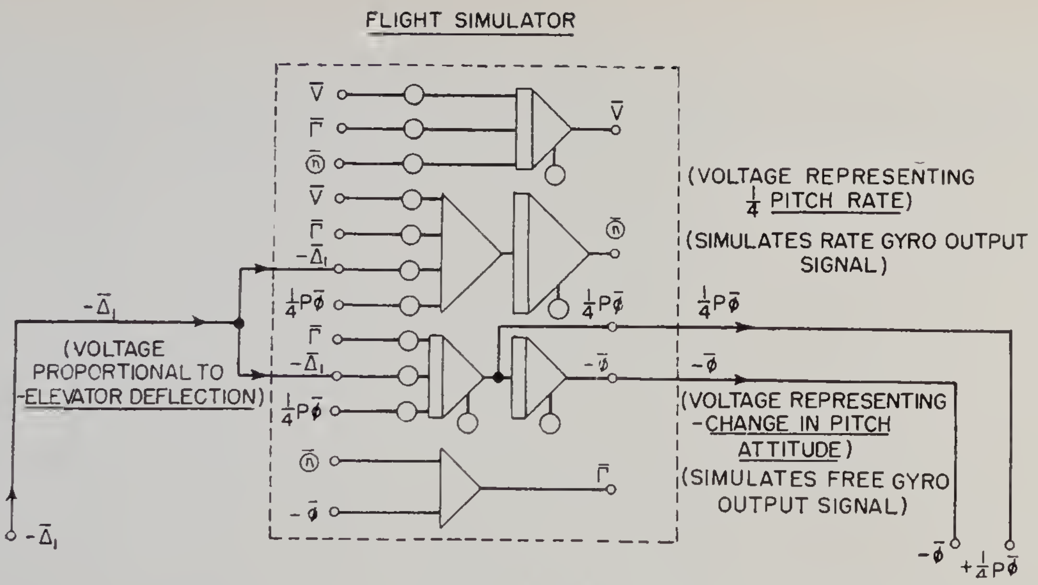


FIG. 3.20*a*. Flight-simulator setup for studying the performance of simulated or real automatic pilots for motion in the pitch plane. The flight simulator computes voltages proportional to pitch-attitude changes and pitch rate continuously by solving the flight equations for small deviations from equilibrium in the pitch plane (see Sec. 3.7).

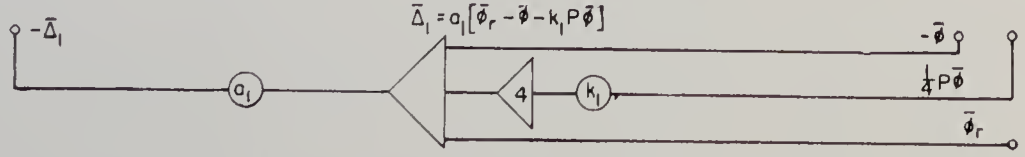


FIG. 3.20*b*. Computer setup simulating the performance of an automatic pilot with perfect servomechanism (first approximation to the performance of a real autopilot).

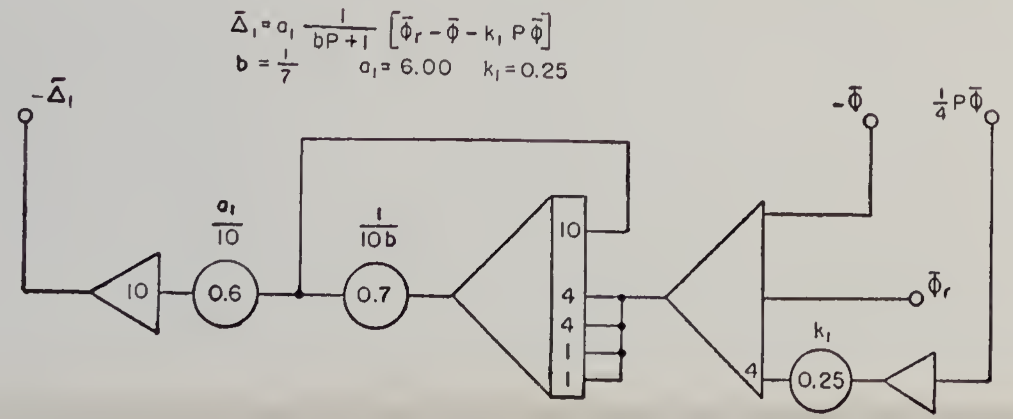

$\bar{\Delta}_1 = a_1 \frac{1}{bP+1}\left[\bar{\Phi}_r - \bar{\Phi} - k_1 P\bar{\Phi}\right]$
$b = \frac{1}{7}$ $a_1 = 6.00$ $k_1 = 0.25$
$-\bar{\Delta}_1$
10
$\frac{a_1}{10}$
0.6
$\frac{1}{10b}$
0.7
10
4
4
1
1
$-\bar{\Phi}$
$\frac{1}{4}P\bar{\Phi}$
$\bar{\Phi}_r$
4
k_1
0.25

FIG. 3.20*c*. Computer setup simulating the performance of an automatic pilot by means of a simple time-delay circuit (second approximation to the performance of a real autopilot).

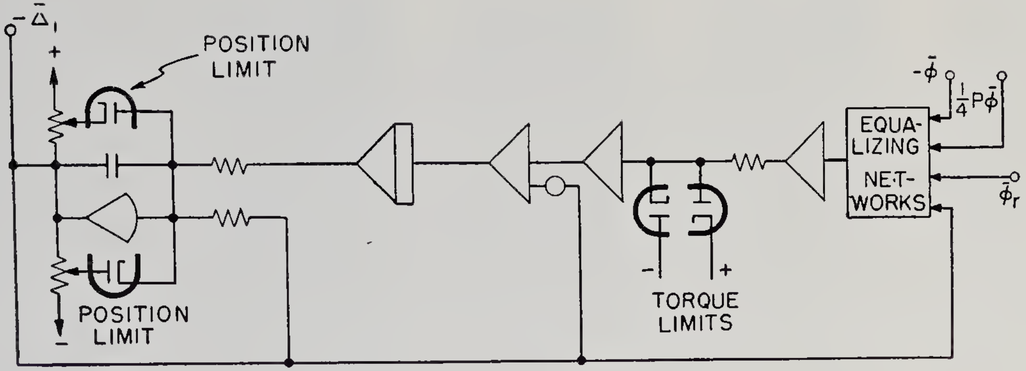


Fig. 3.20*d*. Computer setup simulating the performance of an automatic pilot having an electric motor and equalizing networks as well as position and torque limits (third approximation to the performance of a real autopilot). Note that the simulation of the equalizing networks will involve the use of a phase-inverting amplifier in the case shown.

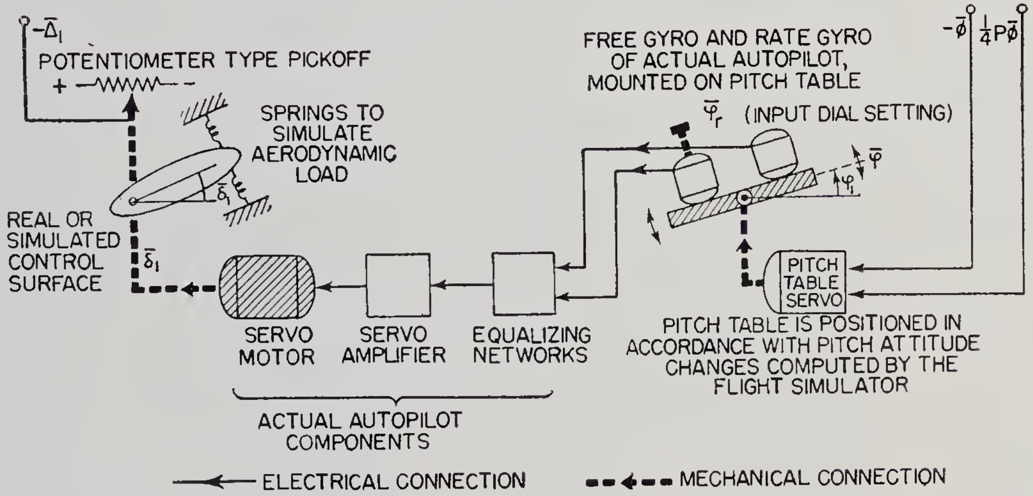

$-\bar{\Delta}_1$
POTENTIOMETER TYPE PICKOFF
+ −
SPRINGS TO SIMULATE AERODYNAMIC LOAD
δ_1
REAL OR SIMULATED CONTROL SURFACE
$\bar{\delta}_1$
SERVO MOTOR
SERVO AMPLIFIER
EQUALIZING NETWORKS
ACTUAL AUTOPILOT COMPONENTS
FREE GYRO AND RATE GYRO OF ACTUAL AUTOPILOT, MOUNTED ON PITCH TABLE
$\bar{\varphi}_r$ (INPUT DIAL SETTING)
φ
φ_i
$-\bar{\phi}$
$\frac{1}{4}P\bar{\phi}$
PITCH TABLE SERVO
PITCH TABLE IS POSITIONED IN ACCORDANCE WITH PITCH ATTITUDE CHANGES COMPUTED BY THE FLIGHT SIMULATOR
ELECTRICAL CONNECTION
MECHANICAL CONNECTION

Fig. 3.20*e*. Flight-table setup for testing actual automatic-pilot components with the flight simulator of Fig. 3.20*a*. The servo feedback connections are not shown.

Fig. 3.20. D-c analog-computer setups for the increasingly detailed simulation of an automatic pilot under study.

The computer setup shown in Fig. 3.20*c* establishes the machine equation (3.42) for

$$a_1 = 6 \qquad k_1 = 0.25 \text{ sec} \qquad b = 0.143 \text{ sec}$$

Figure 3.21 shows a set of typical computer records for an automatic-pilot-controlled maneuver specified by giving the automatic-pilot reference setting $\bar{\varphi}_r$ as a function of the time t.

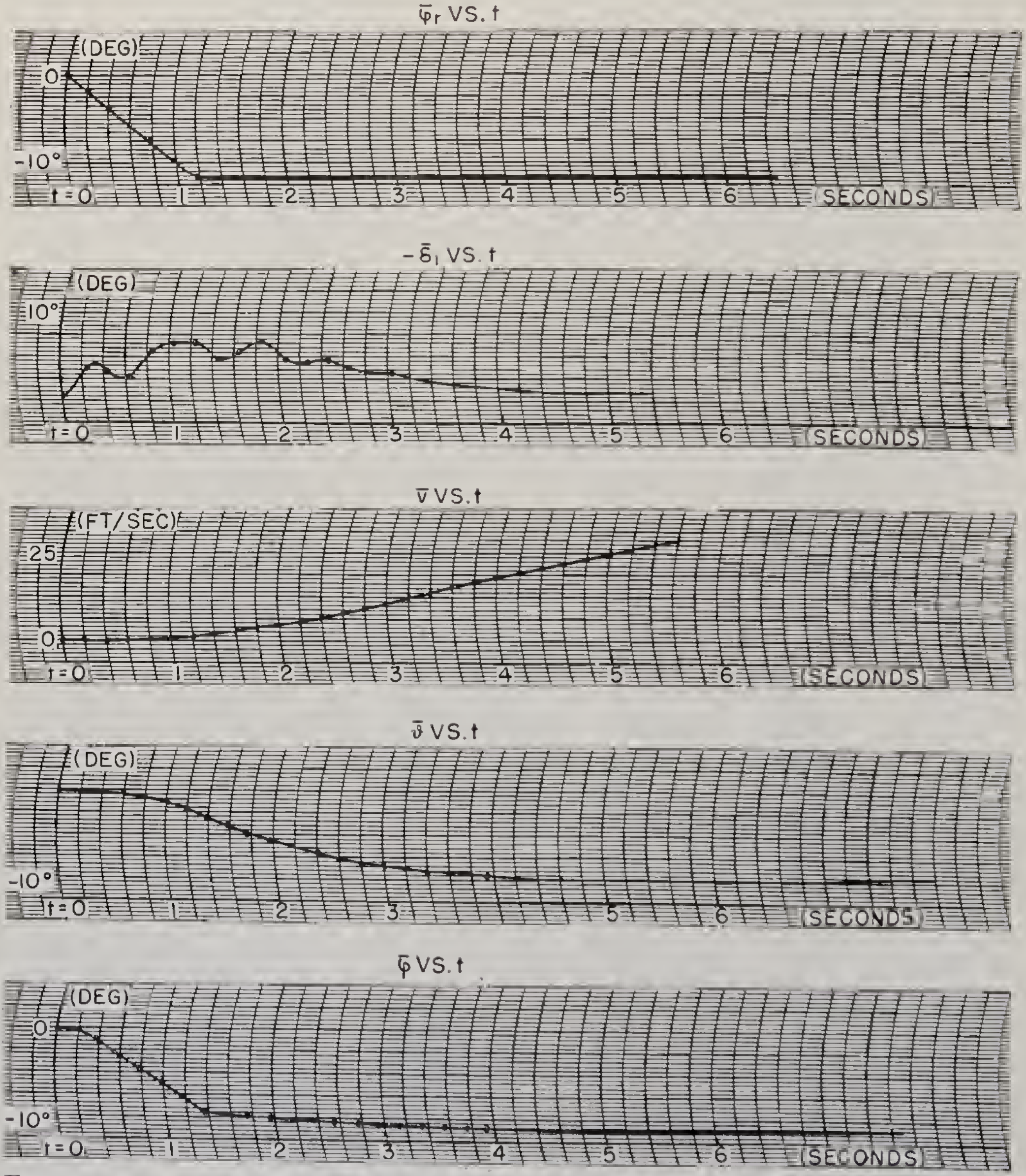


FIG. 3.21. Computer record for a typical "push-over" maneuver controlled by an automatic pilot (Goodyear Aircraft Corporation, *op. cit.*) A multichannel recorder was used to present five variables plotted on the same time scale.

Third Approximation to the Performance of an Actual Automatic Pilot. In the computer setup of Fig. 3.20*d*, the performance characteristics of an actual automatic pilot are simulated even more closely by computing elements capable of solving the performance equation of the automatic-pilot servomechanism. A servomechanism comprising special equalizing networks, an electric motor, and a spring-inertia load has been simulated in the manner described in Sec. 3.4. In addition, inelastic stops on the control surface and the torque limitations of the servomotor are simulated by suitable diode circuits (see also Secs. 3.3 and 3.4). All important design parameters of the simulated automatic pilot can be varied by changes in potentiometer settings or changes in the computer setup.

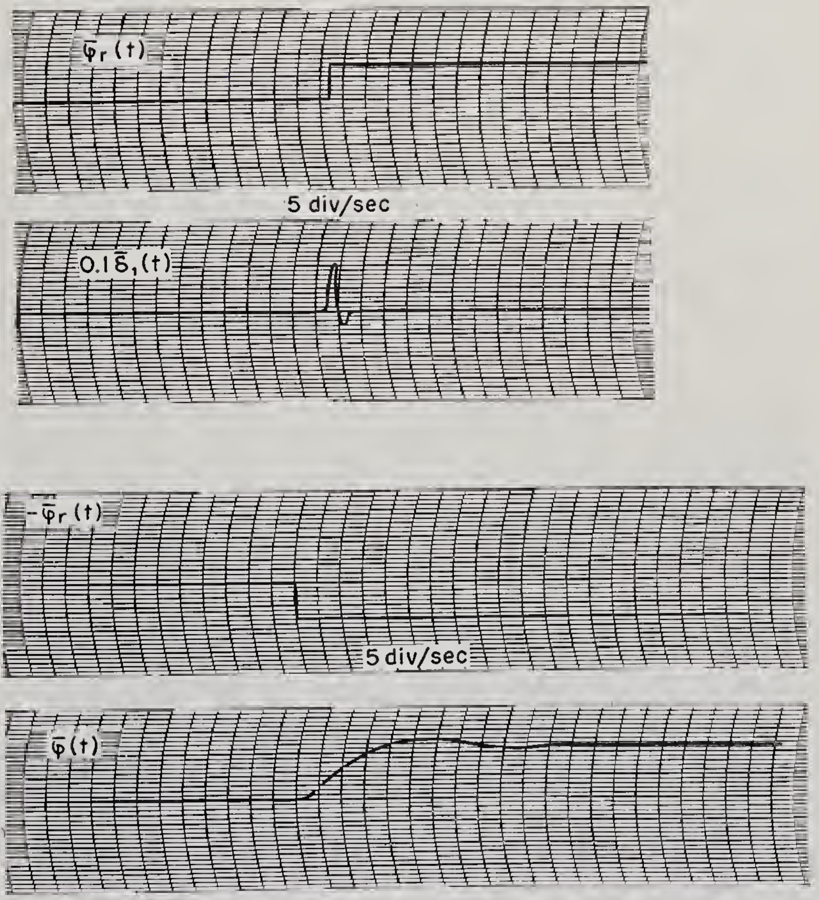


FIG. 3.22*a*. D-c analog-computer records (Hagelbarger, Howe, and Howe, *op. cit.*) showing the transient response of the elevator deflection $\bar{\delta}_1$ and of the pitch-attitude change $\bar{\varphi}$ of a simulated autopilot-controlled aircraft to a step-function change in the reference attitude $\bar{\varphi}_r$.

There is almost no limit on the refinements possible in simulator setups of this kind. Among other interesting variations one may introduce the effects of motor-field time delays, dry friction and gear backlash, disturbing torques on the aircraft or the control surfaces, dead spaces in the gyro response (Sec. 3.4), and errors in the gyro output signals. The simple electric motor simulated in Fig. 3.20*d* could also be replaced by computer setups simulating clutch servos, hydraulic actuators, etc.

The results of parameter studies of automatic-pilot-controlled aircraft can be evaluated exactly in the manner shown for servomechanisms in Sec. 3.4.

As a further example, Fig. 3.22 shows the transient response of another automatic-pilot-controlled aircraft. The effect on the transient response of changing the value of the automatic-pilot gain is shown. Figure 3.23 shows the amplitude and phase of the transfer function $\bar{\varphi}/\bar{\varphi}_r$ for steady-state sinusoidal oscillations of a simulated automatic-pilot-controlled aircraft plotted vs. the circular frequency ω.

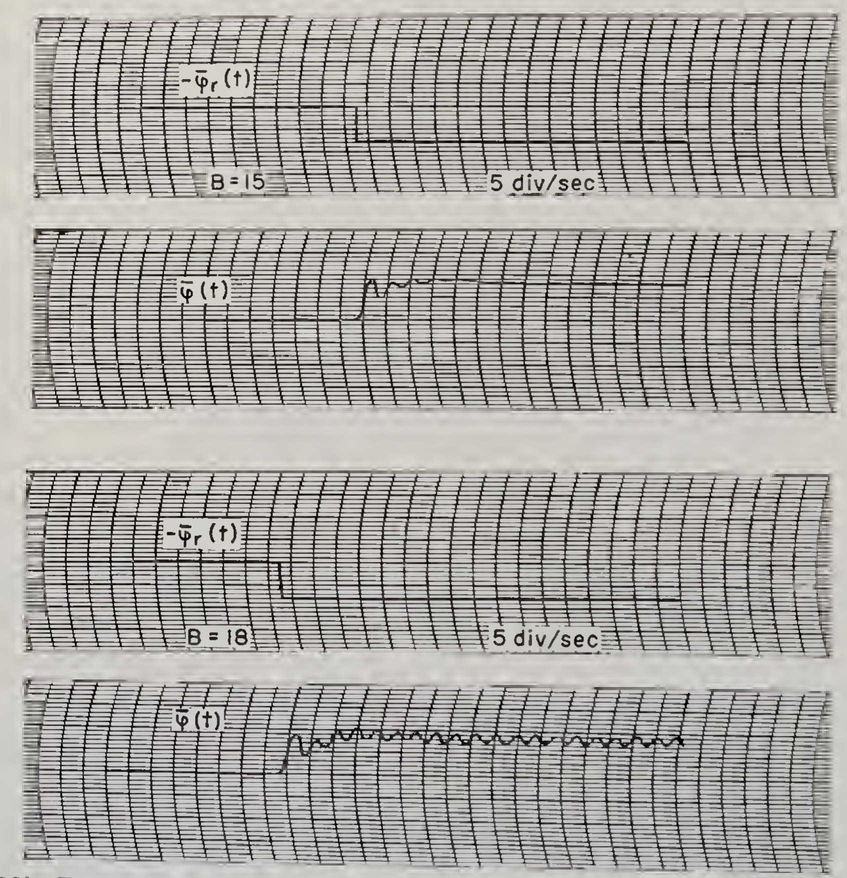


FIG. 3.22*b*. D-c analog-computer records (Hagelbarger, Howe, and Howe, *op. cit.*) showing the effect of increasing the forward gain of the simulated autopilot used to obtain the computer records of Fig. 3.22*a* by a factor of 5 and 6, respectively.

Flight-simulator Setup for Partial System Testing of Actual Automatic-pilot Components. Figure 3.20*c* shows how the flight-simulator setup of Fig. 3.20*a* may be used for testing actual automatic-pilot components. The "output voltages" $-\bar{\Phi}$ and $\frac{1}{4}P\bar{\Phi}$ of the flight simulator are fed to a servomechanism which positions a *flight table* (pitch table). The angle between the pitch-table surface and the horizontal is thus kept equal to the pitch attitude angle $\varphi = \varphi_1 + \bar{\varphi}$ of the simulated aircraft as the attitude change $\bar{\varphi}$ about the equilibrium attitude φ_1 varies in accordance with the flight simulator output voltage $-\bar{\Phi}$. The free and rate gyros of the automatic pilot under test are mounted on the pitch table, and their output signals cause the actual automatic-pilot elevator servo to position an

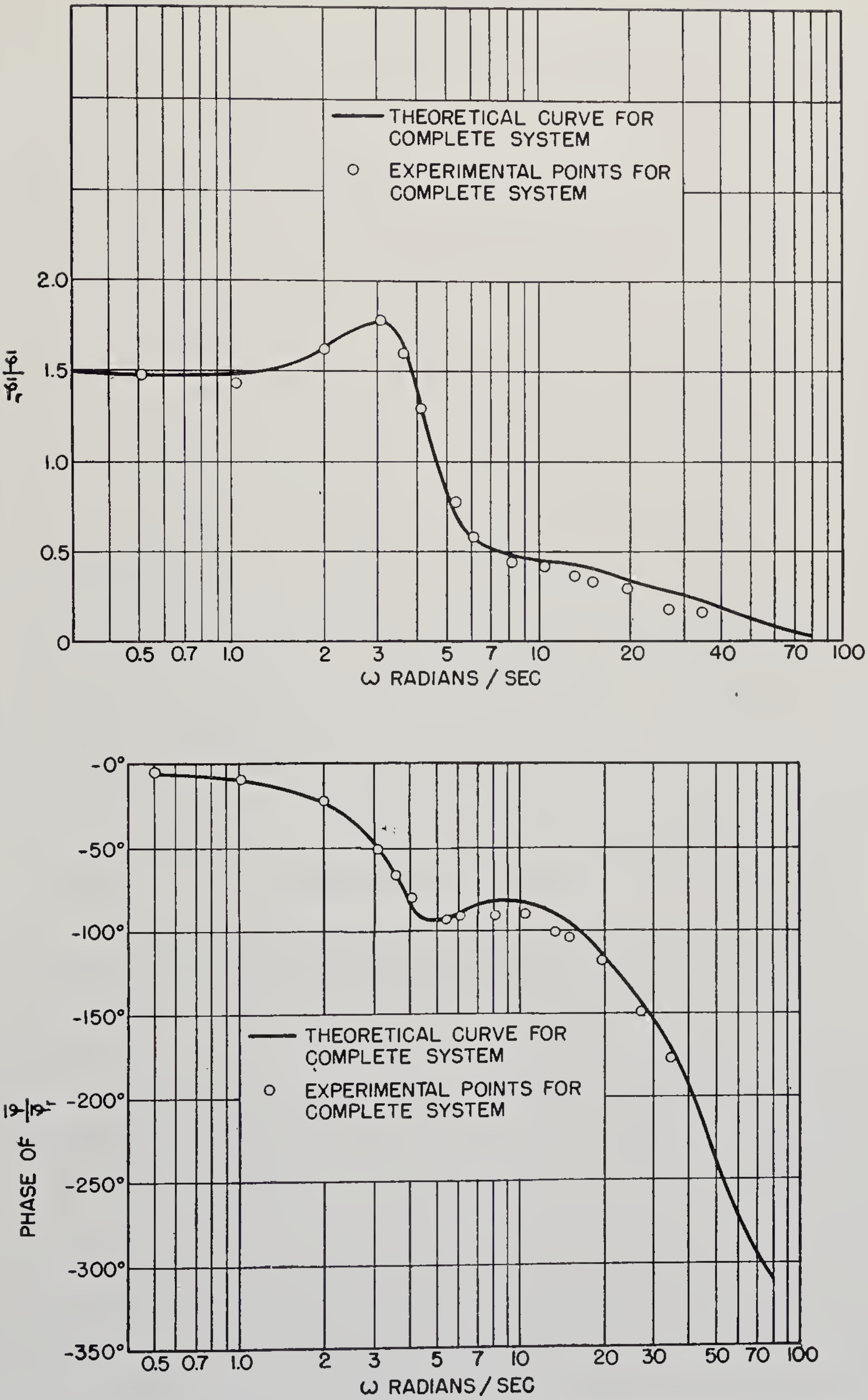


Fig. 3.23. Amplitude and phase response of an autopilot-controlled aircraft simulated by d-c analog-computing elements (Hagelbarger, Howe, and Howe, *op. cit.*).

actual or simulated elevator surface. The latter is loaded by springs simulating the aerodynamic forces on the control surface for each aircraft speed under consideration. A potentiometer-type pickoff on the control-surface shaft is used to generate a voltage $-\bar{\Delta}_1$ proportional to the control-surface displacement $\bar{\delta}_1$ and representing the latter in the flight simulator.

Utilization of Computer Servomechanisms in Flight Simulators. Flight simulators intended for partial system testing of more complicated automatic pilots, such as those used for preventing stalls, may require the use of computer servomechanisms for solving nonlinear flight equations (see Sec. 3.6). Since a d-c analog computer used as a flight simulator must operate on a 1:1 time scale, it may be necessary to consider the possible effects of limitations in the high-frequency response of the computer servomechanisms used. In this connection, it is useful to remember that the frequency-response limitations of computer servomechanisms apply only to one of the two machine variables to be multiplied. It is frequently possible to choose a problem variable which varies slowly to correspond to the machine variable appearing as a servo-shaft displacement in the computer. Thus, for most flight simulators, the inertia associated with aircraft attitude and position changes is, as a rule, by far larger than the inertia associated with the rotations of computer-servomechanism shafts whose displacements are to simulate the attitude and position changes of the aircraft. This usually applies even to the fairly large servomechanisms positioning the flight tables used for simulating aircraft rotations in the laboratory.

3.9. REPRESENTATION OF RANDOM PROCESSES. USE OF NOISE GENERATORS[1]

Problems involving random processes (processes which can be described in terms of sets of probability distributions)[2] require the generation of randomly varying voltages with appropriate statistical properties to represent randomly varying variables and/or parameters. In addition, one requires means for measuring or computing statistical properties of output data. *One may, thus, mix a suitable noise voltage with the servo input in Fig.* 3.10 *and compute the resulting mean square output error for a set of typical input signals; it is then relatively easy to vary the servo parameters so as to minimize the mean square error.*[1] The effects of load disturbances, *e.g.*, the effects of gusts on autopilot-controlled aircraft, can be similarly studied. These techniques are by no means restricted to linear systems; it is, for instance, possible to study the transmission of signals and noise through complicated modulator-detector systems simulated with the aid of appropriate diode-rectifier circuits (Sec. 6.7 and Table 6).

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[1] Bennett, R. R., and A. S. Fulton, The Generation and Measurement of Low-frequency Random Noise, *J. Appl. Phys.*, **22**: 1187, 1951; Laning, J. H., and R. H. Battin, On an Application of the Use of Analog Computers to Methods of Statistical Analysis, *Project CYCLONE Symposium II*, Reeves Instrument Corporation, New York City, 1952; Follin, J. W., G. F. Emch, and F. M. Walters, Modifications and Additions to the REAC, *ibid.*; Duncan, D. B., Response of Linear Systems to Random Inputs, *J. Appl. Phys.*, May, 1953; Bennett, R. R., Analog Computing Applied to Noise Studies, *Proc. IRE*, **41**: 1509, 1953; Random Noise Generator for Simulation Studies, *Report GER*-6434, Goodyear Aircraft Corporation, Akron, Ohio, Dec. 12, 1954; Smith, F. B., Analog Equipment for Processing Randomly Fluctuating Data, *Aeronaut. Eng. Rev.*, **14**: 113, May, 1955; Aseltine, J. A., The Use of Electronic Analog Computers in the Solution of Certain Radar Noise Problems, lecture presented at the West Coast Electronics Convention and Show, San Francisco, Calif., August, 1955.

[2] See, for instance, W. Feller, *Probability Theory and Its Applications*, Wiley, New York, 1950; and James, Nichols, and Phillips, *op. cit.*, Chap. 7.

A typical noise generator designed for use with d-c analog-computing equipment employs a noise source, such as a noisy vacuum tube or gas tube, or a radioactive source monitored by a Geiger-Mueller counter. The noise voltage is amplified and passes through a set of filters and/or limiters yielding the desired statistical properties in the frequency and amplitude domains.

Figure 3.24 shows typical waveforms obtained with the aid of a commercially available noise generator. Substitution of different types of filters (operational amplifiers) yields noise having various power spectra to represent different types of electrical noise, load disturbances, etc. The circuit of Fig. 3.24*b* generates random pulses which can initiate sequences of events in simulated random processes. This circuit has been used, for instance, to portray the breakdown of industrial machinery in a Monte-Carlo-type computation of down-time vs. failure rate and repair time.[2]

Statistical parameters describing the results of suitable[3] simulated random processes can be approximated as time integrals (time averages) over a sufficiently long time interval. In particular, one may obtain the *power spectrum* of a random variable by integrating the squared output of a narrow-band filter, or by Fourier analysis (Fig. 3.28*a*) of the autocorrelation function computed by the scheme of Fig. 3.28*b*. The *amplitude distribution*[4] of a random variable can be obtained through integration of the output voltage of a suitable comparator circuit (Sec. 6.7).

Other types of random processes require repeated computer runs under randomly varying conditions; the machine can take statistics over a set of such runs by storing the values of appropriate variables or time averages on integrators placed in the HOLD condition (Secs. 7.1 and 7.7) between runs.

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[1] James, Nichols, and Phillips, *op. cit.*, Chaps. 7 and 8.

[2] Two Applications of GEDA Computers to Statistical Problems of Operations Research, *Report GER*-6729, Goodyear Aircraft Corporation, Akron, Ohio, June 4, 1955. The second problem referred to in the title deals with a typical waiting-line process (traffic-light timing).

[3] Refer to James, Nichols, and Phillips, *op. cit.*

[4] *Ibid.*

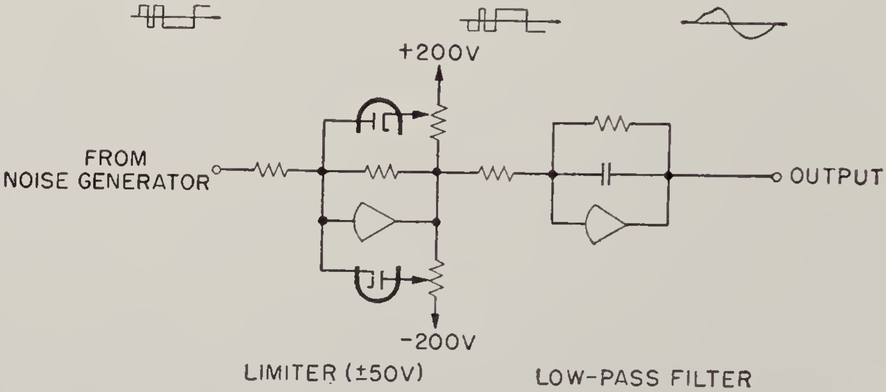

FIG. 3.24*a*. Use of a noise generator whose output is a square wave with Poisson-distributed zero crossings. The circuit approximates white Gaussian noise; for more accurate results, a low-pass filter having a sharper cutoff may replace the one shown. Substitution of other linear filters will yield noise with various types of power spectra to simulate electrical noise, disturbing torques, etc.

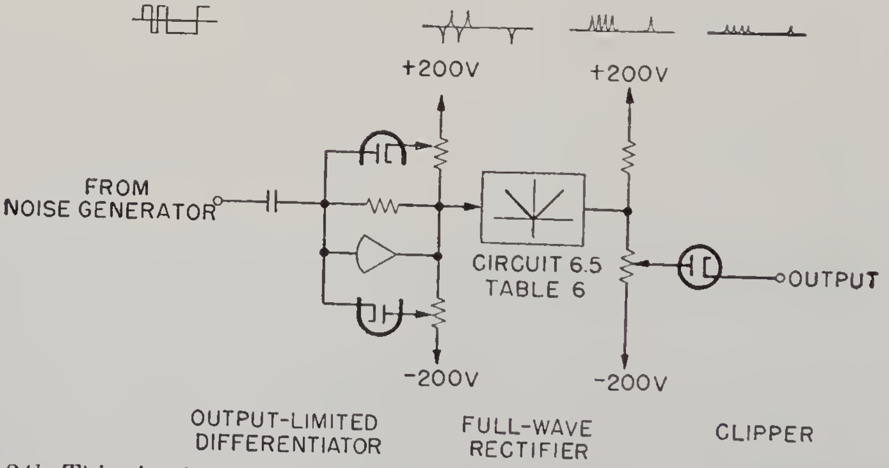

FIG. 3.24*b*. This circuit yields random pulses which may be used to initiate sequences of events in simulated random processes. One may also count or integrate such pulses for a predetermined time interval to obtain Poisson-distributed random numbers (Goodyear Aircraft Corporation *Report GER*-6729).

3.10. BOUNDARY-VALUE PROBLEMS, EIGENVALUE PROBLEMS, AND SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Solution of Boundary and Eigenvalue Problems by Means of D-c Analog Computers. In certain problems involving the solution of differential equations, the number of initial conditions (values of variables and derivatives for $t = 0$) given is not sufficient to determine the solution completely. Instead, the value of dependent variables or derivatives for some finite value t_1 are given. The solution of such *boundary-value problems* on a d-c analog computer proceeds by a trial-and-error process: the initial-value settings left unspecified by the problem are varied until the solution satisfies the desired boundary conditions for $t = t_1$. Excellent examples of d-c analog-computer solutions of boundary-value problems such as the static deflection of elastic beams are available in the literature.[1] Figure 3.25, taken from the first reference, shows several steps in the d-c analog-computer solution of a beam-deflection problem involving

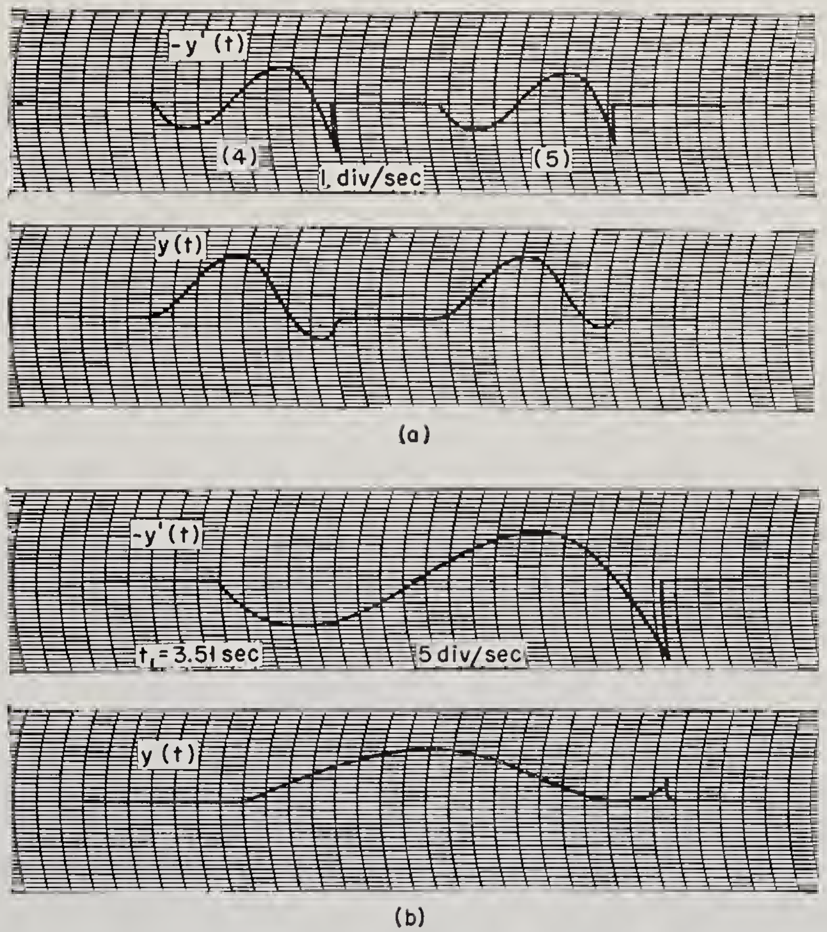

FIG. 3.25. Three steps of a trial-and-error process for the solution of the boundary-value problem given by Eq. (3.43). The final solution, shown in (*b*), is presented with a faster recorder speed for greater accuracy (Hagelbarger, Howe, and Howe, *op. cit.*).

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[1] Hagelbarger, Howe, and Howe, *op. cit.*; Corcos, G. M., R. M. Howe, L. L. Rauch, and J. R. Sellars, Application of the Electronic Differential Analyzer to Eigenvalue Problems, *Project CYCLONE Symposium II*, Reeves Instrument Corporation, New York City, May, 1952.

the solution of a differential equation

$$p^4 y = c \qquad \text{with } y\Big]_0 = py\Big]_0 = 0 \qquad \text{and} \qquad y\Big]_{t=t_1} = py\Big]_{t=t_1} = 0 \tag{3.43}$$

Automatic iteration methods (Sec. 7.7) can be applied to the d-c analog-computer solution of certain boundary-value problems.[1]

Many *eigenvalue problems,* such as those encountered in the study of elastic vibrations and quantum mechanics, can be solved on d-c analog computers by trial-and-error procedures analogous to those used for the solution of ordinary boundary-value problems. The values of the unknown eigenvalues must be varied until the desired boundary or normalization conditions are satisfied by the solutions of the given differential equations. It is often useful to obtain approximate solutions of eigenvalue problems on a d-c analog computer for use in powerful numerical methods involving successive approximations.[2]

Analog computers of the repetitive type (Sec. 8.5) are particularly suitable for the rapid solution of boundary-value and eigenvalue problems, since such machines permit the effects of parameter variations to be observed directly on a cathode-ray oscillograph.

D-c Analog-computer Solution of Partial Differential Equations. To solve a given partial differential equation, such as the heat-conduction equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{3.44}$$

with given boundary conditions on a d-c analog computer, one can reduce the problem to a set of boundary-value problems involving ordinary differential equations through *separation of variables.*[3] Again, if it is sufficient to determine the values of the unknown function $u(x,t)$ for discrete sets of values $x_1, x_2, \ldots ; t_1, t_2, \ldots$ of the variables x and t, the partial differential equation may be reduced to a set of simultaneous equations (*difference equations*) relating the function values $u(x_i,t_k)$. These equations can be solved by means of analog computers,[1] although various numerical methods may be preferable.[2]

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[1] Wadel, L. B., Automatic Iteration on an Electronic Analog Computer, lecture presented at the Western Electronics Show and Convention, Los Angeles, Calif., August, 1954.

[2] For a discussion of such problems, see H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry,* Van Nostrand, Princeton, N.J., 1946.

[3] The method of separation of variables is discussed, for instance, in Margenau and Murphy, *op. cit.* The solution of the resulting ordinary differential equations for a number of special cases is described in Hagelbarger, Howe, and Howe, *op. cit.*; C. E. Howe, R. M. Howe, and L. L. Rauch, Application of the Electronic Differential Analyzer to the Oscillation of Beams, Including Shear and Rotary Inertia, *Memo UMM*-67, University of Michigan Engineering Research Institute, Ann Arbor, January, 1951; C. E. Howe, R. M. Howe, and L. L. Rauch, Solution of Linear Differential Equations with Time Varying Coefficients by the Electronic Differential Analyzer, *Project CYCLONE Symposium II,* Reeves Instrument Corporation, New York City, May, 1952; Corcos, Howe, Rauch, and Sellars, *op. cit.*

A third method replaces the partial differential equation to be solved by a set of ordinary differential equations (*difference-differential equations*). This method, which is relatively well suited to d-c analog techniques, is illustrated by the following simple example.

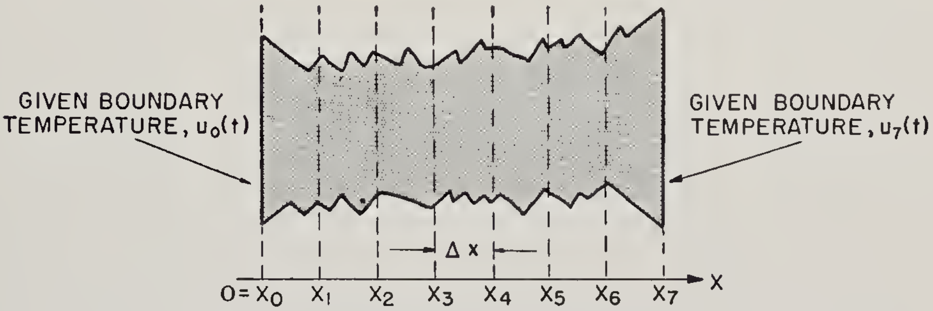
GIVEN BOUNDARY TEMPERATURE, $u_0(t)$
GIVEN BOUNDARY TEMPERATURE, $u_7(t)$
Δx
$0 = x_0$
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x

FIG. 3.26*a*. The heat-conduction problem with eight sample depths.

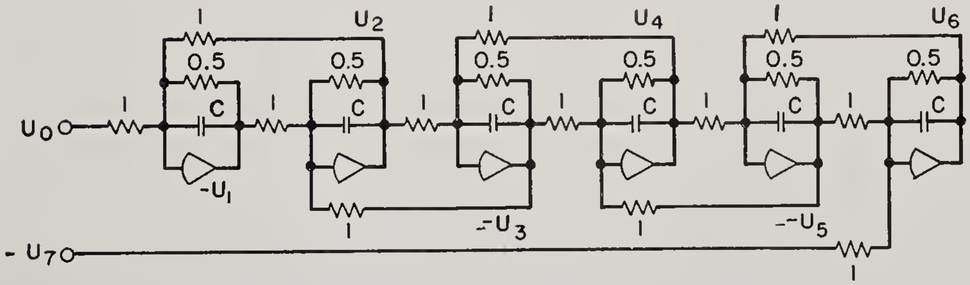

FIG. 3.26*b*. A d-c analog-computer setup establishing the machine equations (3.46) for $n = 7$. $n - 1 = 6$ integrators are required; the boundary values V_0 and V_7 are given as functions of the machine time τ. The initial-condition circuits are not shown.

A Simple Heat-conduction Problem. Figure 3.26*a* illustrates the *transient heating* of a large uniform wall of thickness a, such as a boiler wall, by heat conduction. It is desired to determine the temperature $u(x,t)$ as a function of the time t and the depth x when the initial temperature distribution $u(x,0)$ and the boundary temperatures $u(0,t)$ and $u(a,t)$ are given. For the sake of simplicity, the conductivity k, the density ρ, and the heat capacity c of the wall material are assumed to be constant. Then the temperature distribution in the wall satisfies the partial differential equation (heat-conduction equation) (3.44). Assume that it suffices to determine the temperature values $u(x_i,t) = u_i(t)$ measured at $n + 1$ depths

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[1] Soroka, W. W., *Analog Methods in Computation and Simulation*, McGraw-Hill, New York, 1954; see also Secs. 2.11 and 7.7.

[2] Hildebrand, F. B., *Methods of Applied Mathematics*, Prentice-Hall, New York, 1952, Chap. 3.

$$0 = x_0 < x_1 < x_2 < \ . \ . \ . \ < x_{n-1} < x_n = a$$

separated by equal distances $\Delta x = a/n$. Then Eq. (3.44) may be replaced by a set of $n - 1$ difference-differential equations,

$$\frac{du_i}{dt} = \frac{k}{\rho c(\Delta x)^2}(u_{i+1} - 2u_i + u_{i-1}) \qquad (i = 1, 2, \ . \ . \ . \ , n - 1) \quad (3.45)$$

with given initial conditions $u_1(0), u_2(0), \ . \ . \ . \ , u_{n-1}(0)$, and

$$u_0 = u(0,t) \qquad u_n = u(a,t)$$

The corresponding machine equations

$$\left.\begin{aligned} &PU_i = \frac{1}{C}(U_{i+1} - 2U_i + U_{i-1}) \\ &\qquad\qquad C = \frac{\rho c(\Delta x)^2 \alpha_t}{k} \qquad (i = 1, 2, \ . \ . \ . \ , n - 1) \\ &U_0 = a_u u\left(0, \frac{\tau}{\alpha_t}\right) \qquad U_n = a_u u\left(a, \frac{\tau}{\alpha_t}\right) \end{aligned}\right\} \quad (3.46)$$

are readily solved on a d-c analog computer if n is not too large (Fig. 3.26*b*). The simple computer setup shown is easily generalized to include complications, such as changes in the conductivity k and various types of heat transfer at the boundaries.[1]

Vibrations of Structures. The procedure just described can be applied to many other partial differential equations; the difference-differential equations in question can often be established by direct-analog operational amplifiers (Sec. 1.5 and Tables 1 to 3).[2]

A particularly interesting application is the analog-computer solution of partial differential equations describing vibrations of beams and structures.[3] Both torsional and lateral bending of uniform or nonuniform beams can be analyzed, and variable forces may be introduced at any station or stations along the beam. The resulting analog-computer setups yield continuous records of displacements, bending moments and shear forces; characteristic-mode frequencies can be determined from the response of the simulated structure to sinusoidal forcing voltages. These techniques may be applied to studies of aircraft structural vibrations and permit, for instance, the real-time simulation of wing flexing and nonlinear flutter and their effects on aircraft control and autopilot performance.

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[1] Analysis of Transient Aerodynamic Heating Problems with GEDA Equipment, *Report GER*-5435, Goodyear Aircraft Corporation, Akron, Ohio, June 12,1 953; Howe, R. M., and V. S. Haneman, Jr., The Solution of Partial Differential Equations by Difference Methods Using the Electronic Differential Analyzer, *Proc. IRE*, **41**: 1497, 1953; see also *Memo AIR*-1, University of Michigan Engineering Research Institute, October, 1951.

[2] Rogers, T. A., Electronic Analog Computers and Partial Differential Equation Solutions, *Memorandum* of the Department of Engineering, University of California, Los Angeles, Nov. 17, 1952; Howe and Haneman, *op. cit.*; Soroka, *op. cit.*

[3] Soroka, *op. cit.*, Chap. 8; Howe and Haneman, *op. cit.*

Normal-mode vibrations obtained from experimental d-c analog-computer studies of simple structures exhibit good agreement with theoretical results, provided that three or more stations or "cells" per half-wavelength of the normal mode are used. Since vibrations of elastic structures often involve weakly damped modes at relatively high frequencies, caution must be exercised to avoid instability due to bandwidth limitations of operational amplifiers (Sec. 4.7).[1] It may be necessary to introduce small artificial damping terms to prevent spurious oscillations.

In the case of large-scale studies of complicated elastic structures, the use of network-type analog computers[2] is usually preferable to d-c analog techniques. In such applications, relatively simple passive networks can replace large numbers of operational amplifiers, so that errors due to amplifier drift and bandwidth limitations are eliminated. It should be remembered, however, that a high-quality operational amplifier costs less than one-half as much as a single precision inductor-transformer used in a typical network-type analog computer. There may, thus, be some incentive for the development of special-purpose operational amplifiers designed specifically as building blocks for the simulation of elastic structures.

3.11. D-C ANALOG-COMPUTER SOLUTION OF LINEAR PROGRAMMING PROBLEMS

So-called *linear programming*[3] *problems* in operations analysis, game theory, and economics require the determination of *a set of n numbers* $x_1, x_2, \ldots, x_n$ *which minimizes (or maximizes) a linear expression*

$$v(x_1,x_2, \ldots ,x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{3.47}$$

---

[1] See also L. G. Walters, Hidden Regenerative Loops in Electronic Analog Computers, *Trans. PGEC, IRE,* **2**: 1, June, 1953.

[2] Network-type analog computers utilize passive networks comprising resistors, capacitors, inductors, and transformers as direct analogs (see also Sec. 1.5) of vibrating structures and other physical systems. Relatively few electronic computing elements are required in most applications. Currents as well as voltages serve as machine variables. The application of such direct-analog techniques to problems involving elastic structures has been brought to a high state of perfection, notably by G. D. McCann and his associates at the California Institute of Technology. A simple account of network-type analog computers as well as a comprehensive bibliography will be found in Soroka, *op. cit.*

[3] Charnes, A., W. W. Cooper, and A. Henderson, *An Introduction to Linear Programming,* Wiley, New York, 1953.

while satisfying $m \geq n$ linear inequalities

$$y_i \equiv a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - a_{i0} \geq 0 \qquad (i = 1, 2, \ldots m) \tag{3.48}$$

In a typical application, the problem is to buy necessarily positive quantities x_k of n types of raw materials ("inputs") so as to minimize the cost (3.47) while keeping the respective quantities y_i of n output products at or above $m - n$ corresponding specified levels a_{i0} $(i = n + 1, n + 2, \ldots, m)$.

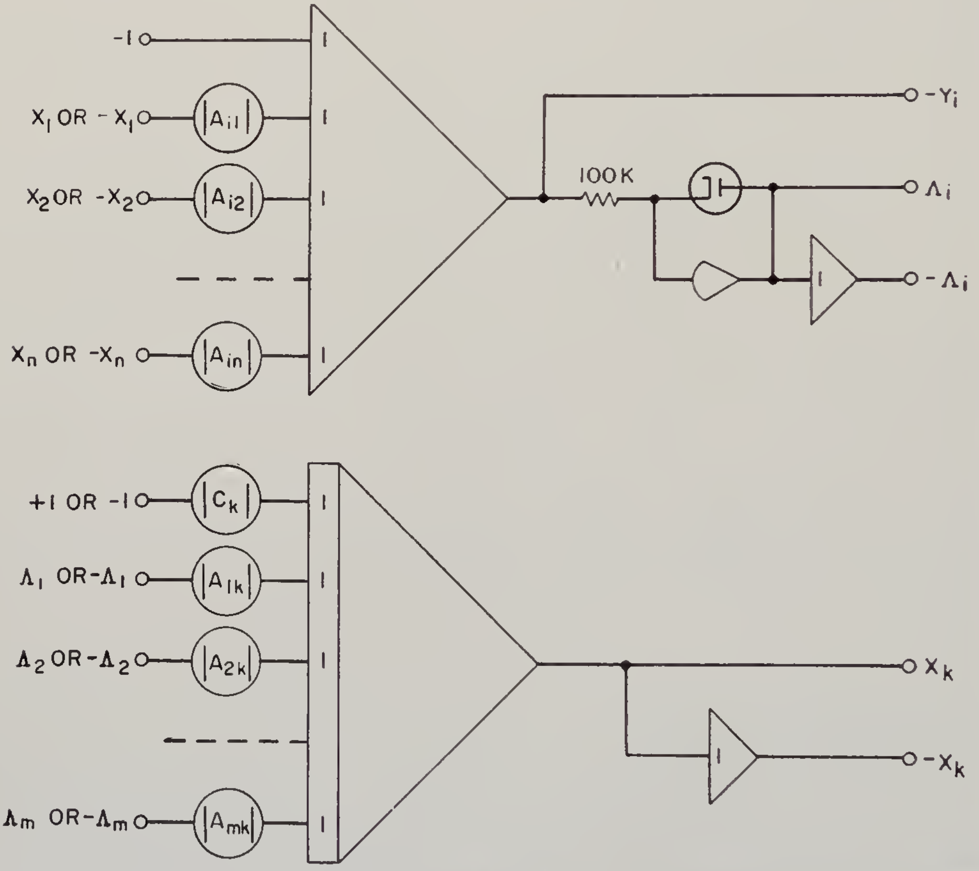

FIG. 3.27*a*. Analog-computer setup for the solution of linear programming problems. The inverted input voltages $-X_k$, $-\Lambda_i$, or -1 are used with negative coefficients A_{ik} or C_k. *If, as in many practical problems, an inequality $Y_i \geq 0$ takes the simple form $X_i \geq 0$, then the entire block of computing elements generating the corresponding voltage Λ_i can be replaced by a simple feed-back diode* (Sec. 6.7) *connected across the integrating capacitor to keep the integrator output positive* (see also Fig. 3.4).

Machine Relations and Scale Factors. A d-c analog computer represents the problem variables $x_1, x_2, \ldots, x_n$ and v by corresponding voltages

$$X_1 = a_1x_1,\ X_2 = a_2x_2,\ \ldots,\ X_n = a_nx_n \qquad V = a_vv \tag{3.49}$$

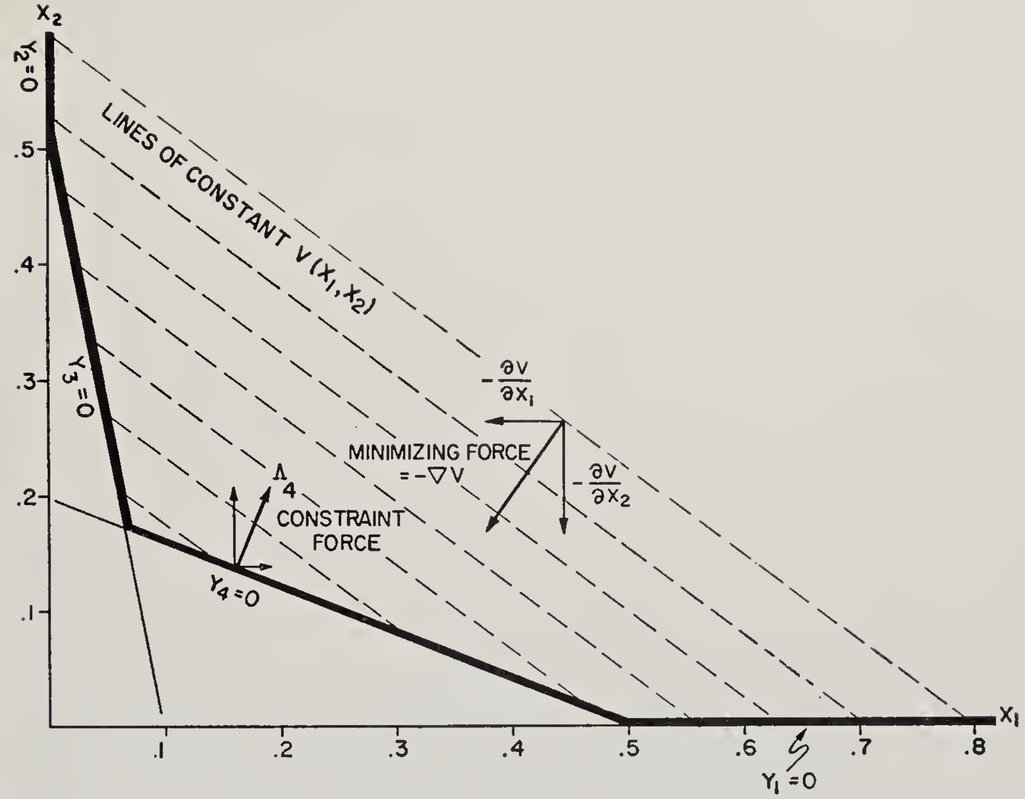

FIG. 3.27*b*. Dynamical analogy for a simple linear programming problem. It is desired to minimize $V(X_1,X_2) = \frac{10}{4} X_1 + \frac{10}{3} X_2$ subject to the conditions

$$Y_1 \equiv X_1 \geq 0, \qquad Y_2 \equiv X_2 \geq 0,$$
$$Y_3 \equiv 10X_1 + 2X_2 - 1 \geq 0, \qquad Y_4 \equiv 2X_1 + 5X_2 - 1 \geq 0$$

which are made to satisfy *differential equations* such that

$$V(X_1,X_2, \ldots ,X_n) = C_1X_1 + C_2X_2 + \cdots + C_nX_n \tag{3.50}$$

converges to a minimum with increasing time τ, while

$$Y_i \equiv \frac{y_i}{a_{i0}} = A_{i1}X_1 + A_{i2}X_2 + \cdots + A_{in}X_n - 1 \geq 0 \qquad (i = 1, 2, \ldots , m) \tag{3.51}$$

where

$$C_k = a_v \frac{c_k}{a_k} \qquad A_{ik} = \frac{a_{ik}}{a_{i0}a_k} \qquad (i = 1, 2, \ldots , m; \quad k = 1, 2, \ldots , n) \tag{3.52}$$

The scale factors a_k and a_v are chosen so that the absolute numerical values of the quantities X_k, Y_i, C_k, and A_{ik} are all less than one.

Analog-computer Setup. Figure 3.27*a* shows how the computer establishes the desired relations. Each voltage $-Y_i$ appears at the output of a summing amplifier and is fed to an output-limited high-gain amplifier whose output voltage Λ_i is zero for $Y_i \geq 0$ but becomes large and negative if $Y_i < 0$. Each voltage X_k appears at the output of a summing integrator, so that

$$-\frac{dX_k}{d\tau} = C_k + A_{1k}\Lambda_1 + A_{2k}\Lambda_2 + \cdots + A_{mk}\Lambda_m \qquad (k = 1, 2, \ldots, n) \tag{3.53}$$

Dynamical Analogy. The n equations (3.53) may be said to describe the motion of a dynamical system with n generalized coordinates X_1, $X_2, \ldots, X_n$, subject to strong viscous damping and having the potential energy $V(X_1,X_2, \ldots ,X_n)$. As long as all inequalities (3.51) are satisfied, the only forces acting are the negative gradient components $-\frac{\partial V}{\partial X_k} = -C_k$; these forces tend to decrease $V(X_1,X_2, \ldots ,X_n)$. If one of the inequalities is violated, say $Y_i < 0$, then a strong corresponding "constraint force" exerts a component $-\Lambda_i A_{ik}$ in the direction of each coordinate X_k so as to restore the inequality (see also Sec. 3.3).

In the example of Fig. 3.27*b*, X_1 and X_2 may be regarded as the rectangular cartesian coordinates of a particle moving in a viscous medium. The constraints are satisfied for all points (X_1,X_2) above and to the right of the convex polyhedron of smooth straight-line barriers indicated in heavy lines. The broken lines are lines of equal potential energy $V(X_1,X_2)$. The particle will move *in the direction of the negative gradient vector* $-\nabla V$ as shown until it strikes a barrier, say, by violating the inequality $Y_4 \geq 0$. The particle then experiences an additional force at right angles to the barrier. As a result, the particle will move along the heavy lines to that barrier vertex which corresponds to the lowest potential energy $V(X_1,X_2)$. The problem is degenerate if ∇V is perpendicular to a barrier; in this case, the solution may not be unique.

Results and Discussion. The computer setup of Fig. 3.27*a* solved the sample problem with better than 0.3 per cent accuracy. The solution converged within a few seconds; one may further reduce this solution time by decreasing the integrating capacitors. The analog computer permits one to read solutions directly on a servo or digital voltmeter and to print or record them as functions of problem parameters. The high computing speeds available are attractive when one desires to investigate the effects of many parameter variations.

The values of the "slack variables" Y_i, which are often of practical interest, may also be read directly. The n variables Y_i closest to zero correspond to those of the inequalities (3.51) which actually determine the solution vertex (Fig. 3.27*b*). It is then possible to obtain solutions of essentially unlimited accuracy by solving the corresponding n simultaneous *equations* $Y_i = 0$ by a suitable iteration method, perhaps one making use of the analog computer in conjunction with a keyboard calculator.[1]

3.12. MISCELLANEOUS PROBLEMS. A LIST OF D-C ANALOG-COMPUTER APPLICATIONS

Computation and Plotting of Functions. A d-c analog computer and its associated recording equipment may be very useful for *computing and plotting functions* of one or several variables. A d-c analog computer with a servo plotting table makes it easy to prepare reproducible graphs of functions like τ^2, $\sin \tau$, e^τ, and many others. Again, if $X(\tau)$ is a given function of the time τ obtained from some signal source or generated by means of a function generator such as an automatic curve follower (Sec. 6.9), a d-c analog computer can compute and plot complicated expressions (transformations) $F = F[X(\tau)]$. A particularly interesting application of this technique is the conformal mapping[2] of a contour

$$X = X(\tau) \qquad Y = Y(\tau) \tag{3.54}$$

in the XY plane into a corresponding contour in the UV plane through a suitable transformation

$$U = U(X,Y) \qquad V = V(X,Y)$$

The functions (3.54) may be generated by means of function generators (*e.g.*, $X = R \cos \omega\tau$, $Y = R \sin \omega\tau$ for a circle in the XY plane), or the contour may be followed by an automatic curve follower. Such conformal mapping techniques are useful for the solution of certain boundary-value problems (*e.g.*, design of airfoil shapes) and for the solution of complex algebraic equations by the isograph[3] method.

Computation of Definite Integrals and Integral Transforms. D-c analog computers can also serve for the computation of definite integrals, including line integrals.[4] In particular, the computation of integral transforms of the form

$$F(x) = \int_a^b K(x,t)f(t)dt \tag{3.55}$$

where K and f are given functions of the variables x and t, is of the greatest importance in a large number of problems (examples: expansion of functions in series of orthogonal functions, Fourier analysis, convolution and superposition integrals, solution of integral equations, boundary value problems, computation of autocorrelation functions and cross-correlation functions; see Fig. 3.28).[1] Some applications of this type are sufficiently important to warrant the design of special-purpose computers.

---

[1] Goldberg, E. A., and G. W. Brown, An Electronic Simultaneous Equation Solver, *J. Appl. Phys.*, **19**: 339, 1948.

[2] Tomlinson, N. P., M. Horowitz, and C. H. Reynolds, Analog Computer Construction of Conformal Maps in Fluid Dynamics, *J. Appl. Phys.*, **26**: 229, 1955.

[3] Marshall, B. O., An Electronic Isograph for Roots of Polynomials, *J. Appl. Phys.*, **21**: 307, 1950.

[4] Evaluation of Definite Integrals on Electronic Differential Analyzers, *Report GER*-6365, Goodyear Aircraft Corporation, Akron, Ohio, Oct. 10, 1954.

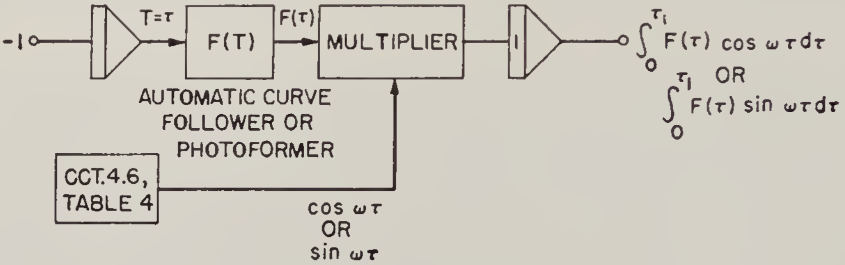
-1
T=τ
F(T)
F(τ)
MULTIPLIER
1
$\int_0^{\tau_1} F(\tau)\cos\omega\tau d\tau$ OR $\int_0^{\tau_1} F(\tau)\sin\omega\tau d\tau$
AUTOMATIC CURVE FOLLOWER OR PHOTOFORMER
CCT.4.6, TABLE 4
cos ωτ OR sin ωτ

FIG. 3.28*a*. A circuit used to compute *Fourier coefficients*

$$\frac{2}{\tau_1}\int_0^{\tau_1} F(\tau)\cos\omega\tau\,d\tau \qquad \text{or} \qquad \frac{2}{\tau_1}\int_0^{\tau_1} F(\tau)\sin\omega\tau\,d\tau$$

for $\omega = \frac{2\pi n}{\tau_1}$, or to approximate *Fourier cosine and sine transforms* as limits of

$$\sqrt{\frac{2}{\pi}}\int_0^{\tau_1} F(\tau)\cos\omega\tau\,d\tau,\ \sqrt{\frac{2}{\pi}}\int_0^{\tau_1} F(\tau)\sin\omega\tau\,d\tau$$

as τ_1 increases. Ordinary computer control circuits are used to start and stop the integration. A repetitive analog computer (Secs. 1.1 and 8.5) can be set up so that ω varies between sweeps; the machine then computes the integral for different values of ω during successive sweeps. If ω is made to vary periodically (as by a rotating potentiometer), the resulting function of ω can be displayed on a cathode-ray oscillograph (MacNee, *op. cit.*).

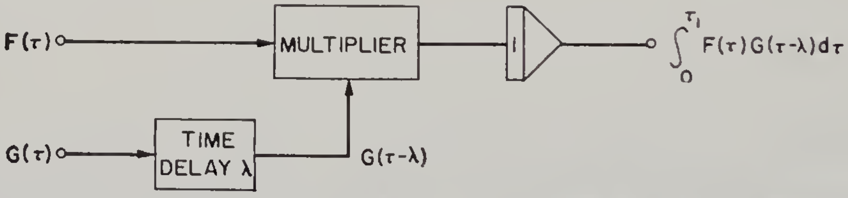

FIG. 3.28*b*. This d-c analog-computer setup approximates the *cross-correlation function* of two suitable functions $F(\tau)$ and $G(\tau)$ [*autocorrelation function* if $F(\tau) \equiv G(\tau)$] as the limit of

$$\frac{1}{\tau_1}\int_0^{\tau_1} F(\tau)G(\tau-\lambda)\,d\tau$$

as τ_1 increases. The time delay λ is introduced by one of the methods described on p. 108; a special-purpose computer may utilize magnetic tape recorders whose writing and reading heads are separated by a variable distance proportional to λ.

A List of D-c Analog-computer Applications. The following partial list of problem types suitable for d-c analog-computer solution may help to suggest new or related applications. Some additional references will be found in the bibliography.

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[1] MacNee, A. B., A High-speed Product Integrator, *MIT-RLE Report* 136, Aug. 17, 1949; Wallman, H., An Electronic Integral-transform Computer and the Practical Solution of Integral Equations, *J. Franklin Inst.*, **250**: 45, 1950.

1. *Aircraft and missile engineering*
    - Design of aircraft controls and automatic pilots (Secs. 3.6 to 3.8)
    - Guidance and landing systems[1]
    - Turbojet and supercharger controllers[2]
    - Helicopter vibrations and control[3]
    - Vibrations and nonstationary flutter, gust loads (see also Sec. 3.10)[4]
    - Landing shocks (see also Sec. 3.2)
    - Rocket burning and flight
    - Trajectory computations (Sec. 3.5)
    - Design of wind-tunnel mounting systems
    - Aerodynamic stability of aerial refueling devices
    - Supersonic flow problems
    - Partial system testing, flight simulation (Secs. 3.6 to 3.8)
2. *Mechanical and automotive engineering*
    - Design and partial system testing of automatic control systems (Secs. 3.4 and 3.8)
    - Ship and submarine control systems
    - Vibrations of structures (Secs. 3.2 and 3.10)
    - Spring systems and vibration absorbers (Sec. 3.2)
    - Torsional vibrations in crankshafts
    - Dynamics of mechanisms (cams, valves, brakes, linkages, etc., Sec. 3.3)
    - Hydraulic transmissions
    - Box-car coupling[5]
3. *Electrical engineering*
    - Analysis of linear and nonlinear circuits[6]

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[1] Bretoi, R. N., and D. L. Markusen, *The Use of the REAC by Minneapolis-Honeywell in the Analysis and Synthesis of Automatic Flight Control: Lateral Case*, Minneapolis-Honeywell Regulator Company, Minneapolis, Minn., 1949; Computer Applications to Pilotless Aircraft Studies, *Report GER*-2572, Goodyear Aircraft Corporation, Akron, Ohio, Apr. 10, 1951; Problems Illustrating the Use of the Beckman EASE Computer, Supplementary data to *Bulletin* 2, 89, Beckman Instruments, Inc. (now Berkeley Division, Richmond, Calif.) May, 1952; Monroe, W. R., Application of Electronic Simulation Techniques to the Development of Airplane Flight Control Systems, *Aeronaut. Eng. Rev.*, **14**: 91, 1955.

[2] Philbrick, G. A., W. T. Stark, and W. C. Schaffer, Electronic Analog Studies for Turboprop Control Systems, *SAE Quart. Trans.*, **2** (No. 2), April, 1948; Schaffer, W. C., Application of Analog Techniques to Control Design for Aircraft Engines, *SAE Symposium*, January, 1952.

[3] Optimizing a Helicopter Autopilot Design, *EASE Application Bulletin* 5, Berkeley Division, Beckman Instruments, Inc., Richmond, Calif. (report prepared by J. B. Rea Company, Los Angeles, Calif.).

[4] Yanowitch, M., Solution of a Flutter Problem, *Project CYCLONE Special Report*, Reeves Instrument Corp., New York City, June, 1955.

[5] Roedel, J., *Application of an Analog Computer to Design Problems for Transportation Equipment*, lecture presented at the International Instrument Conference, Sept. 22, 1954.

[6] GEDA Analysis of an Electrical Circuit, *Report GER*-5455, Goodyear Aircraft Corporation, Akron, Ohio, Mar. 18, 1953; Han Chang, R. C. Lathrop, and V. C. Rideout, Study of Oscillator Circuits by Analog Methods, *NEC Proc.*, **6**, 1950. Larrowe, V. L., Direct Simulation Bypasses Mathematics, *Control Eng.*, November, 1954.

Studies of modulation systems
Studies of the effects of noise in communications and control systems (Sec. 3.9)
Electron optics
Feedback amplifiers
Analysis of rotating machinery[1]
Design of loudspeaker systems and enclosures
Relay design (Sec. 3.4)
Analysis of transmission lines
Power distribution problems (Sec. 3.4)

4. *Chemical and petroleum engineering*
   Process control in chemical plants (Sec. 3.4)
   Chemical reaction rate and diffusion problems
   Geophysics
   Oilfield recovery problems
   Pipe-line flow and vibration problems
5. *Others*
   Operations research (Secs. 3.9 and 3.11)
   Economic system studies[2]
   Flood control[3]
   Water-hammer problems in the selection of dam sites[4]
   Ballistics and fire control (Sec. 3.5)
   Nuclear physics and nuclear engineering[5]
   Solutions of Schrödinger's equation[6]
   Heat flow (Sec. 3.10), freezing and thawing of soils
   Research in nonlinear mechanics (see also Secs. 2.8 and 3.3)[7]
   Curve plotting and conformal mapping (see also p. 151)
   Partial system testing involving human operators (see also Sec. 3.4)
   Production testing

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[1] Fuller, J. L., and A. P. DiVincenzo, *Analog Computer as a Tool in Motor Applications*, Reliance Electric and Engineering Company, Cleveland, Ohio, Oct. 28, 1952.

[2] Smith, O. J. M., Economic Analogs, *Proc. IRE*, **41**: 1514, 1953.

[3] GEDA Application to Flood-control and Water-distribution Problems, *Report GER*-6053, Goodyear Aircraft Corporation, Akron, Ohio, May 4, 1954; Paynter, H. M., Methods and Results from MIT Studies of Unsteady Flow, *J. Boston Soc. Civil Engrs*, **39**: 224, 1952.

[4] Paynter, H. M., Electrical Analogies and Electronic Computers: Surge and Water Hammer Problems, *ASCE Proceedings*, Separate 146, **78**, August, 1952.

[5] Bell, P. R., and H. A. Straus, *op cit.;* Schultz, M. A., *Control of Nuclear Reactors and Power Plants*, McGraw-Hill, New York, 1956.

[6] Garwin, R. L., A Differential Analyzer for the Schrödinger Equation, *Rev. Sci. Instr.*, **21**: 411, 1950.

[7] Markey, H. G., and V. C. Rideout, Analog Computer Solution of a Nonlinear Differential Equation, *Trans. AIEE*, Paper 51–171, April, 1951.

CHAPTER 4

THEORY AND DESIGN OF LINEAR COMPUTING ELEMENTS. COEFFICIENT-SETTING POTENTIOMETERS AND OPERATIONAL AMPLIFIERS.

4.1. MULTIPLICATION BY CONSTANT COEFFICIENTS: POTENTIOMETERS

Introduction. Construction of Potentiometers. A potentiometer consists of a resistance having terminals at each end and provided with a sliding contact or brush arranged so that it can be moved over the resistance. In the most common type of potentiometer, the resistance winding is arranged in cylindrical form while the brush is mounted on an arm which revolves on a central shaft[1] (Figs. 4.1 and 4.2).

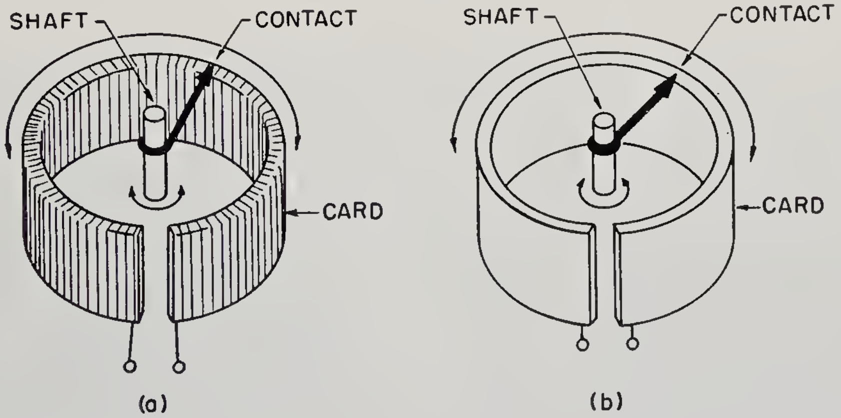

FIG. 4.1. Rotary potentiometers with wire-wound resistance element (*a*) and with deposited-film element (*b*).

In d-c analog computers, potentiometers are most often used to multiply a machine variable by a constant coefficient a $(0 \leq a \leq 1)$ in the manner described in Sec. 1.3. This subject will be discussed in somewhat more detail here. The use of rotating motor-driven potentiometers for multiplication and function generation is described in Sec. 1.3 and in Chap. 6.

A potentiometer must be designed to certain specifications covering its *total resistance* and *power rating.* The resistances of potentiometers used in d-c analog computers vary between 10,000 and 100,000 ohms. The lower limit of the potentiometer resistance is determined by the power of

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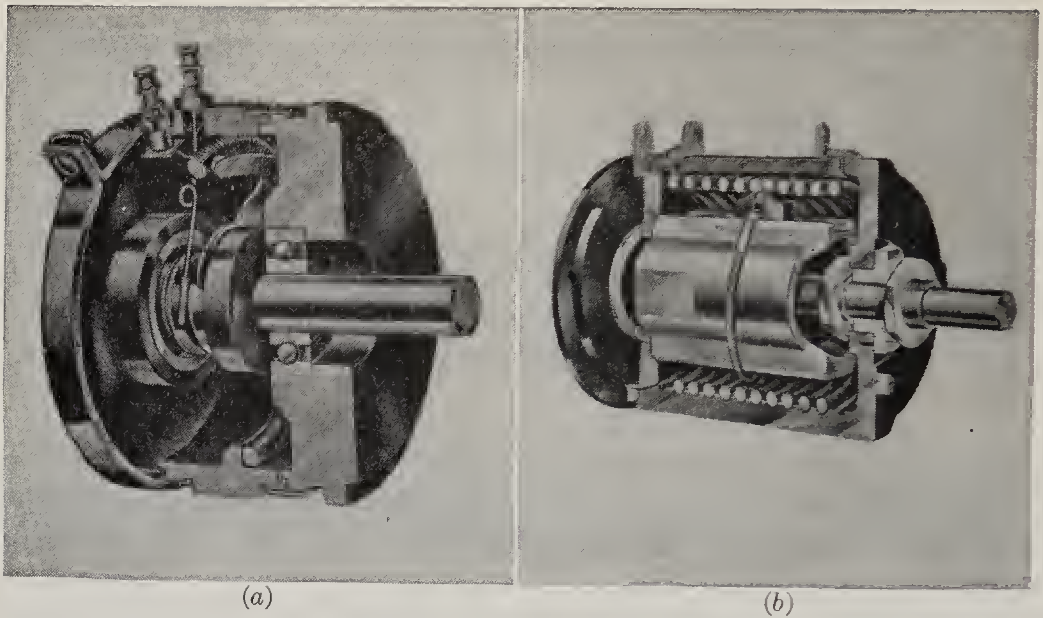
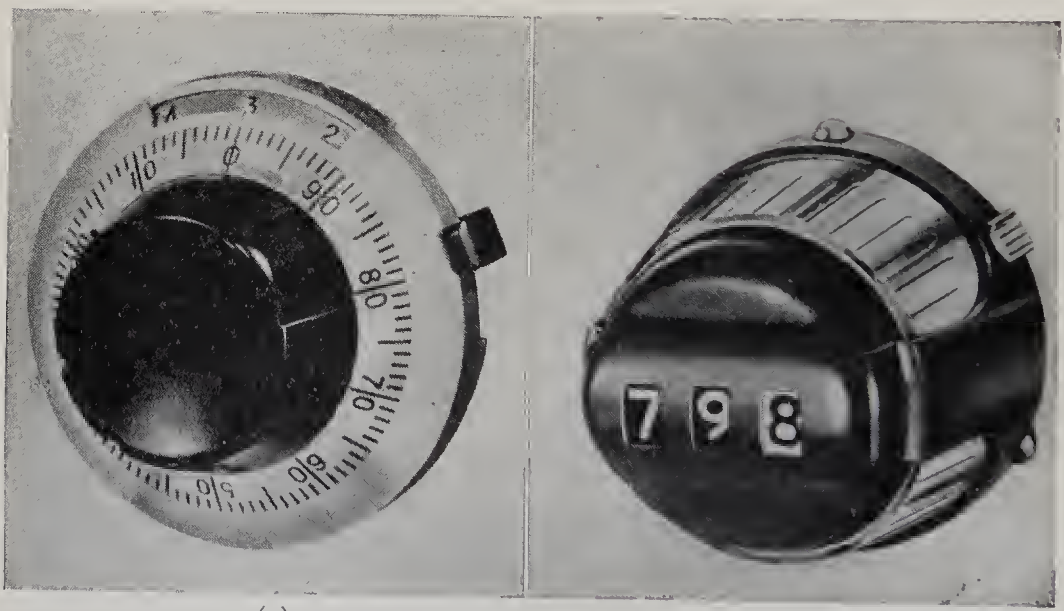
[1] Nettleton, L. A., and F. E. Dole, Potentiometers, *Rev. Sci. Instr.*, **17**: 356, 1946.



(a) (b)

FIG. 4.2. Single-turn (a) and multiple-turn (b) potentiometers with helical resistance elements and precious-metal contacts (HELIPOT® precision potentiometers). The single-turn potentiometer shown is capable of continuous rotation and has ball bearings for servo applications (Sec. 6.9).



(a) (b)

FIG. 4.3. Typical dials for multiple-turn coefficient-setting potentiometers: Type RB DUODIAL® turn-counting dial (a) and Borg Model 1301 MICRODIAL® (b). Note the dial-lock levers.

(Figures 4.1, 4.2, and 4.3a copyright 1954, HELIPOT Corporation—reprinted by permission; Fig. 4.3b courtesy of the George W. Borg Corporation.)

the amplifier driving the potentiometer and by the required resolution (see also below). The upper limit is fixed by loading considerations (see page 158) as well as by the availability of wire-wound resistance elements. Computing potentiometers should be capable of having a steady voltage of about 100 volts applied across a resistance of 10,000 ohms without appreciable heating; 4 watts is a commonly used power rating.

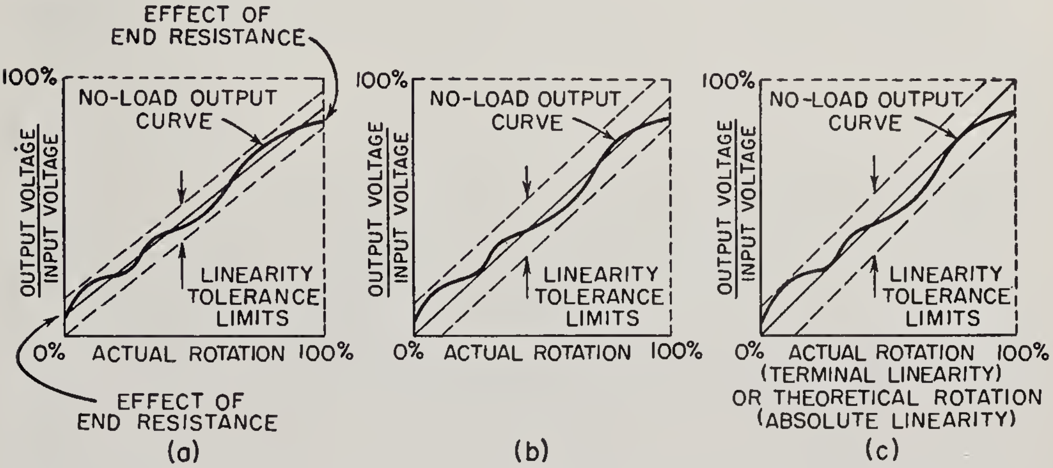


Fig. 4.4. Specification of potentiometer linearity. (Errors are exaggerated.)
Fig. 4.4*a*. *Independent linearity* of a linear potentiometer is the least maximum deviation of the actual output curve from a straight line.
Fig. 4.4*b*. *Zero-based linearity* is the least maximum deviation from a straight line through the origin.
Fig. 4.4*c*. *Terminal linearity* is the maximum deviation from a straight line through the origin and a point corresponding to 100 per cent input voltage and the *actual* potentiometer rotation between end taps. *Absolute linearity* is the maximum deviation from a straight line through the origin and a point corresponding to 100 per cent input voltage and the *theoretical* rotation (between the theoretical end-tap locations).

For accurate computation, the following properties are desirable in potentiometers used for setting constant coefficients:

1. *Mechanical ruggedness*, which will permit a potentiometer setting to be maintained in spite of vibrations, shock, etc.
2. *Freedom from microphonism and electrical noise.*
3. *Accuracy of calibration.* It should be possible to set the transfer function of the potentiometer to any desired value by means of a calibrated dial. If the dial scale is linear, the (absolute) linearity of the potentiometer will be important (Fig. 4.4).
4. *Resolution.* It should be possible to set the potentiometer to any desired coefficient. Accordingly, voltage "steps" due to movement of the sliding contact over the resistance element should be kept within a specified small value.
5. *Stability.* The resistance of the potentiometer, and especially the accuracy of the calibration, should not change appreciably with temperature and age.

The design and construction of potentiometers having such desirable properties have practically reached a state of perfection and have been described in great detail by F. Dole[1] and others (see also Sec. 6.9).

Most of the above requirements will be satisfied only by potentiometers using wire-wound resistance elements, although some types of resistance elements consisting of metallized ceramic materials may be satisfactory.

The potentiometers used most frequently for coefficient settings are the helical-card types, such as *Gibbs Micropots* and *Beckman Helipots* (Fig. 4.2*b*). The entire scale of such a potentiometer is covered in 5 to 20 turns of the specially constructed dial (Fig. 4.3). The latter can be read to 1 part in 10,000, which is better than the 0.1 to 0.05 per cent linearity and resolution of many such potentiometers.

Corrections for Potentiometer Loading. Ordinarily, the dial reading a of a computing potentiometer refers to the fraction ar of the total potentiometer resistance r across which the output voltage X_o is measured. If such a potentiometer is connected to a load of infinite impedance (see Fig. 4.5*a*), the potentiometer transfer function will be

$$\frac{X_o}{X_1} = \frac{ar}{r} = a \tag{4.1}$$

which corresponds to the dial reading.

If a load of resistance r_L is connected to the potentiometer output terminals (see Fig. 4.5*b*), Eq. (4.1) holds no longer. From the nodal equations for the output terminal,

$$(X_o - X_1)\frac{1}{(1-a)r} + \frac{X_o}{ar} + \frac{X_o}{r_L} = 0 \tag{4.2}$$

it follows that

$$\frac{X_o}{X_1} = \frac{a}{1 + a(1-a)\dfrac{r}{r_L}} = a - \frac{a^2(1-a)\dfrac{r}{r_L}}{1 + a(1-a)\dfrac{r}{r_L}} \tag{4.3}$$

This is the correct relation between the potentiometer setting a and the transfer function X_o/X_1. The quantity

$$a - \frac{X_o}{X_1} = \frac{a^2(1-a)\dfrac{r}{r_L}}{1 + a(1-a)\dfrac{r}{r_L}} \qquad \left[= a^2(1-a)\frac{r}{r_L} \text{ for } \frac{r}{r_L} \ll 1\right] \tag{4.4}$$

is a measure for the error made by applying the simplified relation (4.1) instead of the correct relation (4.3).

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[1] Blackburn, J. F., *Components Handbook*, MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949, Chap. 8.

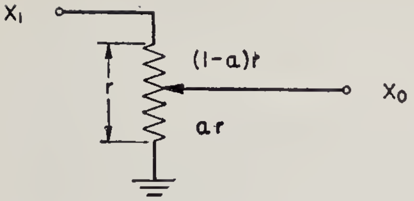

FIG. 4.5*a*. Coefficient-setting potentiometer.

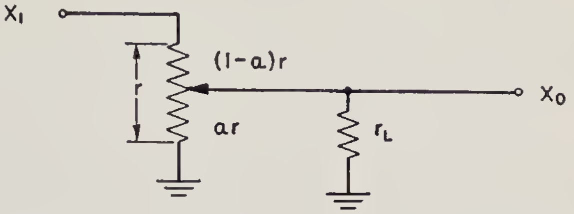

FIG. 4.5*b*. Coefficient-setting potentiometer with load.

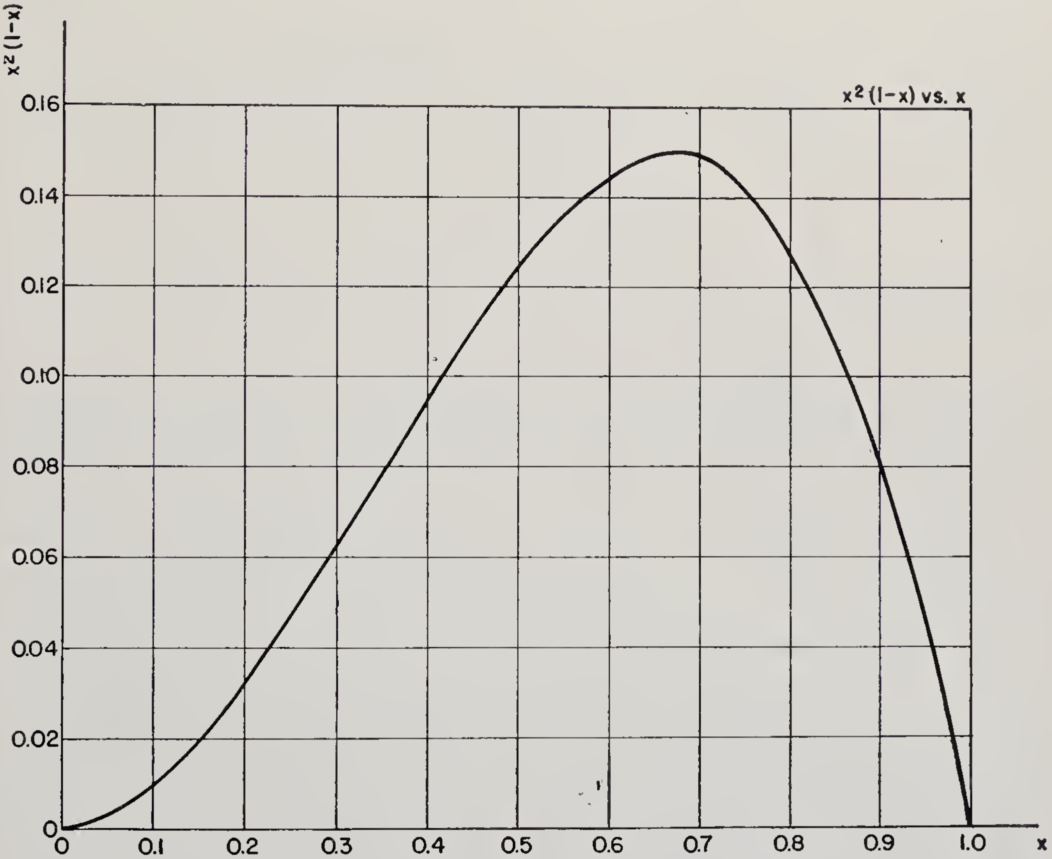

FIG. 4.5*c*. Potentiometer-loading correction chart based on the relation

$$a - \frac{X_o}{X_1} \approx a^2(1 - a)\frac{r}{r_L}$$

Multiply the ordinates corresponding to a desired value of $x = \frac{X_o}{X_1}$ by the value of $\frac{r}{r_L} \leq 0.3$; this quantity must be added to the desired value of the transfer function $\frac{X_o}{X_1}$ to obtain the correct potentiometer setting a. The use of charts of this type is not recommended if setting accuracies better than 0.1 per cent are desired; in this case tables based on the exact formula (4.4) should be used.

Tables or graphs based on the relation (4.3) between a and X_o/X_1 (see Fig. 4.5c) make it relatively easy to introduce the proper *loading corrections* in the potentiometer-dial settings.

The most common load for a computing potentiometer is an input resistor R_i of a summing amplifier, phase inverter, or parallel-feedback integrator. Since the grid of a high-gain operational amplifier is practically at ground potential, one has, in all such cases,

$$r_L = R_i \tag{4.5}$$

Example. It is desired to use a 20,000-ohm potentiometer to multiply an input voltage by 0.4. The only potentiometer load is the 1-megohm input resistor of an integrator. Accordingly, $r_L/r = 50$; reference to Fig. 4.5c shows that the correct dial setting is

$$a = 0.4 + 0.002 = 0.402$$

which may be verified by Eq. (4.3). If the loading correction is neglected, an error of 0.5 per cent results.

Errors due to loading may be serious in the case of motor-driven variable coefficient potentiometers, since it is difficult to apply loading corrections in such cases. Special circuit arrangements for the reduction of loading errors are discussed in Sec. 6.9.

External Calibration. The cost of a number of high-precision linear potentiometers together with their associated dials may account for an appreciable part of the total computer cost. One can sometimes effect a considerable saving by using inexpensive, rugged, wire-wound radio volume controls for coefficient settings. Such potentiometers may be set accurately to any desired coefficient by comparison of their setting with that of a precision-calibrated potentiometer. Figure 4.6 shows the potentiometer to be set connected across the same test-voltage source (direct current or low-frequency alternating current) as the comparison potentiometer. The latter is first adjusted to the desired setting. The low-cost potentiometer is then adjusted until its output voltage equals that of the comparison potentiometer, as indicated by a null on the galvanometer, oscilloscope, etc., used as a null detector. *Since this setting procedure may be performed with the potentiometer load (or an equivalent resistance) connected (see Fig. 4.6), no loading corrections are necessary.* The comparison circuit, which is, incidentally, capable of performing many other useful functions (see, for instance, Secs. 6.7 and 6.10), should be arranged so that it can be instantly connected in turn to any of the potentiometers in the computer proper. This may be accomplished conveniently by means of jacks or push-button switches associated with each potentiometer (see also Sec. 7.7).

Fine Adjustments. If the resolution of a potentiometer (especially one of low cost) is found to be insufficient, a *vernier adjustment* can be provided by means of a second potentiometer or variable resistor of smaller resistance, as shown in Fig. 4.7.

Other Schemes for Multiplying by Constant Coefficients. In some cases, particularly in low-cost computers, it is desirable to avoid the use of the power amplifiers necessary to drive low-impedance computing potentiometers. In these instances, one can perform multiplications by constant coefficients by varying impedances (usually resistors) in the operational amplifiers. Figure 2.2 shows how such multiplications can be realized through the use of combinations of resistors in a summing amplifier. In some computers (see Sec. 8.3), all summing resistors are removable and may be chosen in accordance with the desired coefficients. Figure 4.8 shows operational amplifiers permitting continuous adjustment of coefficients.

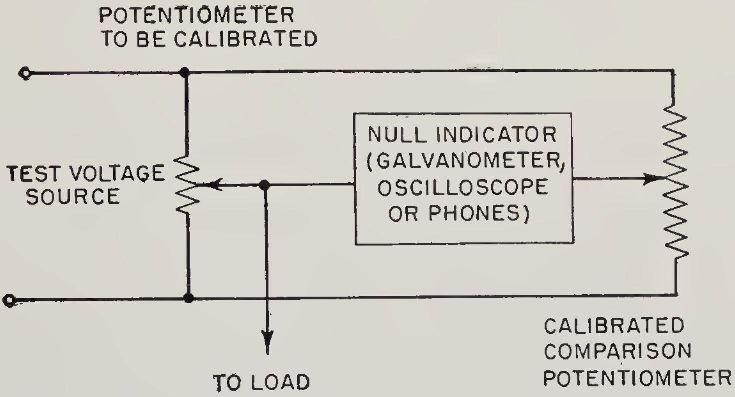
POTENTIOMETER TO BE CALIBRATED
TEST VOLTAGE SOURCE
NULL INDICATOR (GALVANOMETER, OSCILLOSCOPE OR PHONES)
TO LOAD
CALIBRATED COMPARISON POTENTIOMETER

FIG. 4.6. Arrangement for setting or calibrating potentiometers by means of a calibrated comparison potentiometer.

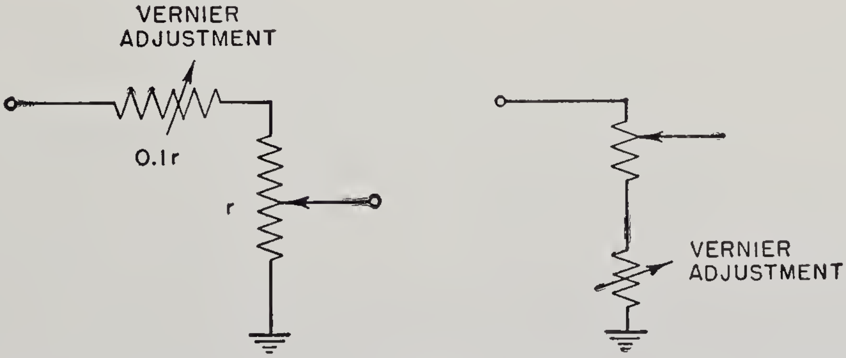

FIG. 4.7. Circuits permitting fine adjustment of potentiometer settings.

Sometimes it is possible to replace potentiometers by special lattice networks having a high input impedance as well as the same output impedance for all coefficient settings.[1] Because of the difficulty of ganging variable resistors accurately, such arrangements are not too suitable if continuously variable coefficients are desired.

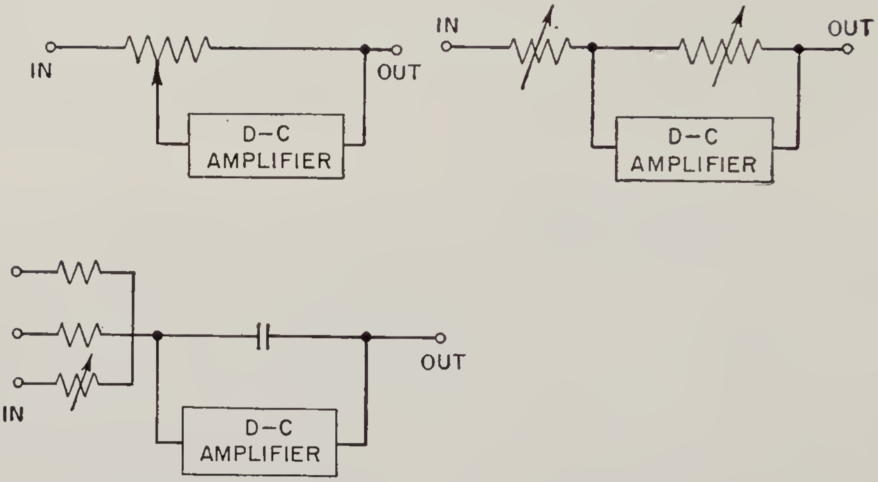

FIG. 4.8. Circuits permitting the introduction of adjustable constant coefficients without the use of low-impedance potentiometers. No loading difficulties are experienced with these circuits, and the use of power amplifiers for driving potentiometers can be eliminated. As a rule, such circuits also have a much lower output impedance (source impedance) than potentiometers.

4.2. ERROR ANALYSIS FOR A CLASS OF OPERATIONAL AMPLIFIERS

Performance Equation and Stability. The parallel-feedback operational amplifier circuit shown in Fig. 4.9*a* has already been introduced in Sec. 1.5. This eminently useful feedback scheme encompasses the phase inverters, summing amplifiers, integrators, and differentiators of Fig. 1.5, as well as the operational amplifier circuit of Fig. 1.14*a* as special cases; the examples of Secs. 3.4, 3.9, and Table 1 indicate a few practical applications.

The performance equation was found in Sec. 1.5 to be

$$X_o = -\left[1 - \frac{1}{1 - A(P)\beta(P)}\right] Z_o(P)\left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right] \qquad (4.6a)$$

with

$$\beta(P) = \left[1 + \frac{Z_o(P)}{Z_1(P)} + \frac{Z_o(P)}{Z_2(P)} + \cdots\right]^{-1} \qquad (4.6b)$$

The operator $A(P)$ describes the *effective forward gain of the d-c amplifier* with the feedback networks and load connected [see also Eq. (4.10)]; for small values of ω, $A(j\omega)$ will be real and negative. If the d-c amplifier is fed from zero-impedance sources, the operator $\beta(P)$ may be called the

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[1] Soroka, W. W., *Analog Methods in Computation and Simulation*, McGraw-Hill, New York, 1954, Sec. 2.6.

feedback ratio of the operational amplifier, and $A(P)\beta(P)$ is its *loop gain*. The feedback circuit is *stable* if, and only if, all roots of the characteristic equation $1 - A(\lambda)\beta(\lambda)$ have negative real parts (see also Sec. 5.4).[1]

Errors Involved in the Use of the Simplified Performance Equation. In practically all applications, the performance of the operational-amplifier circuit of Fig. 4.9*a* is described by the convenient approximation

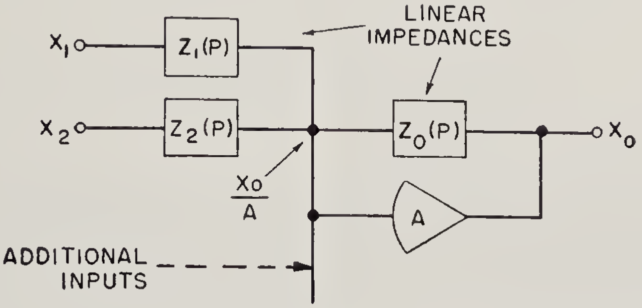

FIG. 4.9*a*. This operational-amplifier circuit encompasses the phase inverters, summing amplifiers, integrators, and differentiators of Fig. 1.5, as well as the operational amplifier circuits of Fig. 1.14*a* and Table 1, as special cases.

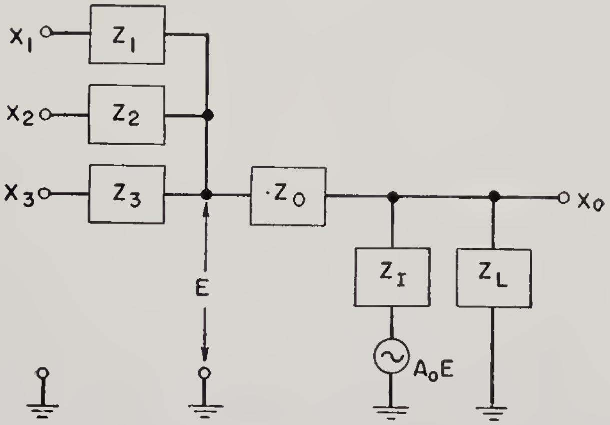

FIG. 4.9*b*. Equivalent circuit of the operational-amplifier circuit of Fig. 4.9*a* connected to a load.

$$X_o = -Z_o(P)\left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right] \tag{4.6c}$$

which permits "calibration" of each operational amplifier in terms of easily measurable impedance values. To determine the error involved in the use of the simplified performance equation (4.6*c*), note that the correct output voltage (4.6*a*) appears as a sum of the expression (4.6*c*) and an *error term*

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[1] See, for instance, H. W. Bode, *Network Analysis and Feedback Amplifiers*, Van Nostrand, Princeton, N.J., 1945.

$$e(\tau) = \frac{1}{1 - A(P)\beta(P)} Z_o(P) \left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots \right] \tag{4.7}$$

Equation (4.7) *yields the error* $e(\tau)$ *for any given set of input voltages* X_1, X_2, . . . *as the solution of a differential equation.*

Amplitude and Phase Errors for Steady-state Sinusoidal Voltages. If X_1, X_2, . . . , and X_o are all sinusoidal voltages (or Fourier components) of circular frequency ω, these voltages may be represented by the corresponding complex voltages[1]

$$\tilde{X}_1 = \bar{X}_1 e^{j\omega\tau},\ \tilde{X}_2 = \bar{X}_2 e^{j\omega\tau},\ \ldots, \qquad \tilde{X}_o = \bar{X}_o e^{j\omega\tau}$$

Substitution of these functions for X_1, X_2, . . . , X_o in the performance equation (4.6*a*) yields

$$\bar{X}_o = -\left[1 - \frac{1}{1 - A(j\omega)\beta(j\omega)}\right] Z_o(j\omega) \left[\frac{\bar{X}_1}{Z_1(j\omega)} + \frac{\bar{X}_2}{Z_2(j\omega)} + \cdots\right] \tag{4.8}$$

The term proportional to $1/[1 - A(j\omega)\beta(j\omega)]$ corresponds to the error term (4.7). The *percentage amplitude error* and the *phase error* involved in the use of the simplified performance equation (4.6*c*) are, respectively,

$$|\bar{\epsilon}_\omega| = \frac{100}{|1 - A(j\omega)\beta(j\omega)|} \quad \text{and} \quad \delta_\omega = -\operatorname{Arg}\left[1 - \frac{1}{A(j\omega)\beta(j\omega)}\right] \tag{4.9}$$

Although the relations (4.9) apply only to pure sinusoidal voltages, they help to indicate the range of working frequencies for which the simplified performance equation is useful. Note that

$$\frac{1}{100}|\bar{\epsilon}_\omega| = \frac{1}{|A(j\omega)\beta(j\omega)|} \geq |\sin \delta_\omega| \quad [\approx |\delta_\omega| \quad \text{if} \quad |A(j\omega)\beta(j\omega)| \gg 1]$$

In a typical "slow" d-c analog computer, the absolute value $|A(j\omega)|$ of the d-c amplifier gain exceeds 50,000 up to 100 cps, so that $|\bar{\epsilon}_\omega|$ will be less than 0.2 per cent up to 100 cps if $|\beta(j\omega)| > 0.01$. The situation is much less favorable in the case of the relatively low-gain amplifiers employed in repetitive computers (Sec. 8.5). The reader may find it interesting to estimate the quantities (4.9) for some of the operational-amplifier circuits of Table 1, using d-c amplifier gains of 50,000 and 1,500.

Example. **Calibration Accuracy of a Summing Amplifier.** For the summing-amplifier circuit of Fig. 1.6,

$$\beta(j\omega) = \left(1 + \sum_{i=1}^{7} \frac{R_o}{R_i}\right)^{-1} = \frac{1}{23}$$

---

[1] The representation of amplitudes and phases of sinusoidal voltages and currents in terms of complex numbers is described, *e.g.*, in W. R. Smythe, *Static and Dynamic Electricity*, McGraw-Hill, New York, 1939.

If the relation (1.20) is used to calibrate the summing amplifier, an accuracy of 0.1 per cent requires a d-c amplifier gain A of at least 23,000 throughout the range of working frequencies.

Effect of the Load. Output and Input Impedances. If a load of impedance Z_L is connected to the operational amplifier (Fig. 4.9*b*), the effective d-c amplifier forward gain is

$$A(j\omega) = \frac{A_0(j\omega) + \dfrac{Z_I}{Z_o}}{1 + \dfrac{Z_I}{Z_o} + \dfrac{Z_I}{Z_L}} \tag{4.10}$$

where $A_0(j\omega)$ and Z_I are, respectively, the open-circuit gain and the internal impedance (output impedance, source impedance) of the d-c amplifier.

According to Thévenin's theorem,[1] the operational amplifier looks to the load like a voltage source in series with an impedance Z_S:

$$X_o = \frac{Z_L(j\omega)}{Z_S(j\omega) + Z_L(j\omega)} X_o'$$

The source voltage X_o' is the operational-amplifier output voltage *without load* given by Eqs. (4.6*a*) and (4.10) for $Z_L = \infty$, and the *internal impedance* (*output impedance, source impedance*) Z_S of the operational amplifier is

$$Z_S = \frac{Z_o Z_I}{[1 - \beta(j\omega)]Z_I + [1 - A_0(j\omega)\beta(j\omega)]Z_o}$$
$$\left[\approx -\frac{Z_I}{A_0(j\omega)\beta(j\omega)} \text{ if } |A_0(j\omega)\beta(j\omega)| \gg 1\right] \tag{4.11}$$

In most applications, Z_I varies between 1,000 and 50,000 ohms, and $|A_0(j\omega)\beta(j\omega)|$ is so large that $|Z_S/Z_L|$, and thus the effect of the load, is negligible.

The *input impedance* seen by the *i*th input voltage source of the operational amplifier of Fig. 4.9*a* is practically equal to the impedance Z_i whenever the absolute value of the loop gain $A(j\omega)\beta(j\omega)$ is large, since the feedback will tend to make the summing-point voltage X_o/A very small.

Effects of Gain Changes and Distortion. The inverse feedback inherent in the operational-amplifier circuit helps to reduce the effects of gain changes in the d-c amplifier on the output voltage X_o; the range of permissible gain changes may be investigated with the aid of Eqs. (4.7) and (4.9).

For pure sinusoidal voltages, the fractional effect of a steady-state gain change $dA(j\omega)$ on the output amplitude is reduced in the ratio $1 : |1 - A(j\omega)\beta(j\omega)|$. A similar reduction applies to the fractional amplitude of each sinusoidal distortion and intermodulation component originating in the d-c amplifier, but the effect of feedback on the *total* distortion error depends on the vector sum of the distortion components and is not so easily computed.[1]

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[1] Smythe, *op. cit.*

4.3. INTEGRATOR PERFORMANCE

The physical basis of most electrical integrators is the basic property of capacitors illustrated in Fig. 4.10. Neglecting the effect of capacitor leakage, the current i through the capacitor is always proportional to the time derivative of the voltage across the capacitor:

$$i = CPE$$

In this chapter, the basic integrating network of Fig. 1.5*e* will be discussed in more detail. In addition, other integrating devices, which constitute attempts to improve on the original circuit, will be presented.

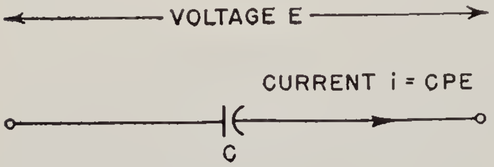

FIG. 4.10. Relation between voltage and current in a capacitive circuit.

Simple RC Integrating Network. In the simple RC integrating network shown in Fig. 4.11, an input voltage X_1 will begin to charge the capacitor C through the resistor R in a manner approximating true integration. The output voltage X_o built up on C during this process will, however, tend to oppose the input charging voltage and may thus make accurate integration impossible. The situation is further complicated through the existence of the leakage resistance R_L which is due to capacitor leakage in addition to any load which may be connected to the integrator output terminals.

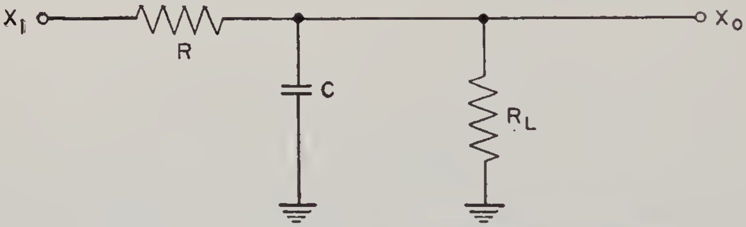

FIG. 4.11. Simple integrating network with load.

The nodal equation of this network leads to the *transfer function* X_o/X_1 of the integrating network,

$$\frac{X_o}{X_1} = \frac{R_L}{R + R_L} \frac{1}{\frac{R_L}{R + R_L} RCP + 1} \tag{4.12}$$

---

[1] Department of Electrical Engineering Staff, Massachusetts Institute of Technology, *Applied Electronics*, 1st ed., Wiley, New York, 1943, p. 528.

If the leakage resistance R_L is infinitely large, Eq. (4.12) reduces to the simplified equation (1.23) given in Chap. 1.

General Integrator Performance Analysis: Time Constant and Attenuation. It should be mentioned at this point that *the performance of most d-c integrators may be described in terms of transfer functions of the form*

$$\frac{X_o}{X_1} = \frac{k}{bP + 1} \tag{4.13}$$

of which Eq. (4.12) is a special case. The coefficient b of P in the denominator is called the *time constant* of the integrator and will be seen to constitute a measure of the integrator accuracy. The parameter k/b corresponds to the *gain* (or rate gain) of the integrator. The practical meaning of integrator time constant and gain will be made clearer by the following analysis.

Output Error and Rate Error of an Integrator. The *output error* of an integrator will be the difference between the integrator output voltage X_o and the output voltage $(k/bP)X_1$ of a "perfect" integrator of equal gain. For an integrator described by the performance equation (4.13), the output error $e(\tau)$ will be

$$\begin{aligned} e(\tau) &= X_o - \frac{k}{bP} X_1 = \left[\frac{k}{bP + 1} - \frac{k}{bP}\right] X_1 \\ &= -\frac{k}{bP(bP + 1)} X_1 = -\frac{1}{bP} X_o \end{aligned} \tag{4.14}$$

for any value of the time τ. It must be remembered that the "linear" output error (4.14) will in general have to be added to other errors due to calibration changes, distortion, and drift.

It is clear that the derivative of the output voltage of any d-c integrator should be as nearly as possible proportional to the input voltage. In fact, for a perfect integrator one would have

$$PX_o = \frac{k}{b} X_1 \tag{4.15}$$

The amount $Pe(\tau)$ by which the time derivative PX_o of the output voltage X_o varies from the value given by Eq. (4.15) shall be called the *rate error* of the integrator under consideration and constitutes a useful measure of the integrator accuracy. As a matter of interest, an integrator whose rate error is always zero will have only errors which can be compensated for by simply changing the initial conditions of the problem.

For an integrator having the performance equation (4.13), the derivative of the output voltage X_o will be

$$PX_o = \frac{kP}{bP + 1} X_1 \tag{4.16}$$

Assuming that the entire error in the derivative PX_o is due to the difference between Eqs. (4.15) and (4.16), the rate error Pe of the integrator is then

$$Pe(\tau) = PX_o - \frac{k}{b}X_1 = -\frac{1}{b}\frac{1}{bP+1}X_1 = -\frac{1}{b}X_o \qquad (4.17)$$

Note again that the effects of drift, distortion, and calibration changes have not been considered here. It is seen that both $e(\tau)$ and $Pe(\tau)$ will decrease as the integrator time constant b is increased.

Step-function Response of a D-c Integrator: Effect of Computing Time on Error. If an input voltage step function $cu(\tau)$* (*e.g.*, from a constant-voltage source suddenly switched into the circuit) is applied to an integrator described by a transfer function of the form (4.13), the circuit response will be

$$X_o(\tau) = kc\,(1 - e^{-\frac{\tau}{b}}) = c\left[\left(\frac{k\tau}{b}\right) - k\frac{\tau^2}{2b^2} \pm \cdots)\right] \quad (\tau \geq 0) \qquad (4.18)$$

This is easily verified by substitution into the differential equation (4.13). It is seen that the output voltage X_o begins to change linearly with time at a rate equal to $(k/b)c$. This would correspond to correct integration of the input step function. With increasing computing time τ, however, the output voltage X_o no longer changes linearly but tends to approach a limiting value[1] kc exponentially, as shown by Eq. (4.18) and Fig. 4.12. (As in the case of the simple RC integrator, this is usually due to a "counter-emf" developed as a result of the charge on an integrating capacitor.)

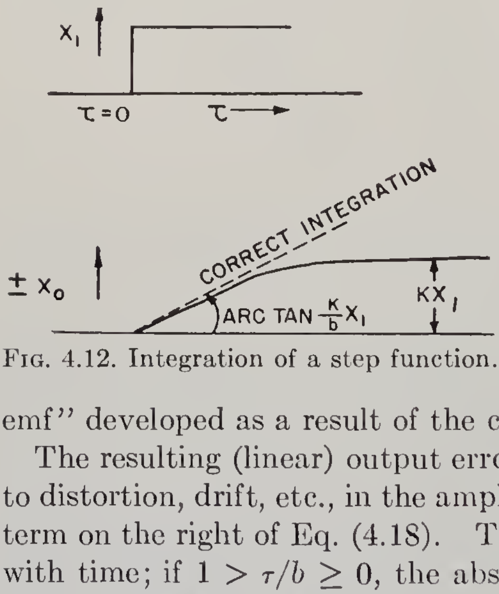

FIG. 4.12. Integration of a step function.

The resulting (linear) output error e (which is in addition to errors due to distortion, drift, etc., in the amplifiers involved) is given by the second term on the right of Eq. (4.18). The integration error is seen to increase with time; if $1 > \tau/b \geq 0$, the absolute value of the error e will be less than, or equal to, the first term of the error series, or

$$|e| \leq \frac{\tau^2}{2b^2}kc \qquad (4.19)$$

* The definition of the unit step function $u(\tau)$ is given by

$$u(\tau) = 0 \text{ for } \tau < 0 \qquad u(\tau) = 1 \text{ for } \tau \geq 0$$

[1] In actual practice, this limiting voltage kc may be higher than the overload or saturation voltage of the computing elements in question, so that the output voltage X_o cannot reach the limiting value kc.

For the *percentage error* ϵ, it follows that

$$\epsilon \leq 50 \frac{\tau}{b} \tag{4.20}$$

Hence, *if the resultant output voltage is to be within* ϵ *per cent of the true integral* $(k/b)c\tau$, *the computing time* τ *must be limited so that*

$$\tau \leq \frac{b\epsilon}{50} \tag{4.21}$$

The *rate error Pe* for step-function input is obtained by substituting Eq. (4.18) into (4.17):

$$-Pe(\tau) = \frac{k}{b} c(1 - e^{-\frac{\tau}{b}}) \tag{4.22}$$

It is seen that

$$|Pe(\tau)| \leq \frac{k}{b^2} c\tau \qquad \left(1 > \frac{\tau}{b} \geq 0\right) \tag{4.23}$$

If the integrator comprises only linear networks and amplifiers, *the relations* (4.20) *and* (4.23) *are not restricted to step-function input voltages*, since the operation of a linear network on *any* input voltage may be considered in terms of a superposition of step-function voltages, with all steps but the first one being integrated for time shorter than τ.

It should be remembered that the relations (4.19) to (4.23) refer only to errors due to d-c integrators as such. The effects of such errors on the over-all accuracy of a differential analyzer must be discussed separately.

Steady-state Sinusoidal Response. It is often important to know the response of an integrator to steady-state sinusoidal voltages, in particular since, again, any input voltage may be represented as a superposition of such voltages. In the steady state, both X_1 and X_o will be sinusoidal voltages represented by[1]

$$\tilde{X}_1 = \bar{X}_1 e^{j\omega\tau} \qquad \tilde{X}_o = \bar{X}_o e^{j\omega\tau} \tag{4.24}$$

where ω is the circular frequency. Substitution of (4.24) into the differential equation (4.13) shows that

$$\bar{X}_o = \frac{k}{bj\omega + 1} \bar{X}_1 = \frac{k\bar{X}_1}{bj\omega} - \frac{1}{bj\omega} \frac{k}{bj\omega + 1} \bar{X}_1 \tag{4.25}$$

where the first term on the right corresponds to true integration and the second term measures the absolute value and phase of the (complex) integration error $\bar{e}_\omega$ for sinusoidal input voltages of circular frequency ω. The *absolute value* of the error is

---

[1] See footnote on p. 164.

$$|\bar{e}_\omega| = \left|\frac{\bar{X}_1}{b\omega} \frac{k}{\sqrt{1 + (b\omega)^2}}\right| \leq \left|\frac{\bar{X}_1 k}{b^2\omega^2}\right| \tag{4.26}$$

whereas the *phase error* [the amount by which the angle $\sphericalangle\, (\bar{X}_o,\bar{X}_1)$ differs from 90 deg] δ_ω is given by

$$\tan \delta_\omega = \frac{1}{b\omega} \tag{4.27}$$

Equations (4.25) to (4.27) further illustrate the importance of the parameters b and k, since any input voltage may be thought of as made up of sinusoidal components. It is seen from Eqs. (4.26) and (4.27) that high-frequency components of any input voltage will be integrated more accurately than low-frequency components.

Time Constant and Gain of the Simple RC Integrator. According to (4.12), the time constant b of the simple RC integrating network shown in Fig. 4.11 is

$$b = \frac{R_L}{R + R_L} RC \tag{4.28}$$

and the gain k/b is

$$\frac{k}{b} = \frac{1}{RC} \tag{4.29}$$

If the effect of leakage and loading is negligible ($R_L \approx \infty$), Eqs. (4.28) and (4.29) reduce to

$$b = RC \qquad \frac{k}{b} = \frac{1}{RC} \tag{4.30}$$

It may be concluded that

1. In the simple RC network, large values of the time constant b desirable for good integration can be obtained only at the expense of large values of R and C and of low gain.
2. The effect of connecting a load to a simple RC integrator is to decrease the time constant b and thus the accuracy of integration.

The RC network has, on the other hand, the obvious advantages of being simple, inexpensive, and free of drift, since it requires no vacuum tubes. RC integrators will often be satisfactory in cases involving relatively short computing times or for input voltages containing only relatively high-frequency components.

4.4. SOME ELECTRONIC INTEGRATORS

Introduction. The Perfect Integrator. The electronic integrators to be discussed in this and the following sections constitute attempts to improve the performance of the basic *RC* network, essentially by reducing the effects of the "counter-emf" which tends to oppose the charging current of an integrating capacitor. Such electronic circuits will, then, permit higher values of the integrator time constant without affecting the integrator gain detrimentally. The integrator circuits discussed in Sec. 4.4 may be of interest for special-purpose computers; the parallel-feedback integrator described in Sec. 1.4 and in more detail in Sec. 4.5 is almost universally used in general-purpose electronic differential analyzers.

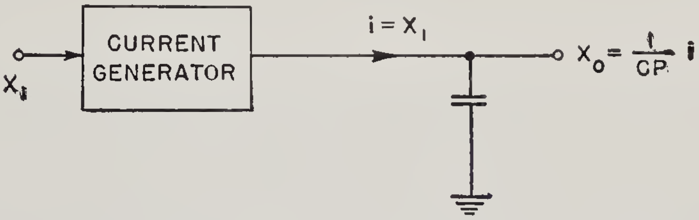

FIG. 4.13. Equivalent circuit of a perfect integrator.

McCool[1] has pointed out that the problem of integrator design may be reduced to that of developing a perfect current generator (with zero shunt admittance) whose current output is accurately proportional to the integrator input voltage X_1. The current integrator is then used to charge an integrating capacitor (Fig. 4.13); according to (4.11), the voltage on the capacitor will be the time integral of the current and thus of the input voltage X_1. The reader will find it instructive to interpret the electronic integrators in this chapter as attempts to realize such an arrangement.

Looking at the question of a "perfect" integrator[2] from the most general viewpoint possible, it can be said that the physical realization of a true integrator with a transfer function

$$\frac{X_o}{X_1} = \frac{k}{b}\frac{1}{P} \tag{4.32}$$

---

[1] McCool, W. A., An Improved Analog Computing Circuit, *NRL Report* P-3423 (unclassified), Feb. 23, 1949.

[2] It should be mentioned in passing that, unlike the true integrator, the "perfect" differentiator is not physically realizable. The ideal differentiator would have a transfer function

$$\frac{X_o}{X_1} = kP \tag{4.31}$$

This would imply infinite gain at infinite frequency, a condition impossible to attain in practice. Reasonable approximations to (4.31) may be obtained for limited frequency ranges. Two differentiating circuits are shown in Fig. 1.5.

is theoretically possible and may be approached in practice by increasing both b and k in transfer functions such as (4.13). Since the ideal transfer function (4.32) implies the use of a device having infinite gain at zero frequency, a true integrator must involve an "active" circuit with critical regeneration; no passive network can satisfy this requirement. Of the electronic integrators discussed in this chapter, the bootstrap integrator (see page 173) and the regenerative feedback integrator (see Sec. 4.7) do make use of regenerative amplifiers and are theoretically capable of integrating perfectly.

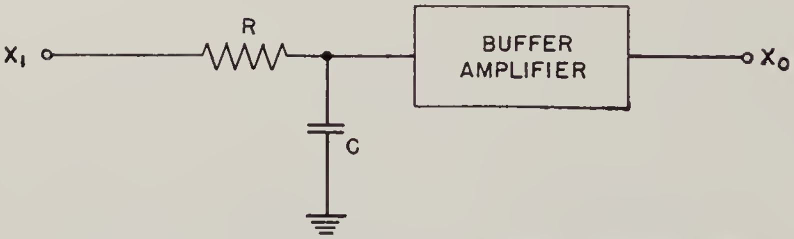

FIG. 4.14*a*. Simple integrating network with buffer amplifier.

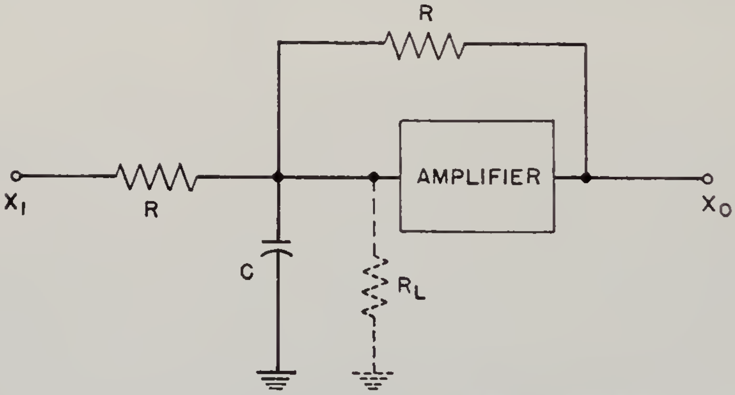

FIG. 4.14*b*. Block diagram of bootstrap integrator.

In general, practical electronic integrating devices are capable of *approximating* the ideal transfer function (4.32), throughout a useful range of frequencies, with or without the use of regenerative amplifiers. In each case, a careful *accuracy analysis* will be necessary to determine the errors involved in the approximation. Such an accuracy analysis is presented in Sec. 4.5 for a frequently used type of integrator.

The question of setting up *initial conditions* in integrators will be taken up in Sec. 7.1.

RC Integrating Network with Buffer Amplifier. The disadvantages which result from loading an *RC* integrating network may be largely overcome through the use of a buffer amplifier, as indicated in Fig. 4.14*a*. The buffer amplifier (which might be a cathode follower or have a cathode-follower input stage) presents a high impedance to the integrating network and isolates it from the load. The amplifier may also be used to increase the integrator gain, but high values of R and C may still be needed to obtain a satisfactory time constant RC. The use of a buffer amplifier cannot, of course, overcome the loading effect of the capacitor leakage resistance. Finally, unless special precautions are taken (see Chap. 5), the grid current of the buffer amplifier may present a further problem by acting as a spurious input source which tends to charge the capacitor and thus to cause drift in the integrator output.

Bootstrap Integrator. Another type of integrator, shown in Fig. 4.14*b*, is similar in principle to the bootstrap integrator, originally used as a pulse integrator to provide linear sweep for fire-control radar. In this integrator, a feedback circuit is used to neutralize the effects of the "counter-emf" which arises across the capacitor as the input voltage attempts to charge it through the resistance R. Specifically, this opposing voltage is amplified by a d-c amplifier which must have a positive forward gain A. The amplifier output voltage is added to the integrator input voltage by means of the summing network shown. The input voltage can thus be "boosted" by an amount sufficient to overcome the opposing "emf" of the capacitor. If the amplifier gain is correctly chosen, the resulting voltage across the capacitor C and thus also the amplifier output voltage X_o will be proportional to the time integral of the input voltage X_1.

It should be noted that this integrator does *not* neutralize the voltage opposing the charging voltage by inverse feedback but rather supplies just enough extra voltage "boost" to permit the capacitor to charge in the manner shown in Fig. 4.13.

If the leakage resistance R_L is neglected, the transfer function of the bootstrap integrator in Fig. 4.14*b* is found (from the nodal equation) to be

$$\frac{X_o}{X_1} = \frac{A}{RCP + (2 - A)} = \frac{A}{(2 - A)\left(\dfrac{RC}{2 - A}P + 1\right)} \tag{4.33}$$

When the leakage resistance is taken into consideration, Eq. (4.33) is replaced by

$$\frac{X_o}{X_1} = \frac{A}{RCP + \left(\dfrac{R}{R_L} + 2 - A\right)}$$

$$= \frac{A}{\left(\dfrac{R}{R_L} + 2 - A\right)\left(\dfrac{RC}{\dfrac{R}{R_L} + 2 - A}P + 1\right)} \tag{4.34}$$

so that

$$b = \frac{RC}{\dfrac{R}{R_L} + 2 - A} \qquad \frac{k}{b} = \frac{A}{RC} \tag{4.35}$$

True integration, or

$$\frac{X_o}{X_1} = \frac{A}{RCP} \tag{4.36}$$

will be obtained if the amplifier gain A and the resistances R and r are adjusted so that

$$\frac{R}{R_L} + 2 - A = 0 \tag{4.37}$$

Little is known about practical applications of this type of integrator at the present time. While the quality of the integration will be sensitive to the gain and resistance adjustments, the use of high-quality resistors and of a stable feedback amplifier should result in good integration. Negative feedback around the amplifier will permit one to obtain a stable, adjustable gain A and will also reduce the voltage appearing on the input grid. Integrators of this type can be used to drive reasonable loads (5,000 to 10,000 ohms) without affecting the quality of the integration.

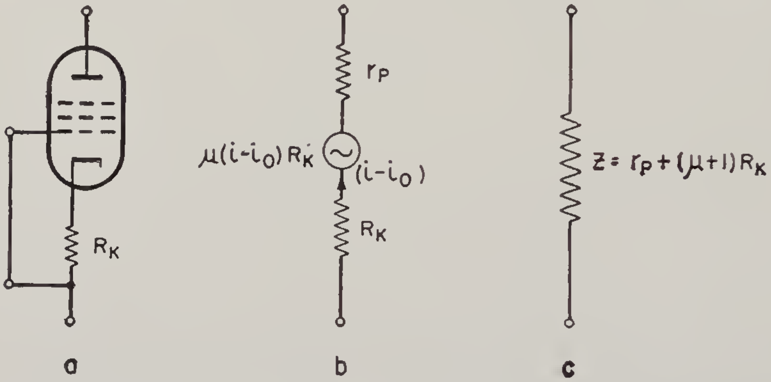

FIG. 4.15. Equivalent circuits illustrating the dynamic plate resistance of a vacuum tube. Because of the current feedback, the circuit tends to oppose changes $(i - i_0)$ in the plate current.

RC Integrators and Other Operational Amplifiers Using "Amplified Impedance" (Series Feedback). Another class of electronic integrators attempts to increase the integrating resistance R of an RC network by replacing the resistor R in Fig. 4.11 with the "dynamic plate resistance" of a vacuum tube. Figure 4.15*a* shows a vacuum-tube circuit with current feedback introduced by means of the cathode resistor R_K. In this circuit, the tube acts like a voltage source tending to oppose any current changes Δi in the cathode resistor, as indicated by the equivalent circuit[1] shown in Fig. 4.15*b*, where μ and r_p are the amplification factor and plate resistance of the tube, respectively. As a result, the tube looks like a resistance ("dynamic plate resistance")

$$Z = r_p + (1 + \mu)R_K \tag{4.38}$$

---

[1] For a discussion of vacuum-tube equivalent circuits, see D. G. Fink, *Engineering Electronics*, McGraw-Hill, New York, 1938.

to any reasonably small voltage change applied at the terminals shown (Fig. 4.15*c*), while the steady-state d-c voltage across the tube and its cathode resistor still has a reasonable value. It may be said that the tube has "amplified" the resistance R_K of the cathode resistor. Such resistance amplification is particularly effective with pentodes because of the high effective value of μ. The application of this principle requires considerable ingenuity, since the tube supply voltages must not be permitted to interfere with the operation of the integrating circuits. Figure 4.16 shows a number of possible integrating circuits; such circuits are severely affected by leakage or load resistances across the integrating capacitor and may require buffer amplifiers.

The principle of impedance amplification has also been applied to the design of operational amplifiers (Sec. 1.5) other than integrators.[1] The circuit of Fig. 4.17 is a generalization of Fig. 4.16*a* and has the transfer function[2]

$$\frac{X_o}{X_1} = -\frac{Z_o(P)}{Z_1(P)} \frac{g_m}{g_m + \left[1 + \dfrac{Z_o(P)}{Z_1(P)}\right]\left(\dfrac{1}{r} + \dfrac{1}{r_p}\right) + \dfrac{1}{Z_1(P)}}$$

$$\left\{\approx -\frac{Z_o(P)}{Z_1(P)} \text{ for } \left|\frac{1}{g_m}\left[\left(1 + \frac{Z_o}{Z_1}\right)\left(\frac{1}{r} + \frac{1}{r_p}\right) + \frac{1}{Z_1}\right]\right| \ll 1\right\}$$

where g_m and r_p are, respectively, the transconductance and plate resistance of the tube used. The relatively high output impedances associated with series feedback tend to make such circuits less useful for conventional applications than the parallel-feedback-type operational amplifiers described in Secs. 1.3 to 1.5.

4.5. THE PARALLEL-FEEDBACK INTEGRATOR: ERROR ANALYSIS OF A D-C INTEGRATOR

Introduction. The feedback integrator shown in Fig. 4.18 and briefly described in Sec. 1.4 is the one most generally used in computer applications because of its ability to perform accurate integration with little attenuation and with reasonable values of resistance and capacitance. Again, the quality of integration is not badly impaired by matching the output to a potentiometer of reasonably low impedance (5,000 to 100,000 ohms).

The integrator shown in Fig. 4.18 consists of an integrating feedback network comprising the resistance R and the capacitance C in conjunction with a d-c amplifier of gain A; since the leakage resistance R_L of the capacitor constitutes an additional feedback path, this must also be taken

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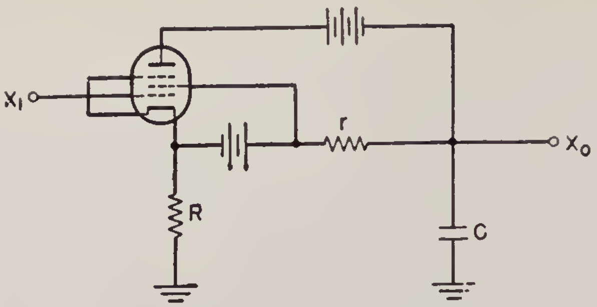
[1] Philbrick, Designing Industrial Controllers by Analog, *Electronics*, June, 1948.

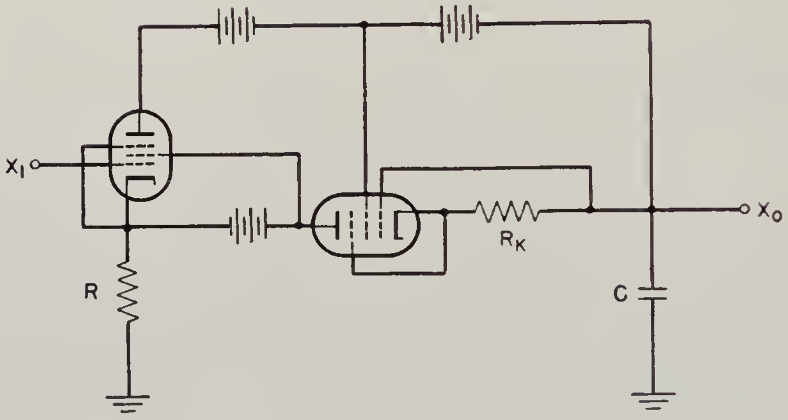
[2] McCool, *op. cit.*



$$b = \frac{\mu + 1 + r_p\left(\frac{1}{R} + \frac{1}{r}\right)}{1 + \frac{r_p}{r}} RC$$

$$k = -\frac{\mu}{1 + \frac{r_p}{r}}$$

(a)



$$b = \frac{\mu + 1 + r_p\left(\frac{1}{R} + \frac{1}{Z}\right)}{1 + \frac{r_p}{Z}} RC$$

$$k = -\frac{\mu}{1 + \frac{r_p}{Z}}$$

(b)

FIG. 4.16. Electronic integrator circuits[1] based on resistance amplification. Both tubes of each two-tube circuit are identical, and $Z = r_p + (1 + \mu)R_k$. If a leakage or load resistance R_L shunts the integrating capacitance C, the gain k/b of each circuit is unchanged, but the time constant b is reduced to the value $\dfrac{b}{1 + \dfrac{b}{R_L C}}$.

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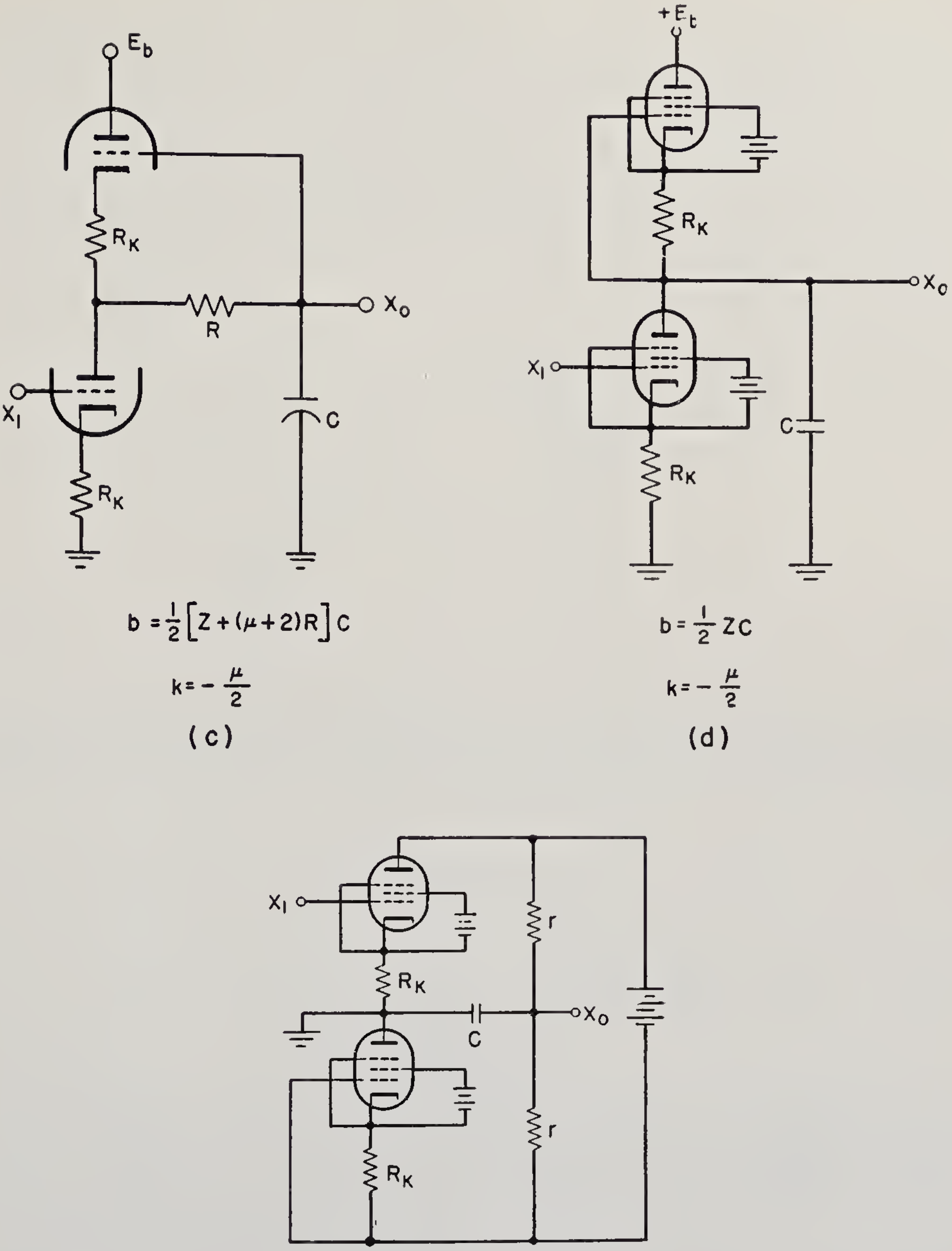
[1] Circuit (*a*) seems to have been described first by G. A. Philbrick, Designing Industrial Controllers by Analog, *Electronics*, June, 1948. Circuit (*b*) is from McCool, *op. cit.*



$b = \frac{1}{2}\left[Z + (\mu + 2)R\right]C$

$k = -\frac{\mu}{2}$

(c)

$b = \frac{1}{2}ZC$

$k = -\frac{\mu}{2}$

(d)

$b = \frac{1}{2}ZC$

$k = -\frac{\mu}{2}$

(e)

FIG. 4.16. (Continued)

into consideration. An input voltage X_1 will attempt to charge the capacitor plates connected to the amplifier grid. The output voltage X_o of the negative-gain d-c amplifier will, however, tend to charge the other capacitor plates so as to neutralize the charge on the input side. The net effect is to keep the grid voltage (which would tend to oppose the input voltage X_1) quite small while the output side of the capacitor is charged to a voltage X_o proportional to the negative time integral of the input voltage X_1.

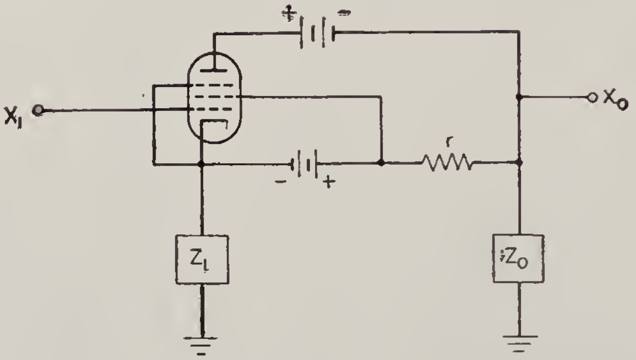


FIG. 4.17. Block diagram of an operational amplifier based on series feedback (G. A. Philbrick, *op. cit.*). This circuit has been improved by McCool (*op. cit.*), who replaced the screen resistor *r* by the dynamic plate resistance of a second pentode (see Fig. 4.16*b*).

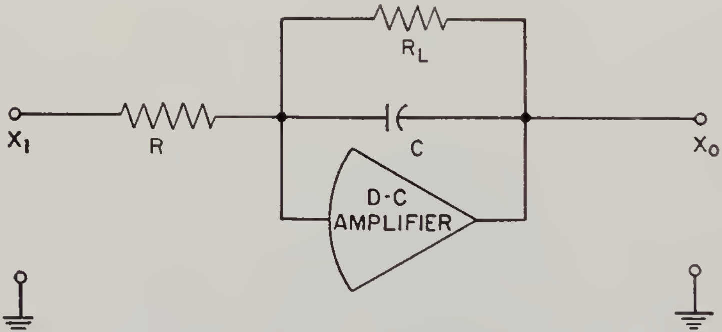


FIG. 4.18. Block diagram of a parallel-feedback integrator.

The purpose of the following analysis is to find integrator designs such that the output voltage X_o of the integrator will be as closely as possible proportional to the time integral X_1/P of the input voltage X_1. Because of the usefulness of the parallel-feedback integrator, the latter has been chosen here as the subject of a rather detailed error analysis. The parallel-feedback integrator may be regarded as a special case of the general operational-amplifier circuit of Fig. 4.9, with

$$Z_1(P) = R \qquad Z_o(P) = \frac{R_L}{R_LCP + 1} \qquad \beta(P) = R\frac{1 + R_LCP}{R + R_L + RR_LCP}$$

Substitution of these expressions into the appropriate relations of Sec. 4.2 will yield the integrator *output impedance* as well as estimates of the *effects of distortion.* On the other hand, the error analysis of Sec. 4.2 does not permit a simple demonstration of the *effect of capacitor leakage on integrator accuracy;* for this reason, the time constant b and the rate gain k/b of the parallel-feedback integrator will be derived directly from its performance equation, so that the analysis of Sec. 4.3 applies.

Performance Equation. In Fig. 4.18, the current flowing through R must be equal to the sum of the currents flowing through R_L and C if grid current is neglected for the time being. If $A(j\omega) \approx A(0) = A$ throughout the range of working frequencies, the nodal equation expressing Kirchhoff's first law is, then,

$$\left(\frac{X_o}{A} - X_o\right)\left(CP + \frac{1}{R_L}\right) + \left(\frac{X_o}{A} - X_1\right)\frac{1}{R} = 0$$

so that

$$X_o = \frac{A}{(1 - A)RCP + (1 - A)\dfrac{R}{R_L} + 1} X_1 \tag{4.39}$$

or

$$\frac{X_o}{X_1} = \frac{A}{(1 - A)\dfrac{R}{R_L} + 1}\,\frac{1}{\dfrac{(1 - A)RC}{(1 - A)\dfrac{R}{R_L} + 1} P + 1} \tag{4.40}$$

The time constant b and the gain k/b are, respectively,

$$b = \frac{(1 - A)\,RC}{(1 - A)\dfrac{R}{R_L} + 1} \qquad \frac{k}{b} = \frac{A}{(1 - A)RC} \tag{4.41}$$

which makes it possible to obtain substantially better results than with a simple RC network. The expressions (4.41) may be substituted into equations like (4.14), (4.17), (4.20), and (4.23) to give further information on the integrator performance.

It may be seen that true integration, or

$$\frac{X_o}{X_1} = \frac{A}{(1 - A)RCP} \qquad \left(\approx -\frac{1}{RCP} \text{ for } |A| \gg 1\right) \tag{4.42}$$

is approximated most closely for small values of the quantity

$$(1 - A)\frac{R}{R_L} + 1$$

In practice, this can be achieved, for instance, by making the gain A large (from 10^3 to 10^9 at low frequencies) and by using capacitors with as high a leakage resistance R_L as possible. Dielectric hysteresis acts somewhat like a leakage resistance which is variable with frequency and should also be kept to a minimum. The actual choice of the type of integrating capacitor is discussed in Sec. 5.9.

Integrator Calibration. If the integrator transfer function is taken to be

$$\frac{X_o}{X_1} \approx \frac{A}{(1 - A)RCP} \tag{4.43}$$

one may find the value of $A/[(1 - A)RC]$ by actually measuring the output-to-input voltage ratio by means of a good potentiometer. Equation (4.43) is usually replaced by

$$\frac{X_o}{X_1} = -\frac{1}{RCP} \tag{4.44}$$

with a percentage calibration error of magnitude

$$\epsilon_c = \frac{100}{A} \tag{4.45}$$

Careful selection of components makes it possible to determine the desired value of RC in (4.44) within 0.01 per cent.

Effects of Finite Gain, Leakage, and Distortion. A quantitative estimate of the accuracy of (4.43) and thus of the quality of integration is found by substituting the values of b and k from (4.41) into Eq. (4.20). The percentage error ϵ due to finite gain and leakage is found to satisfy the relation

$$\epsilon \leq \left| \frac{50\tau}{(1 - A)RC} \left[(1 - A)\frac{R}{R_L} + 1 \right] \right| \tag{4.46}$$

If finite gain and leakage are the only cause of error, X_o must be within ϵ per cent of the integral form (4.43) if

$$\tau \leq \frac{1}{50} \frac{\epsilon(1 - A)RC}{(1 - A)\dfrac{R}{R_L} + 1} \tag{4.47}$$

which is a special case of (4.21).

As in Sec. 4.3, this important criterion is *not* restricted to the case of unit-step-function input voltages, since the networks in question are linear, and any input voltage $X_1(\tau)$ may be represented as a sum of suitable step functions. *The condition given by (4.47) constitutes a limitation on the maximum allowable computing time if the integrator errors are not to exceed a given percentage, ϵ, of the output voltage.*

Changes of the amplifier gain A may affect the error estimates given by Eqs. (4.46) and (4.47). The results of (4.46) and (4.47) may also be affected by changes in the leakage resistance R_L (due to changes in atmospheric moisture, etc.). The relations (4.41), (4.46), and (4.47) should, therefore, always be computed for the most unfavorable values of A and R_L. In any case, errors due to changes in gain and leakage are minimized by making the amplifier gain A as large as possible.

The Effect of Stray Input Voltages. In d-c analog computers, the output of each amplifier is balanced to zero by some biasing arrangement before computation, so that, in the ideal case, only the desired signal voltages can act on the integrator input terminals. Actually, certain stray voltages, due to noise pickup, d-c leakage, and unbalances in the d-c amplifier resulting from grid current and drift, do occur. They are integrated and may lead to errors in the integrator output voltages.

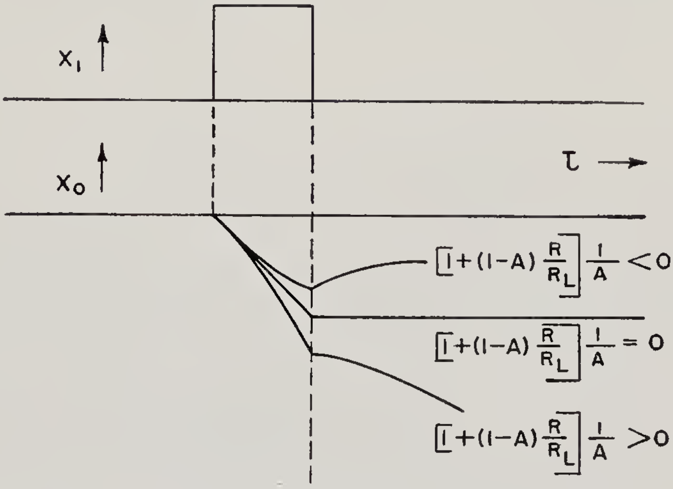

x_1
x_0
τ
$\left[1+(1-A)\frac{R}{R_L}\right]\frac{1}{A}<0$
$\left[1+(1-A)\frac{R}{R_L}\right]\frac{1}{A}=0$
$\left[1+(1-A)\frac{R}{R_L}\right]\frac{1}{A}>0$

FIG. 4.19. Integration of a square voltage pulse.

A noise pulse will be integrated in the manner shown in Fig. 4.19. As long as

$$\left[1 + (1 - A)\frac{R}{R_L}\right]\frac{1}{A} \leq 0$$

high-frequency random noise will tend to cancel itself during the integration process.

D-c integrators are fairly susceptible to errors resulting from pickup of 60-cycle a-c hum. Hum voltages should be kept out of the equipment by reasonable shielding and placement of leads and power supplies which carry alternating currents.

A steady unbalance voltage E_e due to grid current, d-c amplifier drift, etc., at the input of an isolated integrator acts like a step-function input and leads to an absolute output error drift approximately equal to

$$E_e \frac{\tau}{RC} \frac{A}{(1 - A)} \tag{4.48}$$

In order to minimize this drift, which can lead to serious difficulties, it may be expedient to balance each amplifier, not simply for zero output voltage but rather for zero drift. This will help to eliminate the error given by (4.48). Above all, reduction of integrator drift must be accomplished by careful design of amplifiers and power supplies, as discussed in Chap. 5.

In many computer applications, the d-c integrators are not isolated but are set up in feedback loops to simulate stable, damped oscillations. In such circuits, any temporary unbalance will tend to damp out, and any steady unbalance voltage will only change the zero point or reference level of the output voltage scale. The amplifiers in such a system need only be balanced for zero output at zero input voltage.

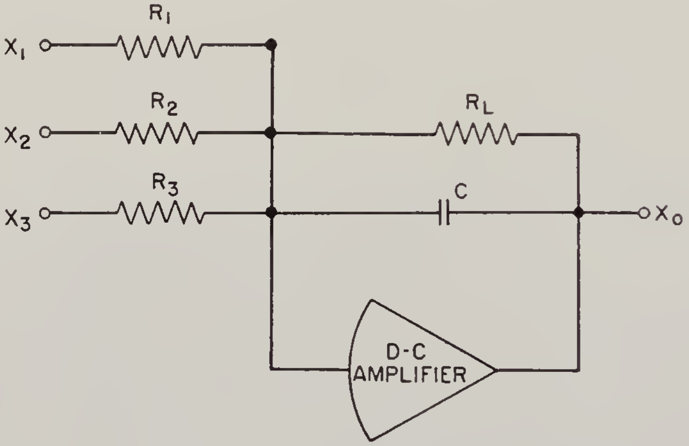


FIG. 4.20. Summing integrator.

Summing Integrators. It was shown in Sec. 1.4 that summing and multiplication by constants may be achieved at the input of an integrator. Such a summing integrator is shown in Fig. 4.20. The output voltage of this circuit is given by

$$X_o = \sum_{i=1}^{n} \frac{X_i}{R_i} \frac{A}{(1 - A)CP + \dfrac{1 - A}{R_L} + \displaystyle\sum_{i=1}^{n} \frac{1}{R_i}} \tag{4.49}$$

or approximately by

$$X_o = - \sum_{i=1}^{n} \frac{X_i}{R_i CP} \tag{4.50}$$

Equation (4.50) *is commonly used for the calibration of summing integrators.* It may be shown that *the error analysis expressed in Eqs.* (4.41), (4.46), *and* (4.47) *will apply to the summing integrator if, in each equation, R is replaced by* $\left(\sum_{i=1}^{n} \frac{1}{R_i}\right)^{-1}$, which is the equivalent resistance of the summing resistors R_i taken in parallel. The equivalent *time constant* for the summing integrator is thus

$$b = \frac{(1 - A)CR_L}{(1 - A) + R_L \sum_{i=1}^{n} \frac{1}{R_i}} \tag{4.51}$$

In particular, Eq. (4.47) for the maximum computing time allowable for a given percentage error ϵ becomes

$$\tau \leq \frac{1}{50} \frac{\epsilon(1 - A)R_LC}{(1 - A) + R_L \sum_{i=1}^{n} \frac{1}{R_i}} \tag{4.52}$$

Effect of D-c Amplifier Input Resistance. The effect of the finite input resistance R_g of the integrating amplifier has hitherto been neglected. This resistance (see Fig. 4.21) may be considered simply as a grounded input terminal in a summing integrator. The results of the last paragraph show that, as far as the quality of integration is concerned, R_g decreases the effective value of R as if it were connected in parallel with R.

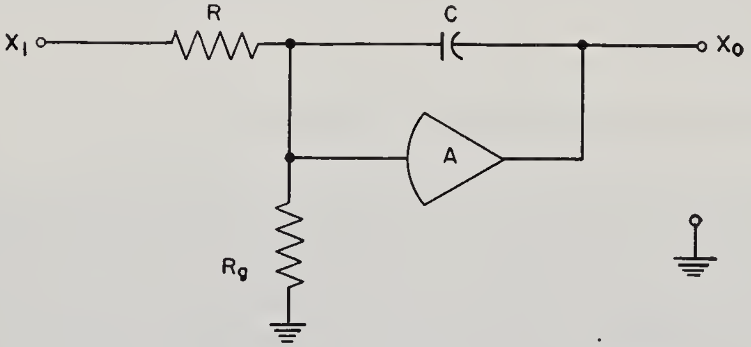


FIG. 4.21. Effect of d-c amplifier input resistance.

High-frequency Performance Requirements. The high-frequency performance of the parallel-feedback integrator is given by Eq. (4.39). The following requirements should be met (see also Sec. 4.7):

1. The absolute value of the d-c amplifier forward gain $A(j\omega)$ should satisfy the condition (4.46) throughout the anticipated range of signal frequencies for each given permissible value of the percentage error ϵ.

2. The integrator must be stable; this will be the case if the roots of the equation

$$[1 - A(\lambda)]RC\lambda + [1 - A(\lambda)]\frac{R}{R_L} + 1 = 0 \tag{4.53}$$

have only negative real parts.

3. The integrator phase shift should not deviate from 90 deg by more than 0.1 deg throughout the frequency range used.

As in the case of summing amplifiers, the gains and phase shifts of several cascaded integrators may add in certain computer setups and lead to instability.

Conclusions. Most errors in the electronic integrators discussed in this section are reduced by using a shorter computing time, high gain, high leakage resistance, and a long integrator time constant.

The errors due to finite gain, gain changes, and distortion can be made very small by using high values of the d-c amplifier gain A and certain types of feedback amplifiers. The most serious errors, then, arise from *drift* due to unbalances in d-c amplifiers. These errors must be minimized by carefully balancing the amplifiers, by using well-regulated power supplies, and by improved amplifier design. Errors may also be reduced by decreasing τ/RC within the limits set by scale-factor considerations. D-c leakage into amplifier input terminals must be kept as low as possible (Secs. 5.9 and 8.3).

4.6. REGENERATIVE OPERATIONAL AMPLIFIERS

Reduction of the Effects of Distortion and Instability through the Use of Feedback. One might think that the application of degenerative feedback to the d-c amplifier would decrease the effects of distortion and gain changes and, on the other hand, regeneration could increase the effective forward gain A of an operational amplifier like that of Fig. 4.9*a* and thus make its performance equation correspond more closely to the desired form

$$X_o = -\left(\frac{X_1}{Z_1} + \frac{X_2}{Z_2} + \cdots\right)Z_o \tag{4.54}$$

A closer investigation shows that at least the first of these possibilities has hardly any practical value, as the decrease in loop gain due to degenerative feedback would tend to offset the possible advantages of the degeneration. The same reasoning shows that a certain amount of overall regeneration can be used to increase the forward gain of an operational amplifier without directly increasing the effects of distortion and gain changes.

Application of Feedback around Sections of the D-c Amplifier. More interesting results can be achieved through the use of feedback around stages or groups of stages of multistage d-c amplifiers.[1] The amplifier shown in Fig. 4.22 will serve as an illustrative example. It comprises a voltage-amplifier section of gain a_1 with a feedback loop of feedback ratio β_1 and a power-amplifier section of gain a_2 with a feedback loop of feedback ratio β_2. The over-all gain A is

$$A = \frac{a_1 a_2}{(1 - a_1\beta_1)(1 - a_2\beta_2)} \tag{4.55}$$

This will be the forward gain of an operational amplifier utilizing the d-c amplifier in question. If ϵ_{a1} and ϵ_{a2} are percentage gain changes in the two sections of the d-c amplifier, respectively, before the application of feedback, the corresponding percentage gain change ϵ_A of the entire d-c amplifier is

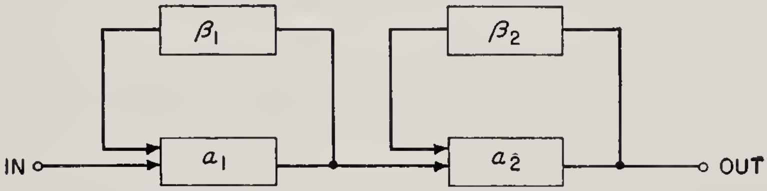


FIG. 4.22. Block diagram of a d-c amplifier with feedback around sections of the amplifier.

$$\epsilon_A = \frac{\epsilon_{a1}}{1 - a_1\beta_1} + \frac{\epsilon_{a2}}{1 - a_2\beta_2} \tag{4.56}$$

so that

$$\frac{\epsilon_A}{A} = \frac{1}{a_1 a_2}[\epsilon_{a1}(1 - a_2\beta_2) + \epsilon_{a2}(1 - a_1\beta_1)] \tag{4.57}$$

Since the fractional effects of gain changes on the output voltage of a feedback amplifier with high loop gain are proportional to ϵ_A/A rather than to ϵ (Sec. 4.2), these effects may be reduced to a remarkable extent through near-critical regeneration ($1 - a_1\beta_1 \approx 0$, $1 - a_2\beta_2 \approx 0$) in one or both of the amplifier stages.

Quite similar improvements may be achieved with respect to the effects of distortion (see also Sec. 4.2).

Qualitatively speaking, the effects of distortion as well as the effects of gain changes in any amplifier stage can often be reduced through the application of near-critical regeneration to *another* stage of the d-c amplifier.

The results just derived indicate the use of regenerative feedback around sections of multistage d-c amplifiers as a very useful method for increasing the loop gain and thus the accuracy of operational amplifiers. Such feedback will tend to make the performance equation correspond more closely to the desired form (4.6*c*), will reduce the output impedance of parallel-feedback operational amplifiers, and will help to minimize the effects of distortion and gain changes in the d-c amplifier.

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[1] Korn, G. A., D-c Integrator Design, *Electronics*, May, 1948.

Before deciding on the application of regenerative feedback, one must ascertain (1) whether the circuits will be stable (see Sec. 5.4) and (2) by how much the "return differences" $(1 - a_1\beta_1)$, $(1 - a_2\beta_2)$, etc., can change from the desired small values as the result of changes in the stage gains and feedback-network parameters. It is highly desirable to design the amplifier so that frequent readjustments of stage gains or feedback ratios are not necessary.

Examples of d-c amplifiers incorporating internal regenerative feedback are shown in Figs. 5.29, 5.30, and 5.32. Other examples have been discussed by Valley and Wallman.[1]

Feedback around one or more stages of the d-c amplifier is also used at times for purposes of stabilizing the high-frequency response of an operational amplifier (see Sec. 5.4). An example of such an application is shown in Fig. 5.39.

Regenerative Integrators. If the absolute value of the negative forward gain A of a parallel-feedback integrator of the type shown in Fig. 4.18[2] is made very large through regeneration or other means, the transfer function

$$\frac{X_o}{X_1} = \frac{A}{(1 - A)RCP + (1 - A)\dfrac{R}{R_L} + 1} \tag{4.58}$$

derived in Sec. 4.5 becomes

$$\frac{X_o}{X_1} = -\frac{R_L}{R}\frac{1}{R_LCP + 1} \tag{4.59}$$

The "leakage time constant" R_LC seems to constitute an upper limit for the time constant (see Sec. 4.5) of the parallel-feedback integrator. This limitation is due to degenerative feedback through the leakage resistance.

An infinitely large integrator time constant or perfect integration could be realized if it were possible to make the quantity

$$(1 - A)\frac{R}{R_L} + 1 \tag{4.60}$$

equal to zero throughout the range of working frequencies of the integra-

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[1] Valley, G. E., and H. Wallman, *Vacuum Tube Amplifiers*, MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

[2] The analysis of this section applies to *summing integrators* of the type shown in Fig. 4.20 if the quantity R, in (4.58) and the following equations, is replaced by

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots\right)^{-1}$$

tor. It is seen that the expression (4.60) can be equal to zero only for positive values of the forward gain A.

Unfortunately, this condition for perfect integration cannot be satisfied by simply using any d-c amplifier whose forward gain A is positive instead of the usual negative-gain amplifiers. Since the phase shift of the integrator feedback network is practically equal to 90-deg lead at high frequencies, *any high-frequency phase lag in an ordinary d-c amplifier of positive gain would result in regeneration and tend to produce instability at high frequencies;* the resulting circuit would act essentially like a multivibrator[1] unless special precautions were taken to equalize the high-frequency response.

It can be shown[2] that *the use of regeneration either around the entire d-c amplifier or (preferably) around a section of the d-c amplifier makes it possible to obtain a positive d-c amplifier gain and thus to make the quantity* (4.60) *equal to zero for perfect integration without making the integrator unstable at high frequencies.* The technique just described can be a striking means for improving the performance of d-c integrators by overcoming the effects of finite amplifier gain and capacitor leakage.

The adjustment of a regenerative integrator will be sensitive not only to changes of the forward gain A of the amplifier but to variations in the leakage resistance R_L as well. Changes in leakage resistance due to changes in atmospheric moisture may be hard to avoid if the values of R_L are very large. It is clear that, due to unavoidable changes in gain and leakage, it will not be possible to achieve perfect integration for an indefinite time. Nevertheless, considerable improvements in integrator performance can be obtained. The error estimates for parallel-feedback integrators derived in Sec. 4.5 hold for regenerative integrators provided the quantity (4.60) is less than or equal to zero. In order to obtain realistic error estimates for regenerative integrators, it is necessary to know only between what limits the gain A and the leakage resistance R_L may vary during computation. The error estimates derived in Sec. 4.5 can then be applied to the most unfavorable combination of A and R_L.

Figure 4.19 shows how a pulse is integrated by a parallel-feedback integrator when the quantity (4.60) is positive, negative, and equal to zero.

It is clear that a perfect d-c integrator can be realized *only* through the use of regeneration, since the perfect integrator must necessarily have infinite gain at zero frequency.

It is possible to apply similar feedback techniques to electronic integrators other than the parallel-feedback integrators discussed here; the bootstrap integrator described in Sec. 4.4 is an example of such a circuit.

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[1] Greenwood, Holdam, and MacRae, *op. cit.*

[2] See Bode, *op. cit.* The method of analysis is essentially similar to that employed in Sec. 5.4.

4.7. ERRORS IN THE SOLUTION OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS[1]

Effect of Finite Integrator Time Constant. Many problems (see, *e.g.*, Secs. 3.2 and 3.4) require differential-analyzer solutions of linear homogeneous differential equations corresponding to a machine equation

$$F(P)Y \equiv (a_0 + a_1P + a_2P^2 + \cdots + a_nP^n)Y = 0 \tag{4.61}$$

where $a_0, a_1, \ldots, a_n$ are constant coefficients. A differential analyzer attempting to establish the relation (4.61) by means of integrators having the transfer function $k/(bP + 1)$ instead of the ideal integrator transfer function k/bP (Sec. 4.3) actually solves the differential equation

$$F\left(P + \frac{1}{b}\right)\hat{Y} \equiv \left[a_0 + a_1\left(P + \frac{1}{b}\right) + a_2\left(P + \frac{1}{b}\right)^2 + \cdots + a_n\left(P + \frac{1}{b}\right)^n\right]\hat{Y} = 0$$

The resulting solution $\hat{Y}$ is related to the correct solution Y by

$$\hat{Y} = Ye^{-\tau/b}$$

The error due to the finite integrator time constant is again seen to increase with the computing time; this effect can, in principle, be compensated for in the recording device, or by regenerative feedback around an integrator or integrators as in Sec. 4.6, or as in the circuit 4.6 of Table 4.

Effects of High-frequency Cutoff in Integrators and Summing Amplifiers. MacNee[2] has similarly derived error estimates for a differential analyzer whose integrators and summing amplifiers have performance equations respectively approximated by

$$X_o = \frac{1}{bP + 1}\,\frac{1}{T_1P + 1}\,X_1, \qquad X_o = (X_1 + X_2 + \cdots)\,\frac{1}{T_2P + 1}$$

Both the integrator time constant b and the circular frequencies $1/T_1$ and $1/T_2$ corresponding to 90-deg phase error must be large compared to the roots s_k of the characteristic equation $F(s) = 0$ associated with the given differential equation (4.61). The errors due to high-frequency cutoff ($T_1 \neq 0$, $T_2 \neq 0$) depend on the characteristic roots s_k and are most serious in the case of characteristic roots near the imaginary axis (sinusoidal modes). In the case of the computer setup 4.6 of Table 4, which is frequently used for testing, one obtains

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[1] MacNee, A. B., Some Limitations on the Accuracy of Electronic Differential Equation Solvers, *Proc. IRE*, **40**: 303, 1950.

[2] *Ibid.;* see also L. G. Walters, Hidden Regenerative Loops in Electronic Analog Computers, *Trans. PGEC, IRE*, **2**: 1, June, 1953.

$$\hat{Y} = Ae^{-\tau/b}e^{\omega^2\left(T_1+\frac{T_2}{2}\right)\tau}\cos\omega\tau \text{ instead of } Y = A\cos\omega\tau \qquad (4.62)$$

Note that the computer setup becomes unstable for large values of ω.

In "slow" d-c analog computers, T_1 and T_2 are usually so small (less than 2×10^{-4} sec) that the effects of high-frequency cutoff are negligible in most problems which do not inherently involve undamped high-frequency oscillations. Errors due to high-frequency cutoff are much more serious in repetitive analog computers (Secs. 1.1 and 8.5), since it is difficult to design high-gain amplifiers having the required high-frequency response. One can often improve the high-frequency response of summing amplifiers and integrators by placing a trimmer capacitor across each input resistor.[1] With proper adjustment, the resulting phase lead may be able to raise the frequency corresponding to 1-deg phase lag by a factor of 10 or more at the expense of a small resonant peak above this frequency.

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[1] Bell, H., Jr., and V. C. Rideout, Precision in High-Speed Electronic Differential Analyzers, *Project CYCLONE Symposium II*, Reeves Instrument Corporation, New York City.

CHAPTER 5

D-C AMPLIFIERS FOR COMPUTER APPLICATIONS

5.1. PRINCIPLES OF D-C AMPLIFIER OPERATION

Introduction. In *direct-coupled or d-c amplifiers,* finite gain at zero frequency is obtained through conductive coupling between stages. Such conductive coupling may involve direct connections or resistive interstage networks; no coupling transformers or "blocking" capacitors are used.[1] The output voltage of a d-c amplifier or of any individual stage will, thus, be sensitive to changes in the input d-c reference level or input voltage for zero input signal. Connections to the amplifier input and output terminals are usually made also through conductive coupling except in a few instances, such as in certain differentiating circuits.

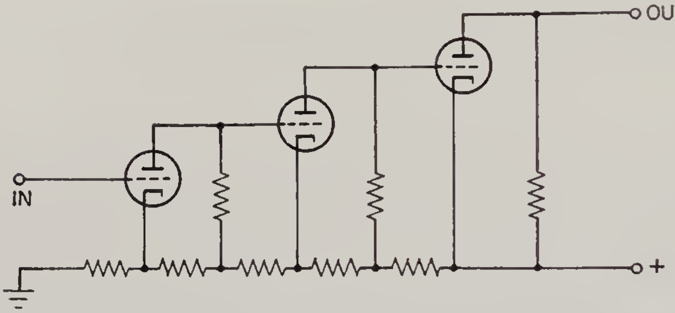

FIG. 5.1. D-c amplifier stages cascaded along a common bleeder.

Cascading of D-c Amplifier Stages. *The special nature of d-c amplifiers gives rise to a number of special design problems.* The first among these is based on the fact that the voltage at the plate of each amplifier tube is necessarily higher than that at the grid. This makes the *cascading of amplifier stages* more difficult than in a-c amplifiers. It is possible to raise the cathode of the succeeding amplifier stage to a potential sufficiently high to obtain a correct grid bias from the plate voltage of the first stage. Such a procedure requires high power-supply voltages if several stages are to be cascaded; Fig. 5.1 shows how the correct supply voltages for several stages may be obtained from a common bleeder resistor.

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[1] Valley, G. E., and H. Wallman, *Vacuum Tube Amplifiers,* Chap. 11, Direct-coupled Amplifiers by J. Gray, MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

A number of alternative designs, all essentially based on the circuit shown in Fig. 5.2, permit *operation of several d-c amplifier stages from the same power-supply terminals.* By means of a resistive summing network comprising the resistances R_1 and R_2 (either of which may be replaced by vacuum or gas tubes in certain designs) a negative bias voltage E_c is added to the voltage appearing at the plate of the amplifier tube. In this manner any desired d-c reference level may be obtained at the output of the stage at the price of some signal attenuation due to the summing network.

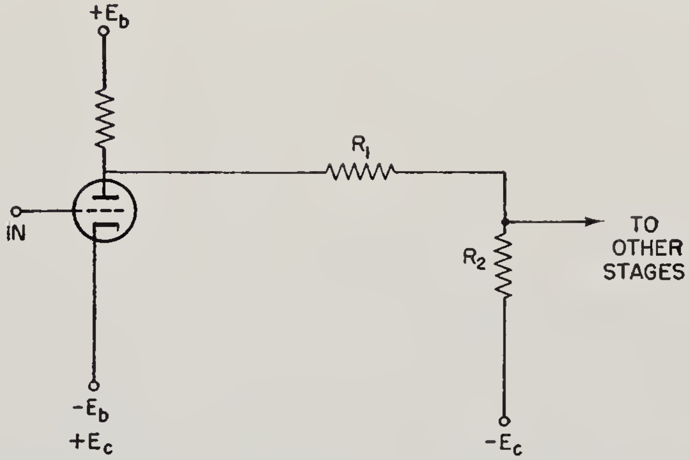

FIG. 5.2. Operation of d-c amplifier stages from common power supplies.

Balancing of Computer Amplifiers. *The output d-c reference level of practically all d-c amplifiers used in d-c analog computing elements must be equal to the input d-c reference level* (*usually zero with respect to ground*) in order to make it possible to apply feedback from amplifier output terminals and to interconnect the various computing elements. The desired output reference level may be obtained by means of circuit arrangements similar to that shown in Fig. 5.2. Accordingly, the d-c amplifiers used in d-c analog computers will in general require at least two power supplies. In each case, it will be necessary to adjust one or more of the resistors determining the voltage levels (see Figs. 5.29 to 5.35) carefully so as to obtain zero output voltage when the input voltage is zero. This process is called *balancing* or *zero adjustment.*

Drift in D-c Amplifiers. The output voltage of an a-c amplifier is zero for zero input voltage (except for the effects of various noise sources). The output level of a d-c amplifier, on the other hand, is sensitive to voltage-level changes throughout the amplifier and tends to change or *drift* with time. Drift will result in an *unbalance* voltage appearing at the amplifier output terminals in addition to the correct output voltage. In computer applications, this unbalance leads to errors in the computation. Such errors may be especially serious if the unbalance voltage is integrated by an integrator.

The amount of unbalance due to drift is conveniently measured and described by the value E of the input d-c voltage needed to cancel the unbalance or to bring the output voltage back to the desired reference level. Figure 5.3 shows a d-c amplifier with three stages of gains a_1, a_2, and a_3 and a feedback ratio β. In the presence of an input voltage E and drift voltages e_1, e_2, e_3, e_4, the output voltage

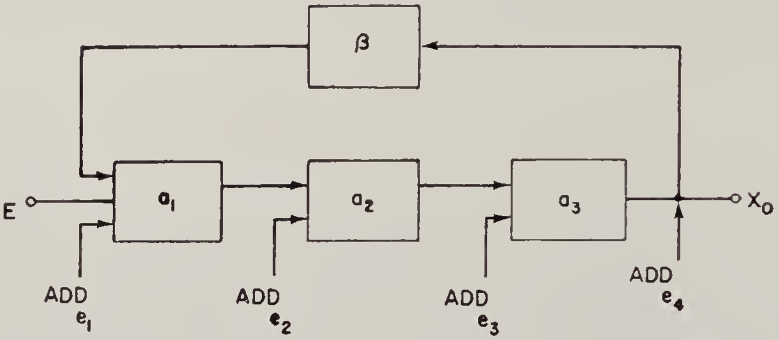

FIG. 5.3. Effects of drift in a multistage d-c amplifier with feedback.

$$X_o = \frac{a_1a_2a_3E + a_1a_2a_3e_1 + a_2a_3e_2 + a_3e_3 + e_4}{1 - a_1a_2a_3\beta} \tag{5.1}$$

will be brought back to zero if

$$E = -\left(e_1 + \frac{1}{a_1}e_2 + \frac{1}{a_1a_2}e_3 + \frac{1}{a_1a_2a_3}e_4\right) \tag{5.2}$$

The value of E is seen to be independent of the feedback networks used. Equation (5.2) also shows that the effects of drift are most serious in the earlier stages of a d-c amplifier, since the resulting unbalances will be amplified by succeeding stages.

The reduction of d-c amplifier drift is one of the most difficult design problems in the development of d-c analog computers and will be discussed in detail later in this chapter. The following outline describes the principal sources of drift together with some possible means of minimizing their effects. A diagram like Fig. 5.3 is usually helpful for analyzing the relative effects of various sources of drift in a d-c amplifier.

1. *Drift due to changes in plate and bias supply voltages.* To reduce drift due to these causes one employs well-regulated power supplies and, possibly, some of the compensating circuits described in Sec. 5.3.
2. *Drift due to changes in filament voltages.* This effect is particularly serious in the input stage of an amplifier. [One may effectively say that filament-voltage changes contribute to the value of e_1 in

Eq. (5.2)!] For most tubes (diodes, triodes, or pentodes) with oxide-coated unipotential cathodes,[1] a 20 per cent change in filament voltage is equivalent to a grid-voltage change of 0.2 volt. The effects of filament-voltage changes should be counteracted at least in the amplifier input stage by means of special compensating circuits (see Sec. 5.3) as well as through the use of regulated filament supplies.

3. *Drift due to changes in resistance values.* This drift can be minimized by using carbon-film or special wire-wound resistors of known quality. All critical resistors should be operated well below their power rating in order to reduce resistance changes due to heating. For best accuracy, the ambient temperature of the entire computer may be kept constant by means of a suitable air-conditioning system.
4. *Drift due to changes in vacuum-tube characteristics.* These changes are mostly due to variations in the cathode emission. The use of compensating circuits (Sec. 5.3) as well as *aging* of the vacuum tubes to be used will be helpful in this respect.

The magnitudes of the unbalance voltages due to the various drift sources will depend on the actual circuits in each case. In addition to d-c amplifier drift in the strict sense of the word, changes in *spurious d-c input voltages* due to *d-c leakage* (from adjacent high-voltage points) and to *d-c ground currents* contribute to the total unbalance voltage.

The Effects of Grid Current in D-c Amplifiers. *Grid currents*, or currents flowing from or into the grid, will cause voltage drops across the input resistors of computing amplifiers and thus lead to errors (distortion) in the machine variables. Even when the input signal is zero, grid current may cause an unbalance voltage which can often be balanced out by means of the balancing control. Changes in this zero-signal grid current constitute an additional source of drift. Since, for instance, a grid current of only 0.01 μa flowing in an input resistor of 1 megohm will give rise to an error voltage of 0.01 volt, it is seen that grid currents may lead to serious computing errors when their effects are amplified and integrated.

Figure 5.4 shows the variation of grid current i_g with grid voltage e_g for a typical triode operated so that the plate current i_p is constant.[2]

The current flowing from the cathode to the grid of the vacuum tube will be *negative* for sufficiently large negative values of the grid voltage e_g because positive ions of the residual gas in the tube are attracted to the negative grid. In some tubes, the latter may even emit electrons if some of the oxide coating from the cathode should be deposited on the grid. The amount of the negative grid current depends chiefly on the amount of residual gas in the tube and varies from tube to tube. The maximum allowable negative grid current of one section of a 6SL7 triode is given in the JAN specifications as 1 μa, but selected tubes may have negative grid currents as low as 0.02 μa. The JAN specifications for the 6AK5 are 0.1 μa and for the 6SU7, 0.02 μa. *It may be helpful to operate the tubes at low plate, screen, and filament voltages in order to minimize ionization;* the grid currents of specially evacuated and insulated electrometer tubes may be kept as low as 10^{-10} μa throughout their operating range in this manner.

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[1] *Ibid.*

[2] *Ibid.*

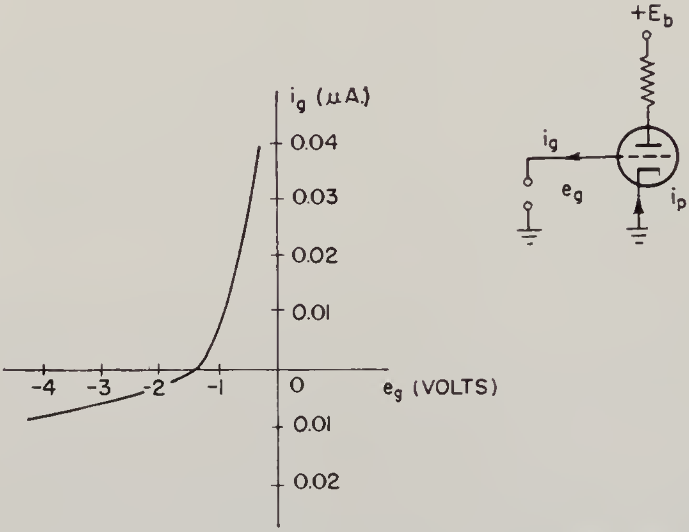
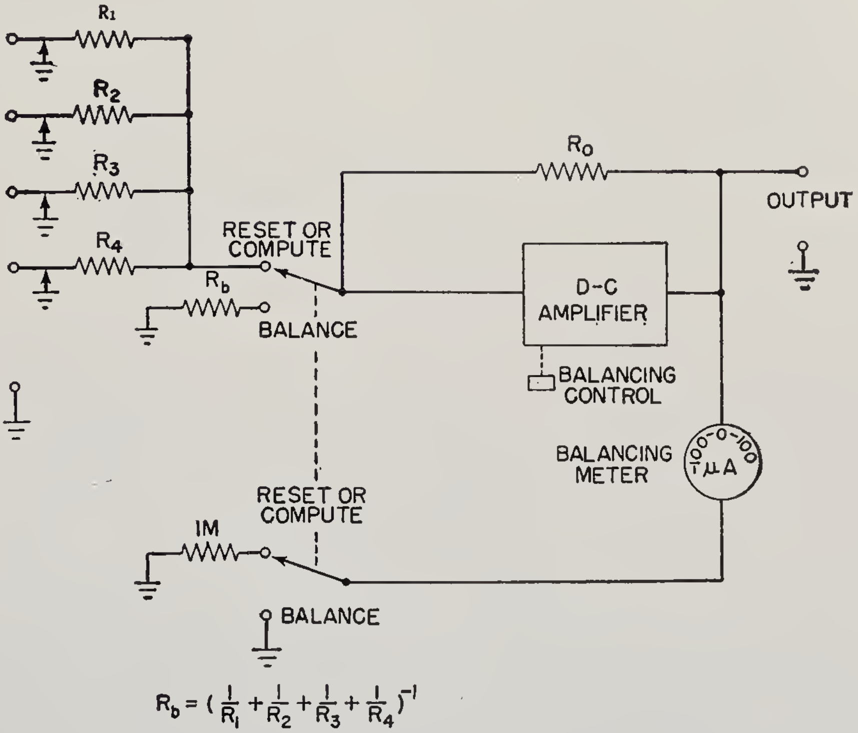


FIG. 5.4. Grid current vs. grid voltage for constant plate current in a triode.

As the value of the negative grid voltage is decreased, the grid current becomes *positive* owing to electrons flowing to the grid from the cathode. This positive grid current tends to increase sharply as the grid voltage approaches zero (see Fig. 5.4) and will be larger for lower plate voltages. The rise in positive grid current determines the limits of the operating range with respect to bias and plate voltage or screen voltage in the case of pentodes.

Practical Balancing Circuits. In order to minimize computing errors arising from unbalances due to drift, *computer amplifiers should be balanced at least every half-hour between computer runs and preferably before each run.*

Figure 5.5 shows a circuit suitable for switching a *summing amplifier* into the BALANCE condition which permits convenient balancing without any necessity for disconnecting the amplifier input and output terminals from the computer setup. The upper section of the *balancing switch* or push button serves to disconnect the amplifier input grid from the computing networks for balancing and to ground this grid through a resistance R_b equal to the resistance of the input resistors in parallel. The resistance between the input grid and ground is then approximately the same in the COMPUTE and BALANCE conditions; for during computation, either each input terminal is connected to a low-impedance voltage source (a preceding potentiometer or amplifier) or it is grounded directly through a closed-circuit jack as shown. Such an arrangement ensures that the zero-signal grid currents of the input stage are approximately equal during balancing and computing. The degenerative feedback through R_o also remains operative; while the feedback decreases the sensitivity of the balance indication, it is necessary to keep high-gain amplifiers and especially regenerative amplifiers stable during the balancing procedure.



$$R_b = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)^{-1}$$

FIG. 5.5. Switching arrangement for balancing a summing amplifier.

The lower section of the balancing switch shorts a resistor in series with a microammeter which can then be used to read the unbalance voltage of the amplifier output. In the RESET and COMPUTE conditions, the same meter serves as a voltmeter which permits one to read the amplifier output voltage continuously.

The balancing circuit for a *summing integrator* shown in Fig. 5.6 is essentially identical with that for the summing amplifier. During the balancing procedure, the integrating feedback network comprising the capacitance C and the resistance R_b is operative, and the integrator will tend to integrate the unbalance voltage. This results in a very sensitive balance indication if a sufficient time interval is allowed after each balancing adjustment. The relay switching the integrator into the COMPUTE and RESET conditions is indicated in Fig. 5.6 (no HOLD circuits are shown). For balancing, the computer is placed in the RESET condition, and each amplifier is balanced in turn. The computer is then still in the RESET condition, ready for computation (see also Sec 7.1).

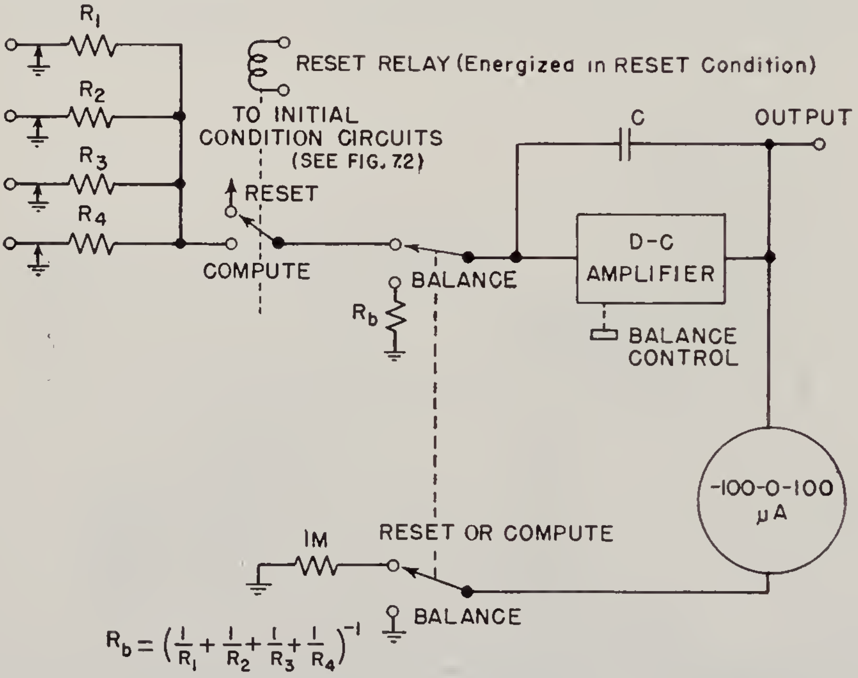
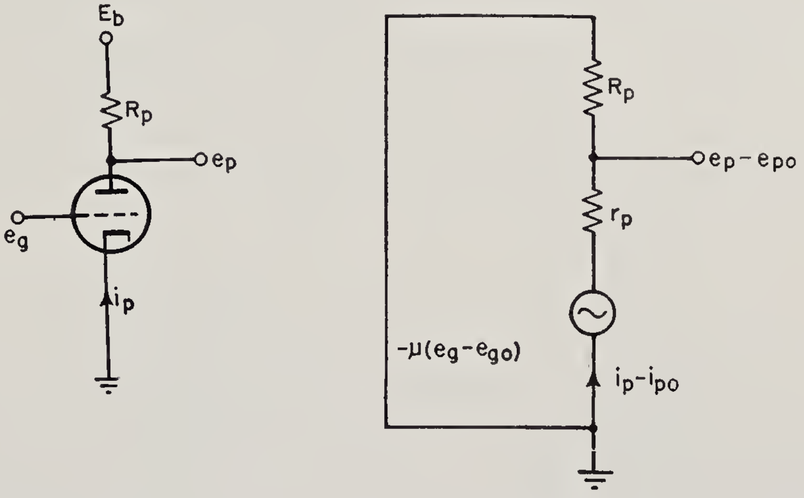

R1
R2
R3
R4
RESET RELAY (Energized in RESET Condition)
TO INITIAL CONDITION CIRCUITS (SEE FIG. 7.2)
RESET
COMPUTE
C
OUTPUT
D-C AMPLIFIER
BALANCE
Rb
BALANCE CONTROL
-100-0-100 μA
IM
RESET OR COMPUTE
BALANCE
$R_b = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)^{-1}$

FIG. 5.6. Switching arrangement for balancing a summing integrator.

Many variations of the balancing circuits shown in Figs. 5.5 and 5.6 are possible. The sensitivity of the balancing meter may be adjustable by means of a range switch. The balancing switches may be replaced by a relay system which permits one to place all d-c amplifiers in the computer into the BALANCE condition by means of a single switch. It is even possible to adjust each amplifier balance control by means of a special servomechanism actuated by the respective unbalance voltage. Generally speaking, such expensive refinements have been rendered unnecessary by the development of the *automatic balancing* circuits described in Sec. 5.6.

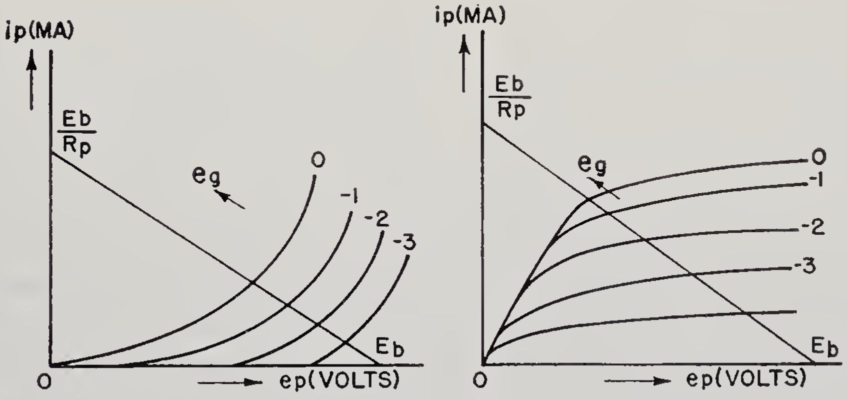
5.2. D-C AMPLIFIER DESIGN

Design Procedure. Ordinarily, the d-c amplifier designer will begin with a rough idea of the types of circuits and the number of stages needed to achieve a specific purpose. The actual design of the amplifier then proceeds stage by stage, usually beginning with the output stage; the values of the output voltage and current will, in general, have been previously specified. After the operating potentials for each tube have been chosen, the resistive networks needed to obtain the correct operating potentials from the given voltage sources must be designed. Finally, the entire design is gone over so as to effect necessary compromises between the characteristics of different stages and also to ensure the high-frequency stability of the amplifier. The amplifier may then be built and tested.



(*a*) (*b*)

FIG. 5.7. Basic vacuum-tube amplifier (*a*) and linear equivalent circuit (*b*).



(*a*) (*b*)

FIG. 5.8. Plate-current and plate-voltage characteristics of a triode (*a*) and of a pentode (*b*).

The operation of a voltage amplifier circuit of the type shown in Fig. 5.7*a* may be conveniently studied by means of the published plate voltage-current characteristics of the particular tube used. Figure 5.8 shows sets of such characteristics for a typical triode and a typical pentode. The *load lines* drawn in Fig. 5.8*a* and *b* for a particular value of the plate load resistor R_p serve to show the variation of the plate voltage e_p as a result of the voltage drop i_pR_p in the plate load resistor.

From such a set of characteristics, one can read the plate voltage e_p and the plate current i_p for all values of the grid voltage e_g for a given tube and load-resistor combination. The technique of designing amplifiers from published characteristics is described in detail in the standard textbooks on electronics.[1]

In the vicinity of any *operating point* given by the values

$$e_p = e_{p0} \qquad i_p = i_{p0} \qquad e_g = e_{g0}$$

corresponding to zero input signal, the characteristics may be considered as straight lines, in which case it follows that

$$\left.\begin{aligned} \frac{i_p - i_{p0}}{e_g - e_{g0}} &= \frac{\mu}{r_p + R_p} = \frac{g_m}{1 + \dfrac{R_p}{r_p}} \\ \frac{e_p - e_{p0}}{e_g - e_{g0}} &= \frac{-\mu R_p}{r_p + R_p} = -g_m \frac{R_p r_p}{R_p + r_p} \end{aligned}\right\} \tag{5.3}$$

where the three derivatives

$$r_p = \left.\frac{\partial e_p}{\partial i_p}\right]_{e_g} \qquad g_m = \left.\frac{\partial i_p}{\partial e_g}\right]_{e_p} \qquad \mu = \left.\frac{\partial e_p}{\partial e_g}\right]_{i_p} = r_p g_m \tag{5.4}$$

are known as the *plate resistance, transconductance,* and *amplification factor* of the tube, respectively. The linear *equivalent circuit* shown in Fig. 5.7*b* is a very convenient aid for deriving approximate linear relations like (5.3) between voltage and current *changes* in the corresponding vacuum-tube circuit of Fig. 5.7*a*. If the voltage and current changes take place sufficiently rapidly, the effects of tube and circuit capacitances must be taken into consideration in the manner shown in Sec. 5.4.

The *choice of the operating point* cannot be made from a consideration of the linear equations (5.3) alone, since the values of r_p, μ, and g_m change with the operating point. Instead, the characteristics shown in Fig. 5.8*a* and *b* must be consulted. The operating point for a voltage amplifier will be selected so that the gain is as high as possible for the particular circuit used, while the gain changes due to distortion still remain below a certain tolerable value, usually about 1 to 2 per cent without feedback.

---

[1] *The Radiotron Designer's Handbook,* 4th ed., edited by F. Langford Smith, Radio Printing Press Pty. Ltd., Sydney, Australia, reproduced and distributed in the United States by RCA Manufacturing Company, Inc., 1954; Terman, F. E., *Radio Engineers' Handbook,* McGraw-Hill, New York, 1948; MIT Electrical Engineering Staff, *Applied Electronics,* The Technology Press, 1943; Fink, D. G., *Engineering Electronics,* McGraw-Hill, New York, 1938; Valley G. E., and H. Wallman, *Vacuum Tube Amplifiers,* MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

For an output stage, both the voltage and current gain must be sufficiently high, while the distortion is kept within tolerable limits.

Particularly in the case of pentodes, a wide variety of operating conditions is possible, since the various grid voltages can all be varied. The references given in the footnote on page 198 should be consulted for more detailed discussions of these questions; an exhaustive treatment of amplifier design is entirely beyond the scope of this book.

The Use of Resistance-coupled-amplifier Tables. In view of the labor involved in designing d-c amplifiers directly from the tube characteristics, it may be expedient to adopt vacuum-tube operating conditions similar to those given in *standard resistance-coupled-amplifier tables.* Such resistance-coupled-amplifier data are given in most tube handbooks and usually refer to an a-c voltage-amplifier stage of the type shown in Fig. 5.9*a*, having a by-passed cathode resistor R_k. The tabulated data may be used to design an "equivalent" d-c amplifier shown in Fig. 5.9*b*. This d-c amplifier will have approximately the same operating point as the original a-c amplifier if the resistance of the load connected to it is high; this is usually the case for voltage-amplifier stages. The equivalent amplifier has fixed bias and, in the case of pentodes, a fixed screen voltage. The gain will, in general, be slightly higher than that given in the tables, since the plate load resistance is higher and also because the resistance of the load connected to the d-c amplifier stage often will be higher than the resistance R_g specified in the tables.

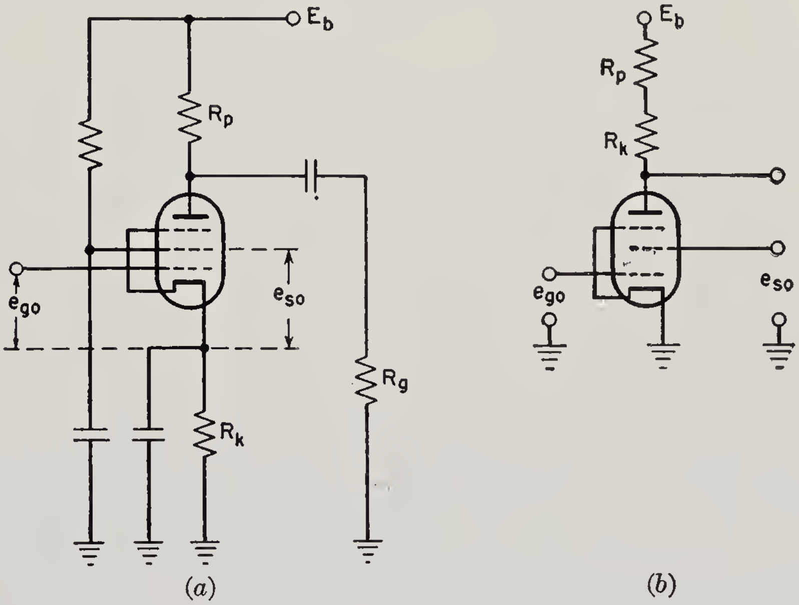
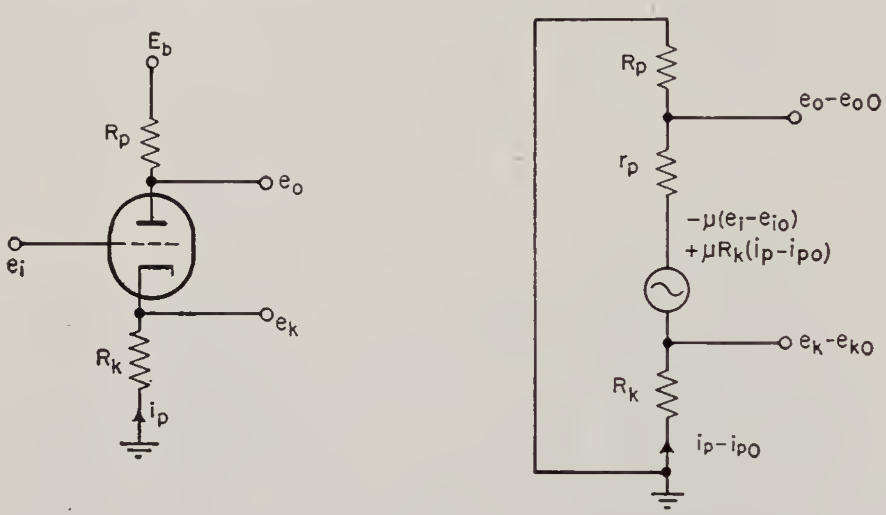


FIG. 5.9. Development of a d-c amplifier stage from a resistance-coupled a-c amplifier. (In the case of triodes, disregard the screen- and suppressor-grid circuits.)

In a-c amplifiers, the value of the plate load resistance R_p is often limited by the high-frequency response required of the stage. In d-c amplifiers, it is often permissible to use rather large values of R_p and correspondingly small plate currents in order to obtain a large voltage gain. In the case of pentodes, maximum stage gain can be obtained by using low screen voltages as well as low plate voltages.[1] The lowest permissible values of plate and screen voltages will be determined by the amount of positive grid current which can be tolerated.



(a) (b)

FIG. 5.10. Amplifier stage with cathode bias and equivalent circuit.

The Effects of Cathode-bias Resistors and Screen Series Resistors. If the grid bias for the operation of an amplifier tube is obtained by means of a cathode resistor in the manner shown in Fig. 5.10*a*, the net voltage between grid and cathode becomes

$$e_g = e_i - i_pR_k$$

The current and voltage changes may again be derived from the equivalent linear circuit as shown in Fig. 5.10*b*. The grid-to-plate gain of the stage is

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{-\mu R_p}{R_p + [r_p + (\mu + 1)R_k]} \tag{5.5}$$

where e_i and e_{i0} are the input and reference input voltages, respectively, and e_o and e_{o0} are the output and reference output voltages, respectively. The grid-to-cathode gain is found to be

$$\frac{e_k - e_{k0}}{e_i - e_{i0}} = \frac{\mu R_k}{R_p + [r_p + (1 + \mu)R_k]} \tag{5.6}$$

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[1] Gray, *op. cit.;* Volkers, W. K., Ultra-high Gain Direct-coupled Amplifier Circuits, lecture presented at the IRE National Convention, New York, 1950; Kaufer, G. E., How to Design Starved Amplifiers, *Tele-Tech and Electronic Industries*, January, 1955.

where e_k and e_{k0} are the cathode voltage and its reference voltage, respectively. All voltages are measured with respect to the point shown grounded in Fig. 5.10.

The values of r_p, g_m, and μ to be used in Eqs. (5.5) and (5.6) are again determined by the effective operating point. The latter may be found from the tube characteristics through successive approximations of the plate current i_p.[1]

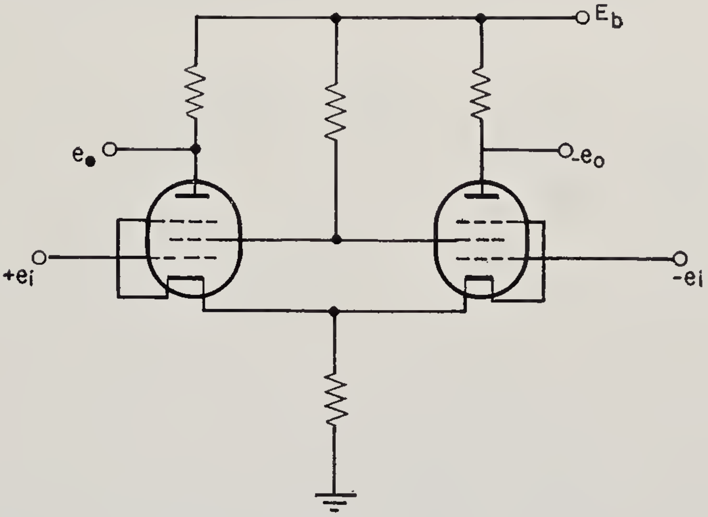

FIG. 5.11. A push-pull amplifier circuit.

A comparison of Eqs. (5.3) and (5.6) shows that a tube in the circuit of Fig. 5.10*a* acts like a tube of "equivalent dynamic plate resistance"

$$Z = r_p + (1 + \mu)R_k \tag{5.7}$$

connected in the simpler circuit of Fig. 5.7*a*. It is also seen that the inverse feedback due to the cathode resistor has decreased the gain of the stage. A similar effect is produced by a screen dropping resistor used with a pentode; the corresponding analysis is quite complicated.[2] Because of the loss in gain, the use of cathode and screen dropping resistors to obtain correct electrode voltages is often avoided in favor of fixed voltage sources. On the other hand, the degenerative effect, especially of the cathode dropping resistor, tends to reduce the effects of amplifier gain changes and distortion.

It should be noted that in push-pull amplifier circuits like the one shown in Fig. 5.11, the degenerative effect of the cathode and screen resistors is canceled or at least reduced because the plate-current changes in the two tubes oppose each other. In such circuits, the full stage gain given by Eq. (5.3) can be realized in spite of the cathode and screen dropping resistors.

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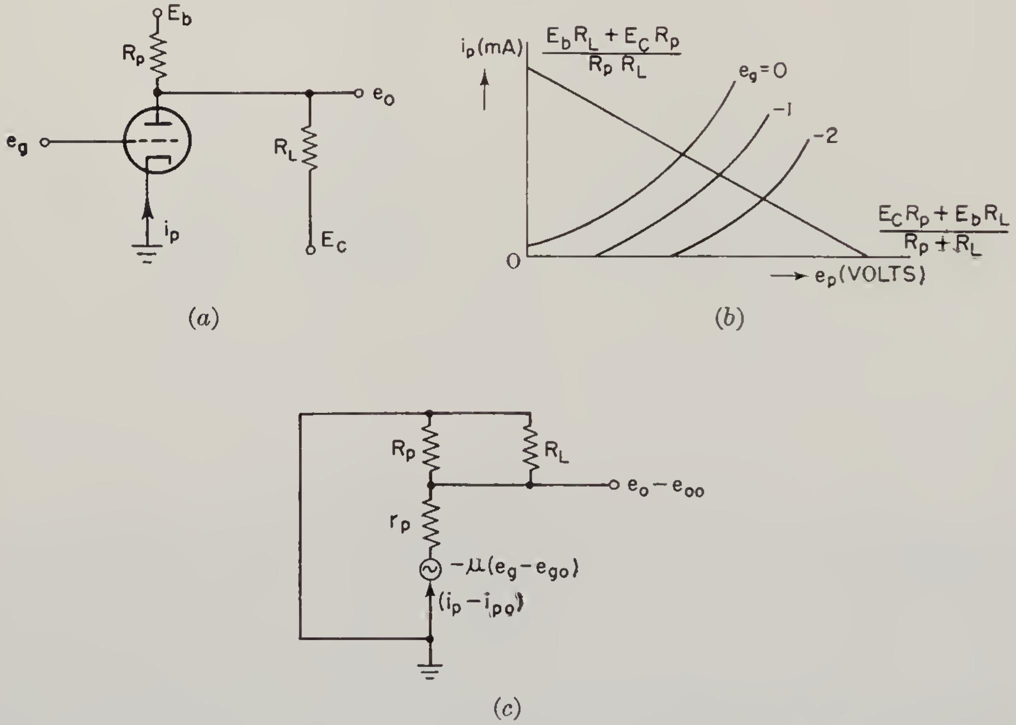
[1] Gray, *op. cit.*

[2] *Ibid.*

The Effect of Load Resistance. Each computer amplifier stage will, in general, be connected to a resistive *load* of resistance R_L, say; the load may be returned to ground or to some fixed voltage source of voltage E_c with respect to ground as shown in Fig. 5.12.

The effect of the load on the amplifier performance is best analyzed by drawing a new load line across the characteristics of the amplifier tube. Such a load line is shown in Fig. 5.12*b* for a triode amplifier and expresses the relation between the plate voltage e_p and the plate current i_p as determined by the values of R_p, R_L, E_b, and E_c. The value of R_L for many voltage-amplifier stages is so large that the effect of the load is negligible, so that the load lines of Fig. 5.8 can still be used. This is not usually true in the case of output stages. When the operating point on some desirable load line has been chosen, it is possible again to use the linear equivalent circuit for the vacuum tube to derive the voltage and current changes. Figure 5.12*c* shows the linear equivalent circuit for Fig. 5.12*a*. The stage gain is now easily derived by substituting the *total load resistance*



(*a*) (*b*)

(*c*)

FIG. 5.12. Illustrating the analysis of a simple amplifier stage with a load of resistance R_L.

$$R'_p = \frac{R_pR_L}{R_p + R_L} \tag{5.8}$$

for the plate load resistance in Eqs. (5.3). Similar equivalent circuits may be used to derive the gain of other loaded amplifier stages.

Interstage Coupling. The design equations for the coupling resistors used to establish the correct d-c reference level at the grid of the succeeding stage may be derived directly from the circuit of Fig. 5.13. Suppose that the positive and negative power-supply voltages E_b and E_c as well as the reference voltage e_{o0} at the plate have been chosen and that it is desired to establish a reference voltage level E_{o0} at the grid of the succeeding tube. Then the required values of the resistors R_1 and R_2 making up the interstage coupling network may be derived from the relations

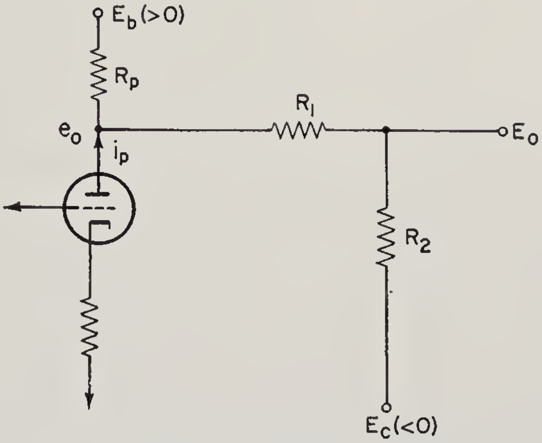


FIG. 5.13. Design of interstage networks.

$$\left.\begin{aligned} R_1 &= \frac{e_{o0} - E_{o0}}{E_b - e_{o0} - i_pR_p} R_p \\ R_2 &= \frac{E_{o0} - E_c}{E_b - e_{o0} - i_pR_p} R_p \end{aligned}\right\} \tag{5.9}$$

The stage gain $(e_o - e_{o0})/(e_i - e_{i0})$ may be derived as shown in the preceding paragraphs. The combined gain of the amplifier stage and coupling network is

$$\frac{E_o - E_{o0}}{e_i - e_{i0}} = \frac{e_o - e_{o0}}{e_i - e_{i0}} \frac{R_2}{R_1 + R_2} \tag{5.10}$$

It may be seen that the loss of gain due to the interstage resistive network can be reduced by increasing E_c and, therefore, R_2.

5.3. SPECIAL D-C AMPLIFIER CIRCUITS

Some Special Applications of Gas-filled and Vacuum Tubes. It is often desirable in d-c amplifier circuit design to have circuit elements whose *incremental resistance* (*i.e.*, the ratio between corresponding voltage and current changes) is different from their *d-c resistance* (corresponding to the ratio of the total voltage and current).

Figure 5.14 shows the use of a voltage-regulator tube or simply a neon bulb as a circuit element of *low incremental resistance* in an interstage network. The gas-discharge tube will tend to maintain a constant voltage across its electrodes. Accordingly, in the circuit[1] shown, the gas tube permits one to bring the voltage at the grid of the second stage to the desired level without any loss in gain due to the interstage networks. The gas-discharge tube acts like an infinitely small resistance with respect to voltage changes while still maintaining a finite d-c voltage across itself. The capacitor shown in Fig. 5.14 serves to bypass any noise which might be generated by the gas-discharge tube.

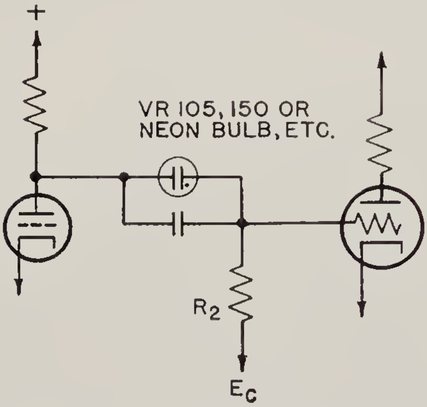


FIG. 5.14. Use of a gas-discharge tube in an interstage coupling network.

Some examples of the use of the dynamic plate resistance

$$Z = r_p + (\mu + 1)R_k$$

of a vacuum tube to obtain a *high incremental resistance* have already been shown in Sec. 4.4. Figure 5.15 shows four examples of the use of vacuum tubes as circuit elements of high incremental resistance in d-c amplifier circuits. Figures 5.15*a* and *b* illustrate the use of a vacuum tube for simulating a large plate load resistance in an amplifier and a large cathode resistance in a cathode follower, respectively. In either case, a large load resistance is simulated, whereas the d-c operating voltages on the electrodes of the amplifying tube can still be reasonably high. Arrangements of this type often result in better linearity than is possible with circuits using resistors. Figure 5.15*c* shows the use of a tube as the common cathode resistor in a differential amplifier. In this circuit, the high effective cathode resistance helps to make the output voltages of the two halves of the amplifier more nearly equal and opposite (see page 210).

Figure 5.15*d* shows how the dynamic plate resistance of a triode may be used to decrease the attenuation of an interstage coupling network. In this circuit the tube acts like a large incremental resistance while still maintaining the correct operating voltage at the grid of the succeeding stage.

Compensation for Supply-voltage Changes: Bridge-balanced D-c Amplifiers. The two vacuum tubes used in the *series-compensated* amplifier[2] shown in Fig. 5.16*a* form two arms of a bridge. If the output

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[1] Miller, S. E., Sensitive D-c Amplifier with A-c Operation, *Electronics*, November, 1941.

[2] Artzt, Maurice, Survey of D-c Amplifiers, *Electronics*, August, 1945.

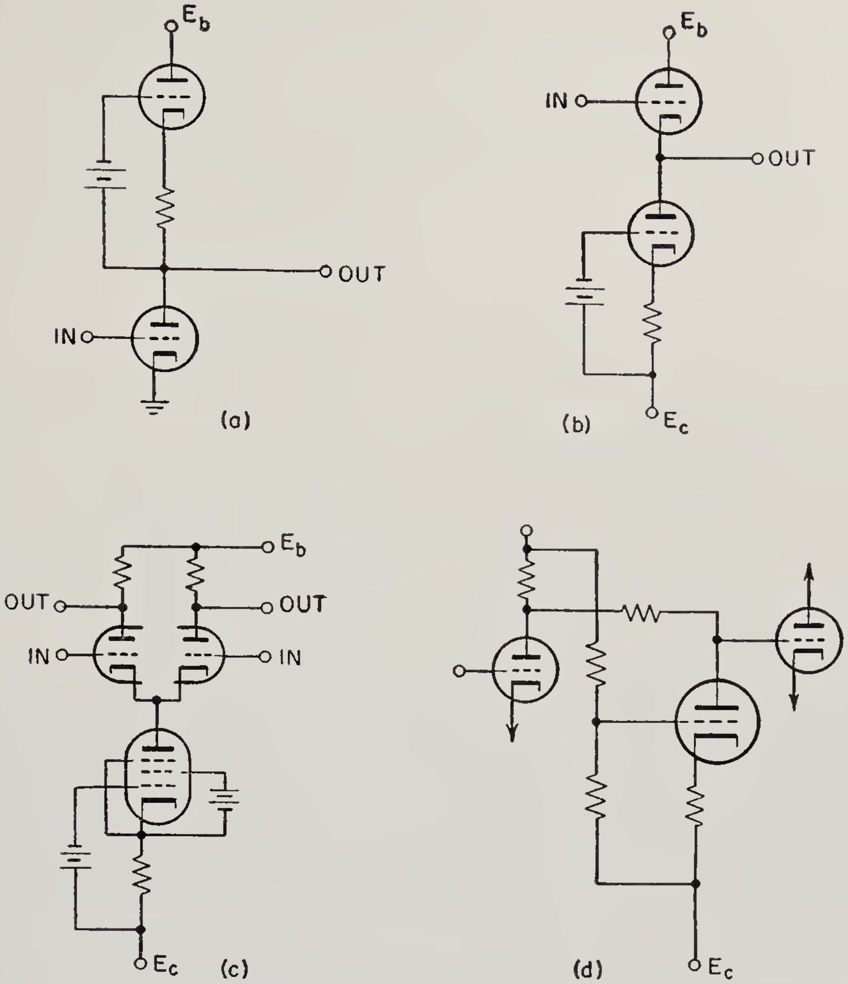

FIG. 5.15. Examples of the use of the dynamic plate resistance of a vacuum tube to obtain high incremental resistance.

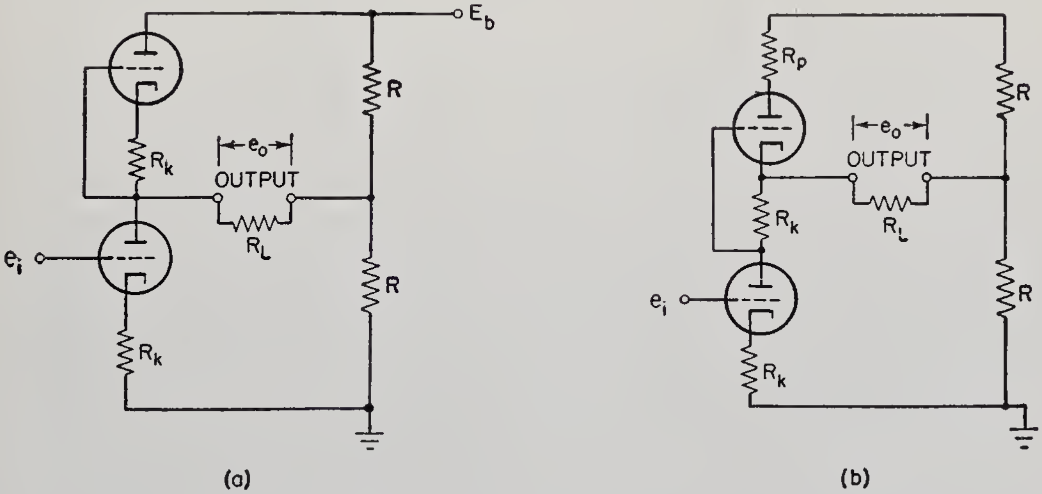

FIG. 5.16. Bridge-balanced (series-compensated) d-c amplifier stages.

voltage is taken off at the terminals indicated in the diagram, the zero-signal output voltage can be balanced to zero by adjusting one of the cathode resistors as shown and will then be, to a first approximation at least, independent of plate and filament supply voltage variations. If a load of resistance R_L is connected between the output terminals, the voltage gain is given by

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{-\mu R_L}{(\mu + 1)R_k + r_p + 2R_L + R} \tag{5.11}$$

assuming that two identical tubes of amplification factor μ and plate resistance r_p are used.

The compensation for cathode-emission changes in such arrangements will be best if the two tubes are as closely identical as possible. Two tubes in one envelope may be used if the cathode-to-filament voltage rating of the tube is not exceeded. It may still be difficult to match the operating conditions for the two halves of the tube accurately, since the two cathode-to-filament voltages cannot be made equal.

The gain given by Eq. (5.11) will assume the value $\mu/2$ if R_L is infinitely large. The value of $\mu/2$ may be as high as 4,000 for pentodes. This high gain is predicated on the high effective load resistance afforded by the dynamic plate resistance of the tube V_2. Equation (5.11) shows that the gain will decrease substantially if V_2 must be by-passed by a finite load resistance R_L. In such cases, the gain of the circuit shown in Fig. 5.16*b* is, if $R_p = 2R_k$,

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{-\mu R_L}{r_p + 2R_k + 2R_L + R} \tag{5.12}$$

which is somewhat higher than that of the circuit of Fig. 5.16*a* for the same circuit constants.

A new type of bridge-balanced d-c amplifier circuit,[1] shown in Fig. 5.17*a*, has all the advantages of series compensation, and, at the same time, its output voltage has the same d-c reference level as the input voltage. In Fig. 5.17*a*, this has been made possible through the use of a single floating power supply. This circuit is particularly suitable for small, portable feedback amplifiers for battery operation, in which case the floating power supply may not be objectionable. The gain of the circuit of Fig. 5.17*a* is the same as that given by Eq. (5.11) for the circuit of Fig. 5.16*a*.

If it is desirable to cascade several stages using common power supplies, the circuit of Fig. 5.17*b* may be used.

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[1] Korn, G. A., and T. M. Korn, Bridge-balanced D-c Amplifier, *Radio and Television News*, Engineering Edition, January, 1950.

Figure 5.18 shows two low-impedance driver circuits (current amplifiers) using series compensation. The gain of each circuit is

$$\frac{e_o - e_{o0}}{e_i - e_{i0}} = \frac{\pm \mu R_L}{r_p + (\mu + 1)R_k + (\mu + 2)R_L + R} \tag{5.13}$$

where the positive sign applies to the circuit of Fig. 5.18*a* and the negative sign to the circuit of Fig. 5.18*b*.

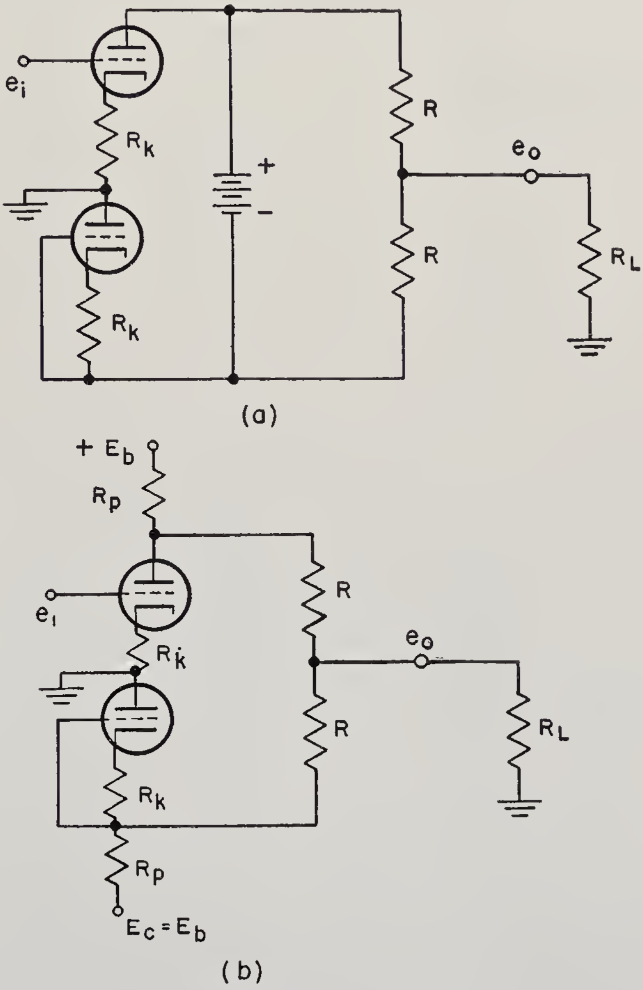

FIG. 5.17. A new type of bridge-balanced d-c amplifier circuit.

The output reference level of a series-compensated amplifier stage is particularly insensitive to changes in the plate-supply voltage. It is possible to turn the plate supply on and off without changing the output reference level. As a matter of fact, in certain applications, bridge-balanced amplifiers have been operated with alternating current on the plate without adverse effects.[1]

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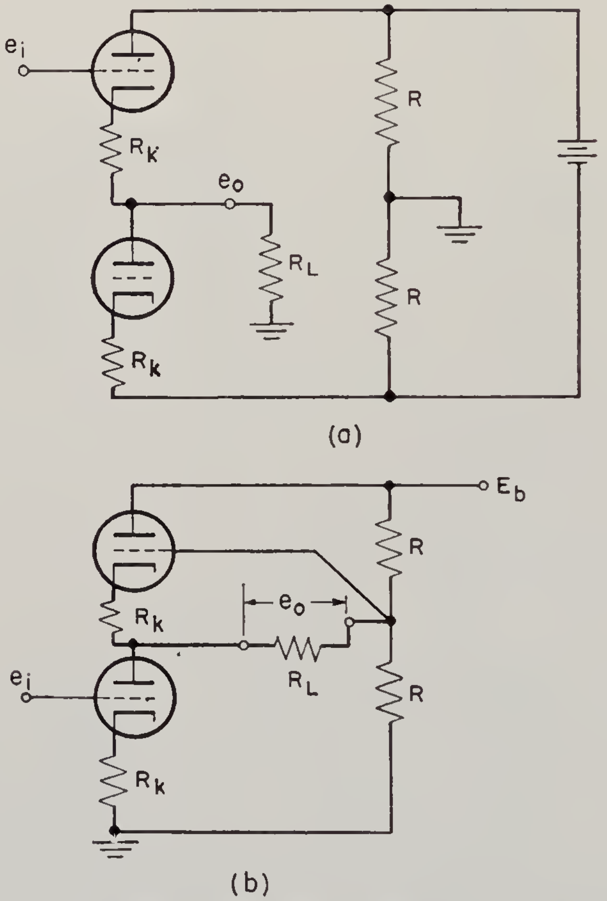
[1] Artzt, *op. cit.*

Special Circuits for the Compensation of Filament-voltage Changes. It has been mentioned in Sec. 5.1 that the d-c amplifier drift due to filament-voltage changes is particularly serious, since filament-voltage changes affect each amplifier stage exactly like input voltage changes. It is, therefore, customary to use some type of *compensating circuit* at least in the input stage of good-quality d-c amplifiers. The technique of *series compensation* has been described above; a number of other compensating circuits will be described in the following paragraphs.



(a)

(b)

FIG. 5.18. Bridge-balanced low-impedance driver circuits.

Figure 5.19*a* shows how the effects of filament-voltage variations in an amplifier tube V_1 may be at least partially compensated by means of a diode V_2 having the same filament voltage as V_1. When the filament voltage changes, the diode plate current flowing through the cathode resistor R_k changes accordingly. The resulting bias change tends to compensate for the effects of the original filament-voltage variation. Tubes with diodes using the same cathode as the triode or pentode amplifier tube V_1 are especially useful in this circuit, since the characteristics of the cathode circuits of the two tubes are more nearly identical in this case.[1] It is also possible to use the control grid of certain beam tetrodes, particularly of the type 6V6, instead of the diode plate; in this case, the screen grid functions as the control grid (see Fig. 5.19*b*). If the circuit constants are chosen correctly, quite effective compensation may be obtained, since the cathode of the diode and the cathode of the amplifier tube are actually identical in this circuit.

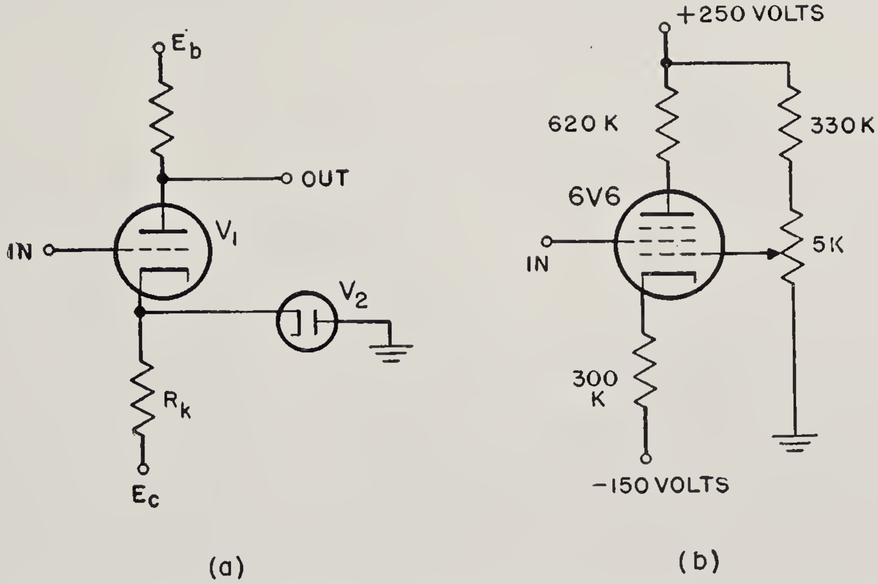


FIG. 5.19. Circuits for the compensation of the effects of filament-voltage changes.

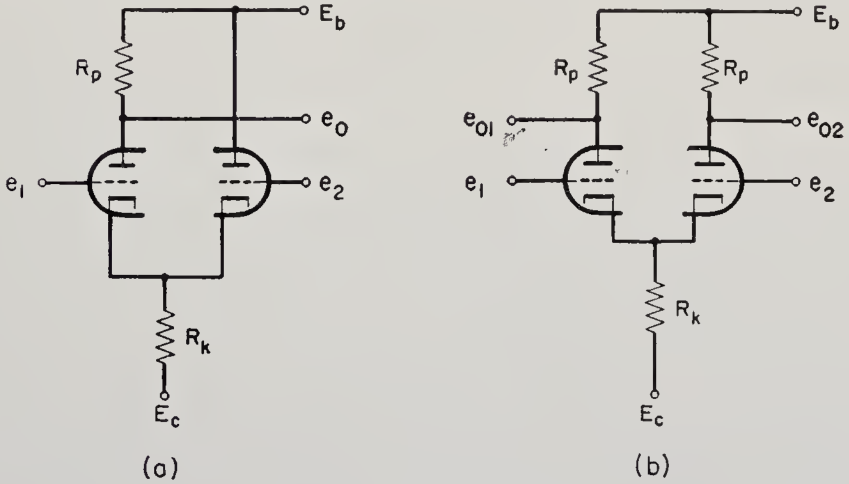


FIG. 5.20. Differential-amplifier circuits.

Probably the most commonly used circuits for compensation of the effects of filament-voltage changes and cathode emission are the differential amplifiers treated below.

The Use of Differential Amplifiers. In the two circuits shown in Figs. 5.20*a* and *b*, each of the two tubes acts as a cathode follower driving the cathode of the other tube. The changes in the output voltages of these circuits may be obtained as functions of the corresponding changes in the input voltages through the use of linear equivalent circuits in the manner outlined in Sec. 5.2. Assuming that the two tubes are identical types in each case, the analysis shows that for the circuit of Fig. 5.20*a*,

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[1] Gray, *op. cit.*

$$\begin{aligned}(e_o - e_{o0}) = {} & \frac{\mu}{1 + \dfrac{2r_p}{R_p} + \dfrac{r_p(r_p + R_p)}{R_pR_k(\mu + 1)}}(e_2 - e_{20}) \\ & - \frac{\mu + \dfrac{\mu}{\mu + 1}\dfrac{r_p}{R_k}}{1 + \dfrac{2r_p}{R_p} + \dfrac{r_p(r_p + R_p)}{R_pR_k(\mu + 1)}}(e_1 - e_{10}) \qquad (5.14)\end{aligned}$$

and for the circuit shown in Fig. 5.20*b*,

$$\left.\begin{aligned}(e_{o1} - e_{o0}) = {} & \frac{\mu R_p}{(R_p + r_p)\left[2 + \dfrac{R_p + r_p}{(1 + \mu)R_k}\right]} \\ & \left\{(e_2 - e_{20}) - (e_1 - e_{10})\left[1 + \frac{R_p + r_p}{(\mu + 1)R_k}\right]\right\} \\ (e_{o2} - e_{o0}) = {} & \frac{\mu R_p}{(R_p + r_p)\left[2 + \dfrac{R_p + r_p}{(1 + \mu)R_k}\right]} \\ & \left\{(e_1 - e_{10}) - (e_2 - e_{20})\left[1 + \frac{R_p + r_p}{(\mu + 1)R_k}\right]\right\}\end{aligned}\right\} \qquad (5.15)$$

In each case, e_{10} and e_{20} are the reference levels of the input voltages e_1 and e_2, respectively. From Eqs. (5.14) and (5.15), it may be seen that the output voltages of these circuits are, for sufficiently high values of μ and R_k, approximately proportional to the differences of the two input voltage changes. For this reason, such circuits are referred to as *differential amplifiers.* The circuit of Fig. 5.20*b* permits output voltages of either sign to be taken off. The flexible properties of differential amplifiers of this type make them extremely useful in many circuit applications. Some of their many interesting properties are listed below:

1. *Compensation of the effects of power-supply voltage changes in the input circuits.* Since the output voltages of differential amplifiers are approximately proportional to the differences between the two input voltages, changes in the input voltages due to power-supply voltage variations common to both *input* circuits will tend to cancel out. In order to make the output of a differential amplifier as closely proportional to the difference between the two input voltages, it is advisable to use tubes having a high amplification factor μ and to use a high value of the cathode resistor

R_k. The latter is made possible through the use of a large negative voltage for the cathode return. Otherwise, the plate currents may become too small, and the amplification factors of the tubes may decrease; it is also possible to replace R_k by the dynamic resistance of a tube in the manner shown in Fig. 5.15*c*. One might think that pentodes would be particularly useful as differential-amplifier tubes because of the high value of μ. The value of r_p for pentodes is, however, so high that Eqs. (5.14) and (5.15) show that pentodes are *not* preferable to high-μ triodes in this application. Specially matched double triodes, like the 6SU7 or 5691, are most suitable and also have a low negative-grid-current rating.

2. *Compensation of the effects of cathode-emission changes.* Changes in the cathode emission due to filament-voltage changes and other causes are equivalent to voltage changes common to both input circuits of differential amplifiers. They are, therefore, at least partially canceled. Figure 5.21 shows a special circuit for canceling the effects of filament-voltage changes more accurately (the so-called Miller compensator circuit).[1] It may be shown that the resistance below the cathode resistor tap in Fig. 5.21 should be equal to r_p/μ. In order to make the compensation reasonably permanent, the tubes should be aged for a while and cycled several times through the extremes of heater voltage before the adjustment is made.[2]

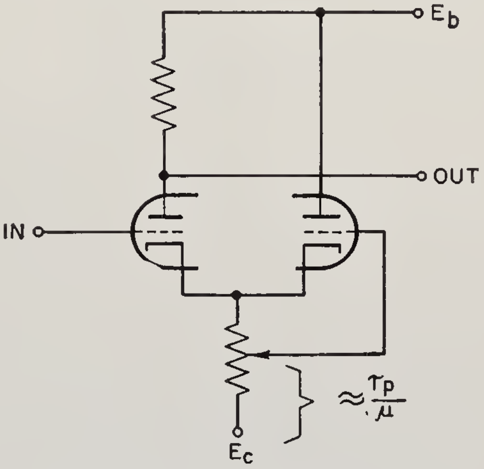

FIG. 5.21. Miller compensator.

The value of R_k is not critical but should be between two and eight times the value of r_p/μ. The adjustment is best made by observing the output voltage while changing the heater voltage.

3. *Compensation for the effects of changes in plate- and bias-supply voltages.* Figure 5.22 shows a differential amplifier arranged for compensation of the effects of changes in plate and bias voltages. If the same power supplies are used for all stages of an amplifier, the circuit of Fig. 5.22 may be arranged so as to serve as a compensating circuit for the entire amplifier. At the same time, the circuit still provides a measure of compensation for the effects of changes in its own cathode emission.

Generally speaking, the plate and bias supplies in d-c analog computers are well regulated for reasons other than the prevention of amplifier drift

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[1] Miller, *op. cit.*

[2] Gray, *op. cit.*

(see Sec. 7.4). Accordingly, circuits permitting compensation for plate-voltage variations are not too frequently used in computer amplifiers.

4. Another significant advantage of differential amplifiers is that *they permit the output voltage to be taken off either with or without a sign reversal referred to the input voltage.* This allows great flexibility in various applications, particularly in circuits involving feedback around one or more amplifier stages.

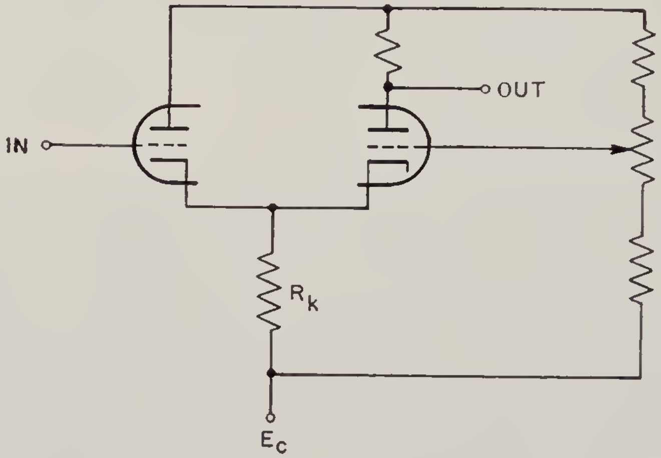

FIG. 5.22. Compensation for the effects of plate- and bias-supply voltage changes.

5. A differential amplifier makes it possible to *mix two voltages by taking their difference.* In this manner, one may, for instance, introduce an adjustable balance voltage for setting the output reference level of a d-c amplifier. Again, feedback voltages may be conveniently introduced at one of the two grids in the sense desired for either degeneration or regeneration.

6. In differential amplifiers, the plate currents of the two tubes are approximately equal and opposite, and for this reason, *cathode bias can be used in a differential stage without any corresponding loss in gain through degeneration.*

7. *Push-pull computer amplifiers* may be arranged by cascading symmetrical differential amplifiers. Push-pull amplifiers permit one to obtain output voltages of either sign; they may thus be used to eliminate phase-inverting amplifiers in computer setups.

Several examples of d-c amplifier designs utilizing these properties of differential amplifiers are presented in Sec. 5.5.

Output Circuits. As mentioned in Sec. 5.1, the output reference level of d-c amplifiers used in computer applications must be the same as the input reference level, usually zero with respect to ground. Figure 5.23 shows a number of output circuits suitable for computer amplifiers. Figure 5.23*a* represents the already familiar arrangement using a bias voltage E_c to bring the reference voltage of the output terminal to ground level.

The trouble with this arrangement is that, in an output stage, a reasonable amount of current may have to flow through the load, so that a considerable amount of power might be wasted in the coupling network. In such cases, the circuit of Fig. 5.23*b* is preferred. In this frequently used arrangement, the reference voltage of the output-tube plate itself is brought to ground level by the bias voltage E_{c2}. In this circuit, a separate bias supply must be provided to supply grid bias for the output tube unless cathode-resistor bias is used. This, in turn, would result in a decrease in gain together with an improvement in the linearity of the output stage.

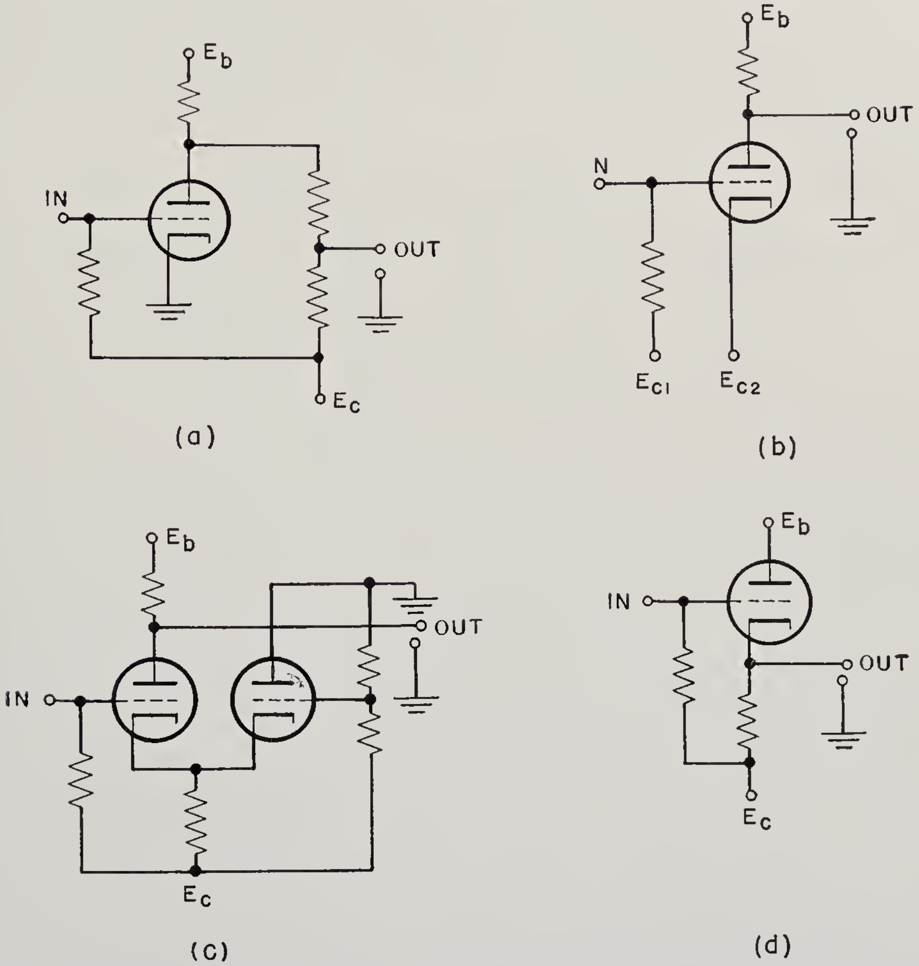

FIG. 5.23. Output circuits with zero d-c reference voltage with respect to ground.

The degeneration due to a cathode resistor may be avoided through the use of a *differential amplifier output stage,* as shown in Fig. 5.23*c*. *Cathode-follower output,* as shown in Fig. 5.23*d*, may be used with good results; such an output stage will, however, not contribute to the voltage gain of the amplifier. Finally, *series-balanced circuits* similar to those in Figs. 5.16, 5.17, and 5.18 are very useful as output stages; their current requirements are relatively low and do not vary appreciably over a wide range of output voltages (Figs. 5.29*c* and 5.41).[1]

5.4. STABILIZATION OF FEEDBACK AMPLIFIERS

Introduction. The operational amplifiers used in d-c analog computers are special types of feedback amplifiers. Thus, the parallel-feedback types of operational amplifiers constitute feedback loops like that shown in Fig. 5.24.

It is convenient to discuss the response of feedback amplifiers in terms of the responses to sinusoidal components of the applied voltages. It is possible to prove that *feedback amplifiers like that shown in Fig.* 5.24 *will break into uncontrolled oscillations if, for some range of frequencies, the phase shift inherent in the loop transfer function* $A\beta$ *deviates from* 180 *deg by more than* 180 *deg while the absolute value of* $A\beta$ *is equal to or greater than unity.*[2] Intuitively speaking, the feedback will then be sufficiently regenerative to make the amplifier unstable. In order to design amplifier circuits which eliminate the possibility of uncontrolled oscillations, it will be necessary to investigate the frequency-response characteristics of typical d-c amplifiers in detail.

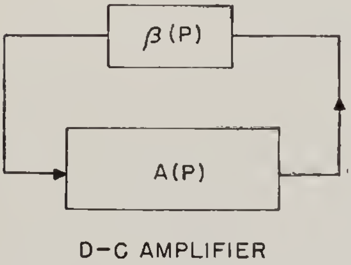

FIG. 5.24. Feedback loop formed by a parallel-feedback operational amplifier.

The Effects of Tube and Circuit Capacitances: Computation of Stage Transfer Functions. With d-c amplifiers, no oscillations due to excessive phase shifts are experienced at low frequencies, but careful design is necessary in order to prevent regeneration at frequencies higher than the actual operating frequencies. At such frequencies, the effects of tube and circuit capacitances must be taken into consideration.

A typical simple d-c amplifier stage like that shown in solid lines in Fig. 5.25*a* will be analyzed by reference to its linear equivalent circuit. It will be possible to analyze the high-frequency behavior of special d-c amplifier circuits such as cathode followers or differential amplifiers in an exactly analogous manner.

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[1] Revis, J. T., High Power from Miniature Tubes, *Electronic Design*, April, 1956.

[2] This may be derived from the general necessary and sufficient condition (given in H. W. Bode, *Network Analysis and Feedback Amplifiers*, Van Nostrand, Princeton, N.J., 1945) that, for stability, all roots λ_k of the characteristic equation must have positive real parts. Assuming that all circuit constants including the load and source impedances are given, the characteristic equation of the feedback loop shown in Fig. 5.24 is simply

$$1 - A(\lambda)\beta(\lambda) = 0 \tag{5.16}$$

The high-frequency equivalent circuit for the amplifier stage of Fig. 5.25*a* is shown in solid lines in Fig. 5.25*b*. This circuit is based on the familiar circuit of Fig. 5.7*b* with the addition of the capacitances C'_o and C'_i. C'_o is the output capacitance of the tube V_1 given in the tube manual plus an allowance of about 5 to 10 $\mu\mu$f for distributed circuit capacitance. C'_i is the input capacitance of the succeeding stage, also plus an allowance of 5 to 10 $\mu\mu$f for circuit capacitances. If V_2 is a triode, a portion of the input capacitance will be due to the Miller effect.[1] In this case

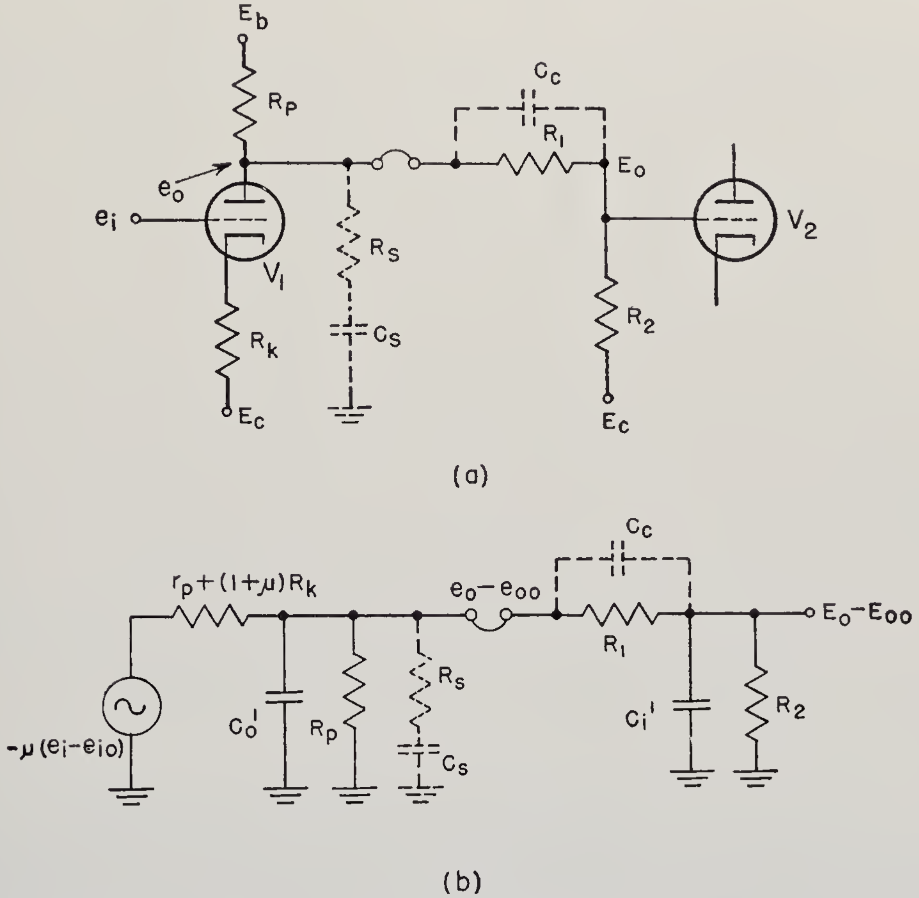

FIG. 5.25. D-c amplifier stage and linear equivalent circuit at high frequencies. The broken lines show high-frequency stabilization networks.

$$C'_i = \text{grid-cathode capacitance} + (\text{1-stage gain})\ \text{grid-plate capacitance} + \text{circuit capacitance}$$

The effects of capacitances in the *cathode and screen circuits* have been considered as negligible.

The circuit of Fig. 5.25*b* will act somewhat like two cascaded integrating networks (Sec. 4.3); *the effect of the tube and circuit capacitances will be to introduce gain changes and phase shift at the higher frequencies.* The circuit of Fig. 5.25*b* consists essentially of four admittances[1]

---

[1] *Miller effect* is a common term for the amplification of the grid-plate capacitance in certain amplifier stages. The effect is analogous to the operation of the parallel-feedback integrator discussed in Sec. 4.5.

$$\left.\begin{aligned} Y_1 &= \frac{1}{r_p + (1 + \mu)R_k} & \qquad Y_2 &= C_o'P + \frac{1}{R_p} \\ Y_3 &= \frac{1}{R_1} & Y_4 &= C_i'P + \frac{1}{R_2} \end{aligned}\right\} \tag{5.17}$$

arranged in the general circuit of Fig. 5.26. Since the currents flowing to any point of the circuit must add up to zero, one obtains the nodal equations

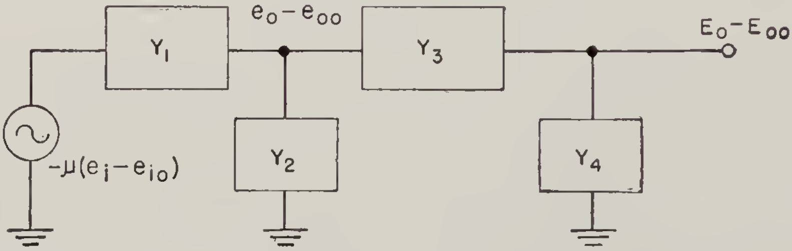

FIG. 5.26. More general equivalent circuit for an amplifier stage having no subsidiary feedback loop.

$$\begin{aligned} [(e_o - e_{o0}) + \mu(e_i - e_{i0})]Y_1 + (e_o - e_{o0})Y_2 + [(e_o - e_{o0}) - (E_o - E_{o0})]Y_3 \\ = 0 \qquad (5.18) \\ [(E_o - E_{o0}) - (e_o - e_{o0})]Y_3 + (E_o - E_{o0})Y_4 = 0 \end{aligned}$$

and hence the transfer function

$$\frac{E_o - E_{o0}}{e_i - e_{i0}} = -\mu \frac{Y_1Y_3}{Y_1Y_3 + Y_1Y_4 + Y_2Y_3 + Y_2Y_4 + Y_3Y_4} \tag{5.19}$$

Computation of the Amplifier Output Voltage and of the Feedback Transfer Function. In the computation of the amplifier output voltage and of the *feedback transfer function* $\beta(P)$, the effects of tube and circuit capacitances must also be taken into consideration. Figures 5.27*a* and *b* show linear equivalent feedback circuits for a *summing amplifier* and a *summing integrator*, respectively; it is seen that, in the computation of the loop transfer function $A\beta$, the output stage and the feedback networks of a parallel-feedback amplifier may be treated together just like the amplifier stages discussed above.

Properties of the Loop Transfer Function. The loop transfer function $A\beta$ will be the product of the transfer functions (5.19) for each stage; the amplifier output stage and the feedback networks will be considered together as one stage. $A\beta$, as a function of P, will be of the general form

---

[1] The *admittance* Y of any two-terminal circuit is the ratio of corresponding current changes through the circuit to voltage changes across the circuit. The admittance is thus the reciprocal of the *impedance* of the circuit.

$$A\beta = \frac{f_1(P)}{f_2(P)} \tag{5.20}$$

where $f_1(P)$ and $f_2(P)$ are polynomials in P. It is possible to show that[1]

1. The degree of the polynomial $f_2(P)$ is always greater than the degree of the polynomial $f_1(P)$.

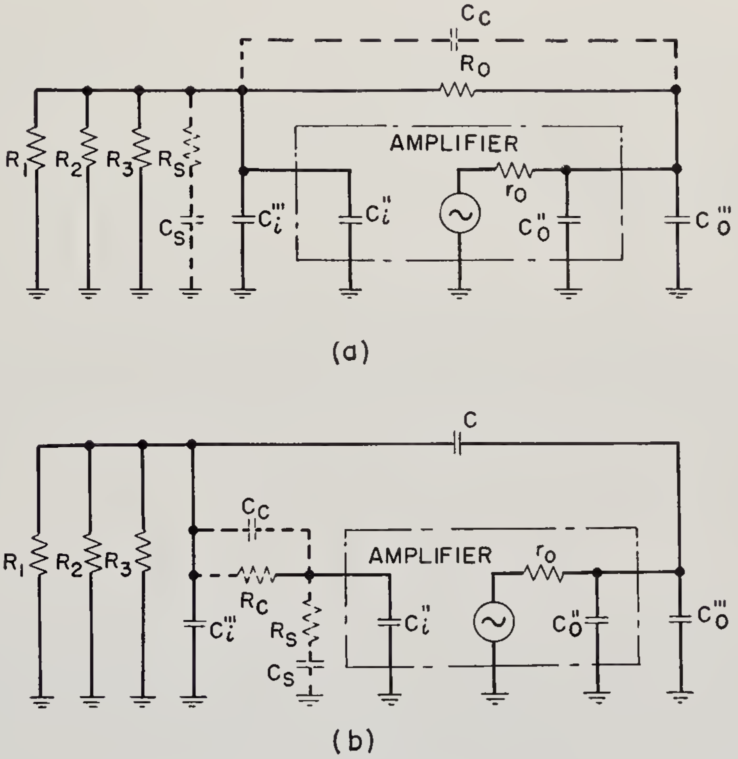

FIG. 5.27. Equivalent circuits for a summing amplifier (*a*) and a summing integrator (*b*). r_0 is the amplifier source impedance, and C_i'' and C_o'' are amplifier input and output shunt capacitances. C_i'', C_i''', and C_o''' include the effects of cables and patch bays, which may give rise to shunt capacitances as large as 1,500 $\mu\mu$f. Possible stabilization networks analogous to those in Fig. 5.25 are indicated in broken lines; sometimes C_i'' is increased artificially to improve stability.

2. If the interstage networks contain only resistances and capacitances, and if no subsidiary feedback loops are used, then the roots s_i of

$$f_1(P) = 0$$

as well as the roots s_k' of

$$f_2(P) = 0$$

will be real and negative. The s_i are the *zeros* and the s_k' are the *poles* of the loop transfer function $A\beta$.

The first theorem simply expresses the fact that the loop gain must eventually decrease as the frequency of the input voltage is increased.

---

[1] Bode, *op. cit.*

Equation (5.20) may be rewritten in the form

$$A\beta = \frac{-k(P - s_1)(P - s_2) \cdots}{(P - s_1')(P - s_2') \cdots} \tag{5.21}$$

where k is a positive real constant. For steady sinusoidal voltages represented by

$$\tilde{X} = \bar{X}e^{j\omega\tau}$$

the complex loop gain $\bar{A}\bar{\beta}$ is obtained by replacing each P in Eq. (5.21) by $j\omega$:

$$\bar{A}\bar{\beta} = \frac{-k(j\omega - s_1)(j\omega - s_2) \cdots}{(j\omega - s_1')(j\omega - s_2') \cdots} \tag{5.22}$$

The absolute value of $\bar{A}\bar{\beta}$

$$|\bar{A}\bar{\beta}| = k\sqrt{\frac{(\omega^2 + s_1^2)(\omega^2 + s_2^2) \cdots}{(\omega^2 + s_1'^2)(\omega^2 + s_2'^2) \cdots}} \tag{5.23}$$

is the rms or peak loop gain for sinusoidal voltages of circular frequency ω. The phase shift Φ corresponding to Eq. (5.22) is

$$\begin{aligned}\Phi = \pi + \left[\text{arc tan}\left(\frac{-\omega}{s_1}\right) + \text{arc tan}\left(\frac{-\omega}{s_2}\right) + \cdots\right] \\ - \left[\text{arc tan}\left(\frac{-\omega}{s_1'}\right) + \text{arc tan}\left(\frac{-\omega}{s_2'}\right) + \cdots\right]\end{aligned} \tag{5.24}$$

The equations (5.23) and (5.24) permit one to plot the loop gain and the corresponding phase shift[1] against the circular frequency ω or against each other in order to determine whether the feedback amplifier is stable. This will be the case if Φ does not differ from 180 deg by more than 180 deg while $|\bar{A}\bar{\beta}|$ is greater than unity.

High-frequency Stabilization Networks. Equation (5.24) shows that the phase shift associated with the loop transfer function (5.21) is intimately associated with the poles s_k' and zeros s_i of that function. Each pole s_k' contributes a phase lag Φ_k' of 45 deg at $\omega = -s_k'$. As ω increases further, this phase lag Φ_k' increases to 90 deg as given by

$$\Phi_k' = \text{arc tan}\left(\frac{\omega}{-s_k'}\right)$$

Although each of the terms $(j\omega - s_k')$ in the denominator of Eq. (5.22) also acts to decrease the loop gain with increasing frequency, this effect may not be sufficient to keep the amplifier stable if the d-c gain (gain for $\omega = 0$) is high. Each zero s_i of the function (5.21), on the other hand, contributes a negative phase lag which can partially cancel the phase lag due to the poles s_k'; each zero s_i contributes a (negative) phase lag

---

[1] The loop gain and the phase shift are not independent of each other; the relations existing between these two quantities have been made the basis of a very elegant graphical analysis method. See H. M. James, N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms*, MIT Radiation Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947.

$$\Phi_i = -\text{ arc tan}\left(\frac{\omega}{-s_i}\right)$$

equal to -45 deg for $\omega = -s_i$ and tending to -90 deg with increasing ω.

The values of the zeros s_i and the poles s_k' are relatively easily computed from the transfer functions of each stage. An inspection of the expression (5.19) for a stage transfer function shows that each stage of the type shown in solid lines in Fig. 5.25*a* can contribute two poles s_k'; zeros can be contributed only by feedback networks such as those of the integrator shown in Fig. 5.27*b*.

A practical method for canceling some of the phase lag due to the poles s_k' of the loop transfer function is to introduce suitable zeros artificially. The dotted lines in the d-c amplifier circuit of Fig. 5.25*a* and *b* show two *high-frequency stabilization networks* (lead networks) used for this purpose. The by-pass capacitor C_c modifies the admittance Y_3 in Eqs. (5.17), which becomes

$$Y_3 = \frac{1}{R_1} + C_cP \tag{5.25}$$

whereas the network made up of R_s and C_s changes the original value of the admittance Y_2 to

$$Y_2 = \frac{1}{R_p} + C_o'P + \frac{C_sP}{R_sC_sP + 1} \tag{5.26}$$

These two networks may also be used separately.

On substituting Eq. (5.25) into Eq. (5.19) it will be seen that C_c introduces a zero into the stage transfer function and thus also into the loop transfer function. The R_sC_s network will introduce a zero but also a new pole; in general, the latter may, however, be made to correspond to so high a circular frequency ω that it will do no harm.

It will be desirable to modify the circuits so that the values of the zeros and the first poles of the loop transfer function $A\beta$ alternate as shown by the sequence

$$s_1' < s_1 < s_2' < s_2 < s_3' \cdot \cdot \cdot$$

If this is true, and if there are less than three more poles than there are zeros, the phase shift will differ by less than 180 deg from 180 deg for all finite values of the circular frequency ω, and the amplifier will be stable for *all* negative values of the d-c gain. This goal cannot always be achieved in practice, but stabilization networks may be employed in the majority of cases to stabilize d-c amplifiers for all practical values of the d-c gain. A practical design method will be presented below.

Practical Design Method. The computation of the poles and zeros of each stage transfer function from the expression (5.19) is straightforward but somewhat tedious. The following is a simplified method for the design of high-frequency stabilization networks.

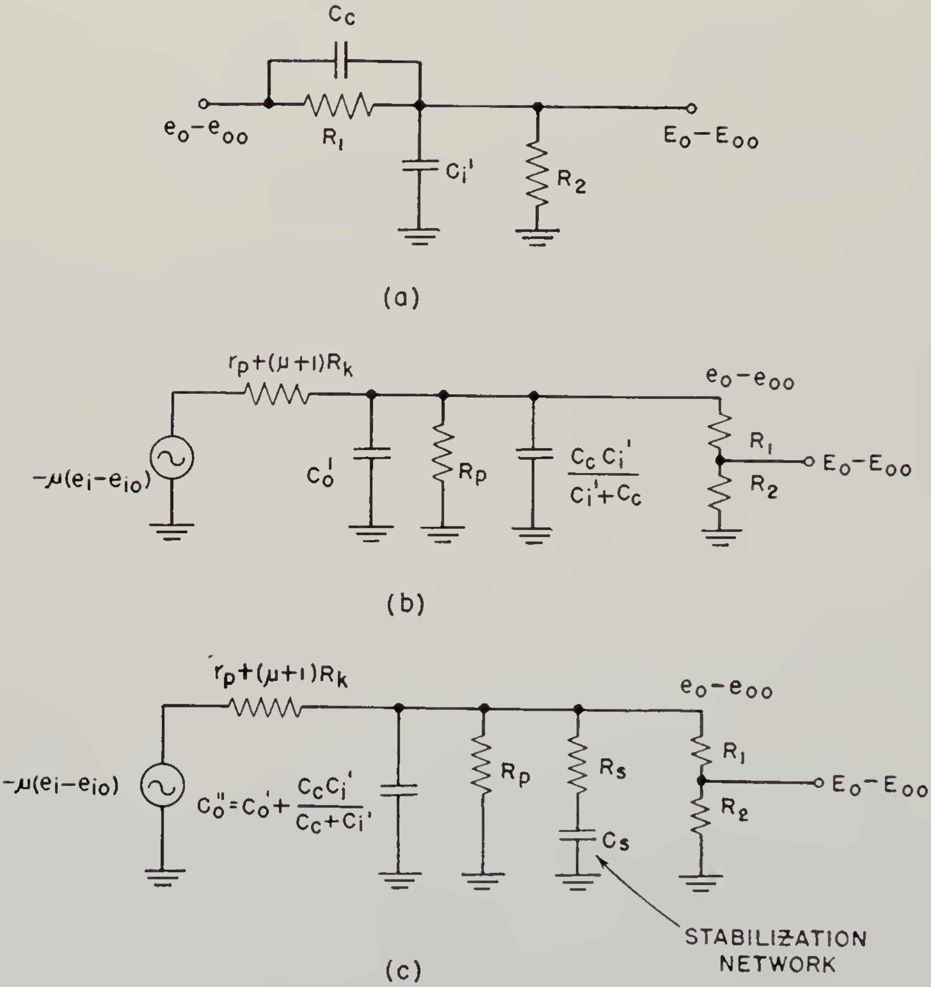

FIG. 5.28. Design of high-frequency stabilizing networks.

In single-stage amplifiers, no stabilization will be needed. In d-c amplifiers having two or more stages, *the first step is to eliminate the phase lag due to the interstage coupling networks* (R_1, R_2, and C_i' in Fig. 5.28*a*) by means of a by-pass capacitor C_c. The modified interstage coupling network, shown in Fig. 5.28*a*, has the transfer function

$$\frac{E_o - E_{o0}}{e_o - e_{o0}} = \frac{R_2}{R_1 + R_2} \frac{1 + R_1 C_c P}{1 + \dfrac{R_1 R_2}{R_1 + R_2} (C_c + C_i')P} \tag{5.27}$$

If the value of C_c is chosen so that

$$C_c = C_i' \frac{R_2}{R_1} \tag{5.28}$$

Eq. (5.27) reduces to

$$\frac{E_o - E_{o0}}{e_o - e_{o0}} = \frac{R_2}{R_1 + R_2} \tag{5.29}$$

The only effect of C_c and C_i' is now to add to the capacitance C_o' loading the first tube. The new simplified equivalent circuit for the stage is shown in Fig. 5.28*b*. In practice, the value of the capacitance C_c should be chosen slightly in excess of that given by Eq. (5.28), so that all phase lag due to the interstage network will be eliminated with certainty.

After this simplification it will be much easier to compute each stage gain. For a two-stage amplifier, no further stabilization will be necessary. For amplifiers having more than two stages, further stabilization may be needed. Figure 5.28*c* shows the simplified equivalent circuit of Fig. 5.28*b* with the addition of a stabilization network comprising the resistance R_s and the capacitance C_s. If Eq. (5.28) is assumed to hold, the stage transfer function is

$$\frac{E_o - E_{o0}}{e_i - e_{i0}} = \frac{-K\left(P + \dfrac{1}{C_sR_s}\right)}{P^2(r_p'R_sC_o''C_s) + P(R_sC_s + r_p'C_o'' + r_p'C_s) + 1} \tag{5.30}$$

where

$$C_o'' = C_o' + \frac{C_cC_i'}{C_i' + C_c}$$

$$r_p' = \frac{R_p[r_p + (1 + \mu)R_k](R_1 + R_2)}{R_p(R_1 + R_2) + [r_p + (1 + \mu)R_k](R_p + R_1 + R_2)}$$

$$K = \frac{\mu r_p' R_2 R_s C_s}{(R_1 + R_2)[r_p + (1 + \mu)R_k]}$$

This is of the form

$$\frac{E_o - E_{o0}}{e_o - e_{o0}} = \frac{-k(P - s_1)}{(P - s_1')(P - s_2')} \qquad \text{with} \qquad s_1 = -\frac{1}{R_sC_s} \tag{5.31}$$

It is seen that one zero and two poles will be associated with each stage incorporating stabilization networks of the type described if (5.28) holds exactly (in general, there will be two zeros and three poles).

If it is permissible to assume that

$$R_s \ll r_p' \qquad \text{and} \qquad C_o'' \ll C_s \tag{5.32}$$

the poles s_1' and s_2' are given by

$$s_1' = -\frac{1}{R_s C_o''} \qquad s_2' = -\frac{1}{r_p' C_s} \tag{5.33}$$

With the location of the poles and zeros given by such relatively simple formulas, it is not too difficult to choose the correct values of the circuit constants for each stage.

Numerical examples showing the design of high-frequency stabilizing networks are found in several textbooks on amplifier design.[1] As an interesting exercise, the reader might compute the zeros and poles of the loop transfer function for a summing amplifier using the circuit of Fig. 5.33.

All stabilization networks for feedback amplifiers must be designed with large safety factors (perhaps 10-db amplitude margin) in order to allow for tolerances in the amplifier components as well as for the usually rather rough estimates of the distributed capacitances. Each circuit must be tested for stability; if the initial design was sound, any remaining oscillations can usually be eliminated experimentally by shunting one or more of the interstage coupling resistors (R_1 and R_2 in Fig. 5.25*a*) with small capacitors.

Whenever stabilizing networks are introduced in operational amplifiers, care should be taken that they do not alter the transfer function of the operational amplifier within the range of expected signal frequencies.

Stability Analysis of More Complicated Circuits. In principle, all linear feedback circuits may be analyzed by means of *equivalent circuits* in the manner outlined above. A number of elegant graphical methods have been developed to facilitate such analyses; the reader is referred to the textbooks[2] on this subject. A detailed discussion, *e.g.*, of the stability of *regenerative amplifiers*, is beyond the scope of this book.[3] One can often combat a tendency to high-frequency instability in regenerative d-c amplifiers by decreasing the amount of regeneration at the higher frequencies.

Over-all Stability of Computer Setups. The stability of individual computing elements is no guarantee of the high-frequency stability of computer setups consisting of cascaded computing elements. Quite frequently, loops of computing elements with loop gains in excess of 20 will break out into uncontrolled high-frequency oscillations. This is due to the addition of small phase shifts associated with each computing element and possibly also to stray coupling between computing elements. These considerations establish limitations of the computer setups possible, especially when fast time scales are used.

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[1] Valley and Wallman, *op. cit.;* Bode, *op. cit.*

[2] *Ibid.*

[3] In amplifiers containing subsidiary feedback loops, the poles and zeros of the loop transfer function need no longer be real but may occur in complex conjugate pairs. Each pair can give rise to phase shifts in excess of 90 deg.

5.5. PRACTICAL EXAMPLES OF COMPUTER-AMPLIFIER DESIGN

The following examples[1] of practical d-c amplifier circuits used in computer applications will serve to illustrate the design principles presented in the preceding sections. Examples of computer amplifiers using *automatic balancing* are shown in Sec. 5.7. A brief discussion of the *physical construction* of computer amplifiers is given in Sec. 5.9.

D-c Amplifiers with a Single Voltage-amplifier Stage. Figure 5.29*a* shows a *simple pentode amplifier* suitable for use in applications which do not require great accuracy, *e.g.*, in simple repetitive computers. The gain of this circuit is about 150. No special circuits for drift reduction are employed.

The *regenerative integrator circuit* shown in Fig. 5.29*b* was developed at the Massachusetts Institute of Technology[2] and is capable of good integration. The circuit may be quite susceptible to drift and distortion; in the version shown, moreover, the output voltage reference level is not the same as the input reference level. But with a suitable clamping device (see Sec. 8.5), this circuit might make a satisfactory integrator for a low-cost repetitive computer; in this application, only one power supply is needed.

The principles of Figs. 5.29*a* and 5.29*b* are combined in the Donner[3] *regenerative d-c amplifier* shown in Fig. 5.29*c*. This simple and useful circuit is a modified version of a design developed at the U.S. Naval Ordnance Test Station, Inyokern,[4] Calif., and widely used as a low-cost operational amplifier at the University of California in Los Angeles and in Beckman computers (see also Sec. 8.4). A new Donner feature is the series-balanced output cathode follower, which reduces current requirements and, to some extent, amplifier drift.

Multistage-amplifier Circuits Requiring Only Two Power Supplies. The simple two-tube d-c amplifier circuit shown in Fig. 5.30*a*, developed at the University of Michigan,[5] incorporates a differential-amplifier input stage for drift reduction, a triode voltage-amplifier stage, and a cathode-follower output stage. Only one tube type is used, and only two power supplies are needed.

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[1] All amplifier specifications presented in this chapter are based on manufacturers' ratings.

[2] Greenwood, Holdam, and MacRae, *op. cit.*

[3] Donner Scientific Company, Berkeley, Calif.

[4] U.S. Patent No. 2,581,456, granted January 8, 1952, to I. H. Swift. See also *NOTS Inyokern Tech. Memo* 4529-23, dated Nov. 16, 1949.

[5] McDonald, D., Analog Computer for Servo Problems, *Rev. Sci. Instr.*, **21**: 154, 1950.

Each of the Philbrick[1] two-tube amplifiers shown in Figs. 5.30*b* and 5.30*c* employs a differential-amplifier input stage and a regenerative output stage. These amplifiers are neatly packaged plug-in units (Fig. 5.31*a*); Fig. 5.31*b* shows four such amplifiers plugged into a larger chassis containing the requisite power supplies, balancing potentiometers, and a simple patch bay. The Philbrick plug-in amplifiers are suitable for repetitive analog computers (Sec. 8.5) as well as for "slow" d-c analog computers and may serve as subassemblies in various control devices such as power-supply regulators (Sec. 7.4).

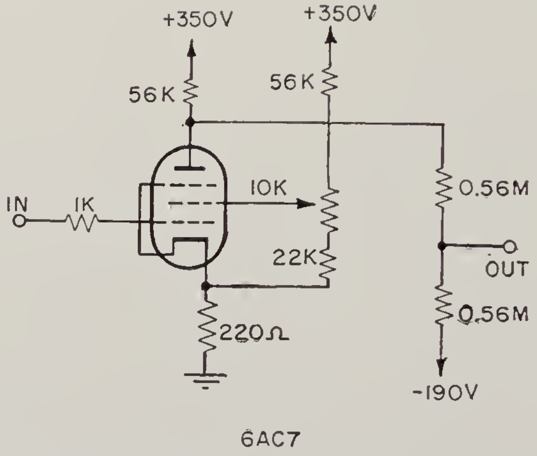

Fig. 5.29*a*. Simple one-tube d-c amplifier.

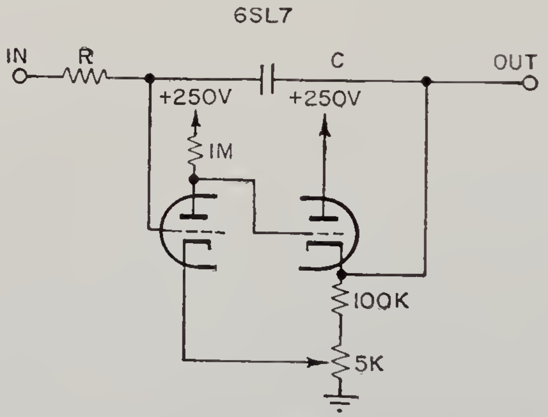

Fig. 5.29*b*. Simple regenerative integrator with cathode-follower output. (Output is not at the same reference level as input.)

The low-cost analog computer of the Boeing Aircraft Company (Sec. 8.4) uses the d-c amplifier circuit shown in Fig. 5.32. This is a particularly interesting design comprising two differential-amplifier stages which permit drift reduction, cathode bias without any loss in gain, and regenerative feedback around the second stage. Cathode-follower output employing only one section of the output 6SN7 is used ordinarily; by using both sections of the 6SN7 in parallel, it is possible to drive as many as twelve 50,000-ohm potentiometers through the output-voltage range of plus or minus 50 volts. The amplifier is said to be capable of driving a Brush recording oscillograph directly (without a recorder driver) with fair accuracy. The designers have made provisions for connecting optional automatic balancing units (see Sec. 5.6) to the amplifier. Especially with the addition of such a device, this d-c amplifier is capable of excellent performance in computer applications. Note that the interstage by-pass capacitors (see Sec. 5.4) are the only stabilization networks needed in this circuit. The frequency response of this all-triode amplifier is said to make it suitable for use in repetitive computers (at a repetition rate just sufficient for eye persistence) as well as in "slow" d-c analog computers (see also Fig. 1.16).

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[1] George A. Philbrick Researches, Inc., Boston, Mass.

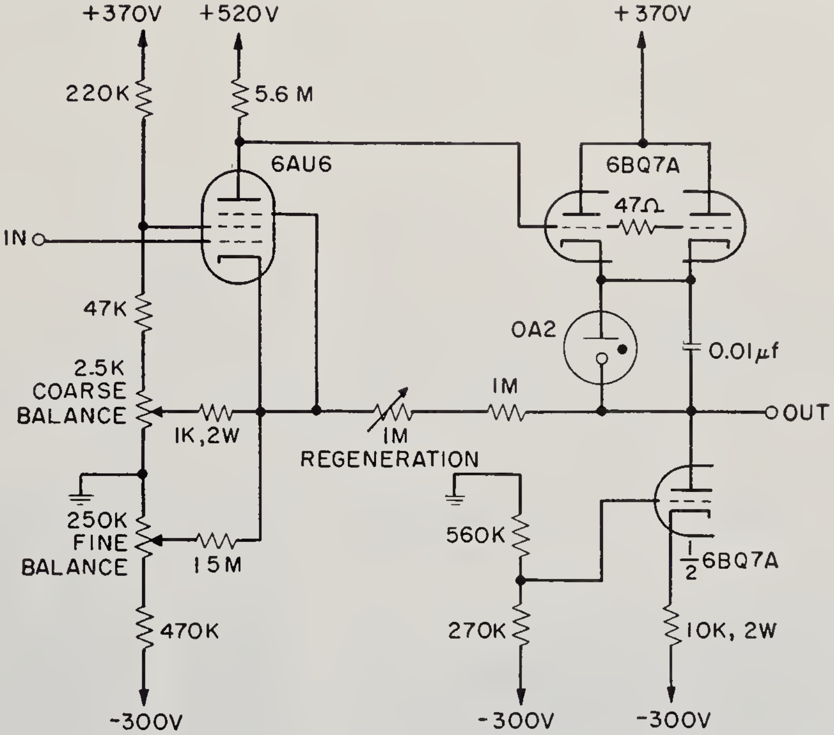


Fig. 5.29*c*. This practical circuit combines the features of Figs. 5.29*a* and 5.29*b*. Gain, 1,000 without regeneration and over 10,000 with regeneration; output range, plus and minus 100 volts at 5 ma; grid current, less than 10^{-3} μa; one-hour offset voltage, 4 mv referred to input (Donner Scientific Company).

Examples of High-gain D-c Amplifiers Using Three Power Supplies. In the amplifier circuit of Fig. 5.33, high voltage gain is achieved without the use of regeneration through the use of a pentode in the second stage and a beam-power-tube output stage which also has some voltage gain. The output-voltage range is plus or minus 100 volts with a load of about 15,000 ohms. A differential-amplifier stage arranged as a Miller com-

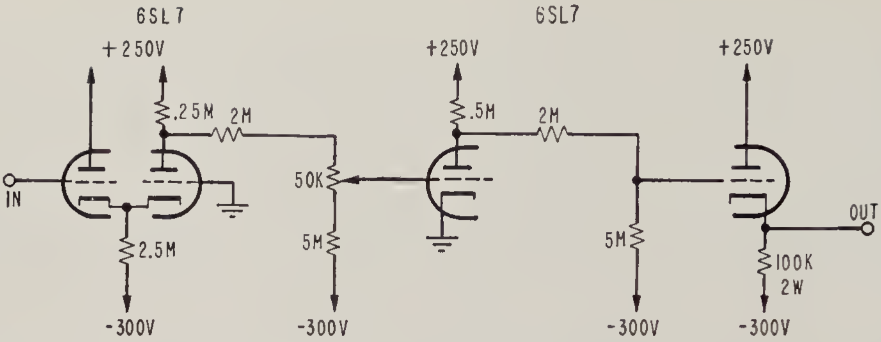


FIG. 5.30*a*. University of Michigan computer amplifier. Gain, about 700; source impedance, 700 ohms; output range, plus and minus 40 volts at 1.6 ma; long-term drift (5 hours), about 1 mv.

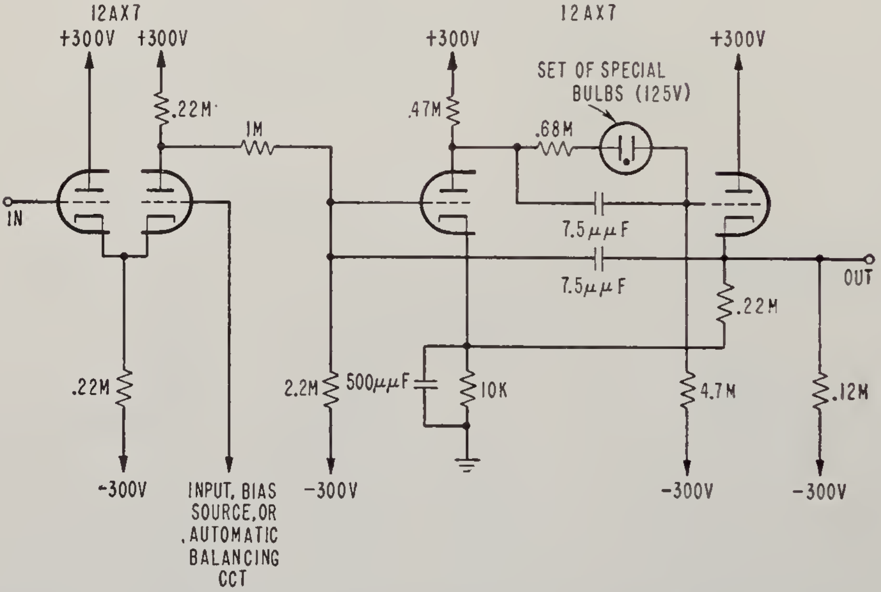


FIG. 5.30*b*. This regenerative computer amplifier is commercially available as a low-cost miniaturized plug-in unit (see also Fig. 5.31). D-c gain, 15,000; output range, plus and minus 50 volts at 1 ma; long-term drift (24 hr), 5 mv without automatic balancing. An accessory automatic balancing circuit of the chopper-stabilization type is available as a similar plug-in package (George A. Philbrick Researches, Inc.).

pensator (see Sec. 5.3) is used as the input stage and serves to minimize the effects of cathode-emission changes. The plate voltages for the input stage are kept low in order to minimize grid current.

Three high-voltage power supplies are required for the operation of this amplifier; the plus 75 volts indicated in the diagram are obtained from the plus 350-volt supply by means of a voltage divider using a VR75 voltage-regulator tube. This amplifier was designed by the Bell Telephone Laboratories for the M-9 fire-control computer; this design has served as a model for many other computer-amplifier circuits. The gain obtainable with such arrangements varies from 10,000 to 75,000 without the use of regeneration.

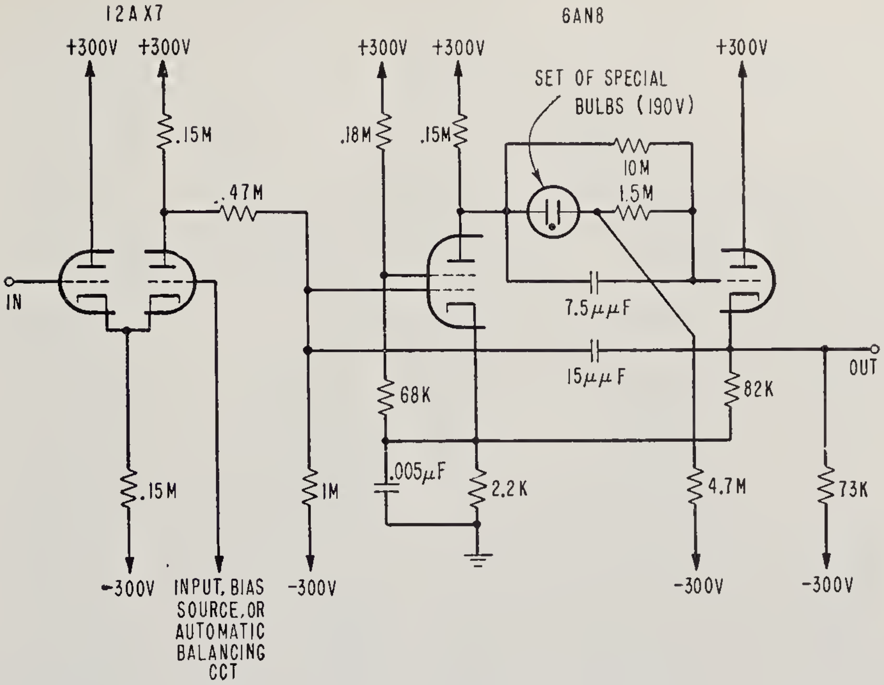
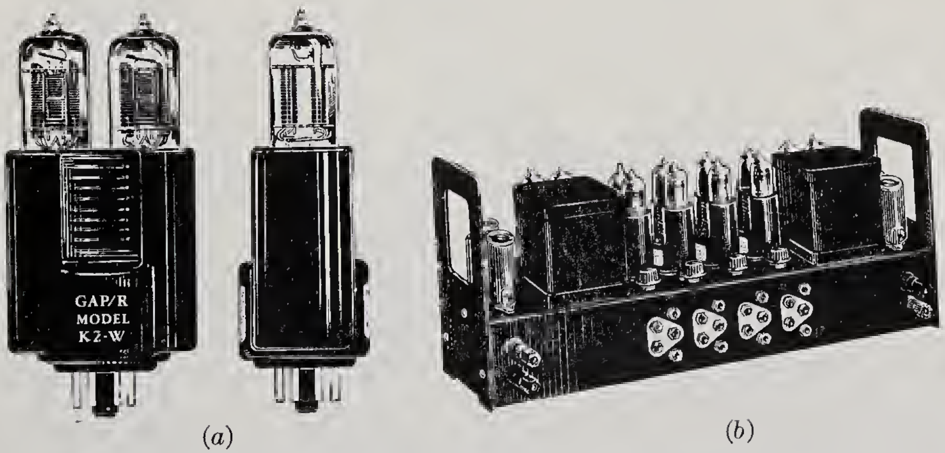


FIG. 5.30*c*. A somewhat higher-powered plug-in amplifier circuit (George A. Philbrick Researches, Inc.). D-c gain, 30,000; output range, plus and minus 100 volts at 3 ma; long-term drift (24 hr), 5 mv without automatic balancing.



(*a*) (*b*)

FIG. 5.31. Philbrick plug-in amplifier (*a*) and operational fourfold (*b*).

The Goodyear[1] d-c amplifier circuit of Fig. 5.34 employs a differential-amplifier input stage with low plate voltages to minimize grid current. Two triode voltage-amplifier stages and a triode output stage take the place of the screen-grid-tube circuits used in Fig. 5.33.

A second Goodyear amplifier, shown in Fig. 5.35, is of interest because of its series-compensated input stage. Two cascaded differential-amplifier stages are used to amplify the difference between the output voltage

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[1] Goodyear Aircraft Corporation, Akron, Ohio.

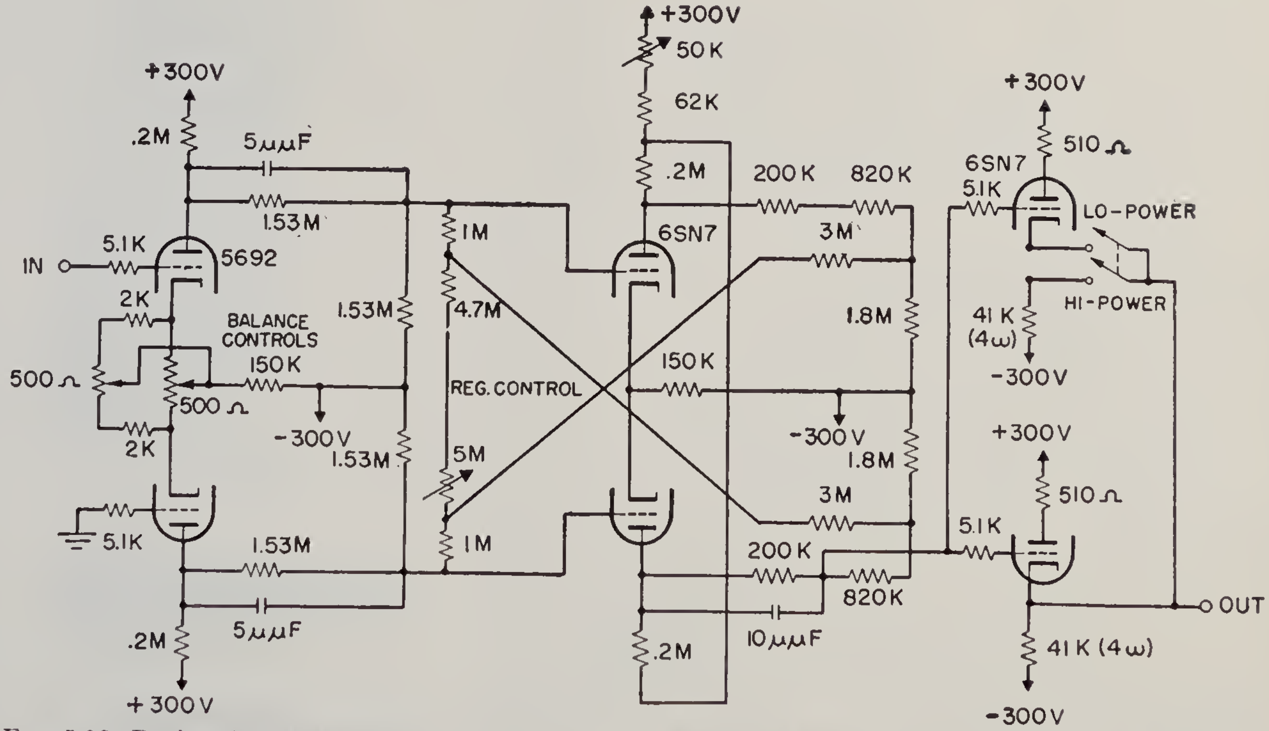

FIG. 5.32. Boeing Airplane Company computer amplifier. D-c gain in excess of 10^5 with regeneration; output range, plus and minus 50 volts at 5 ma (low-power setting) or 8 to 10 ma (high-power setting); grid current less than 10^{-4} μa; long-term drift (8 hr) 10 mv without automatic balancing. Automatic balancing units using either chopper or commutator stabilization (Sec. 5.6) are available as accessory units.

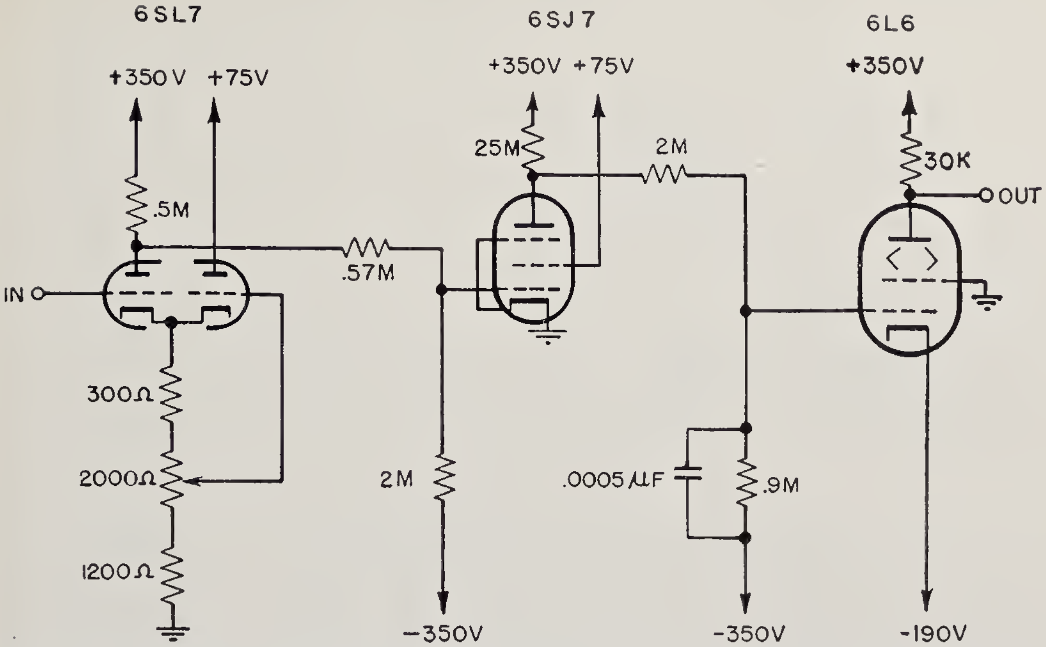

FIG. 5.33. Computer amplifier designed by the Bell Telephone Laboratories for the M-9 fire-control computer.

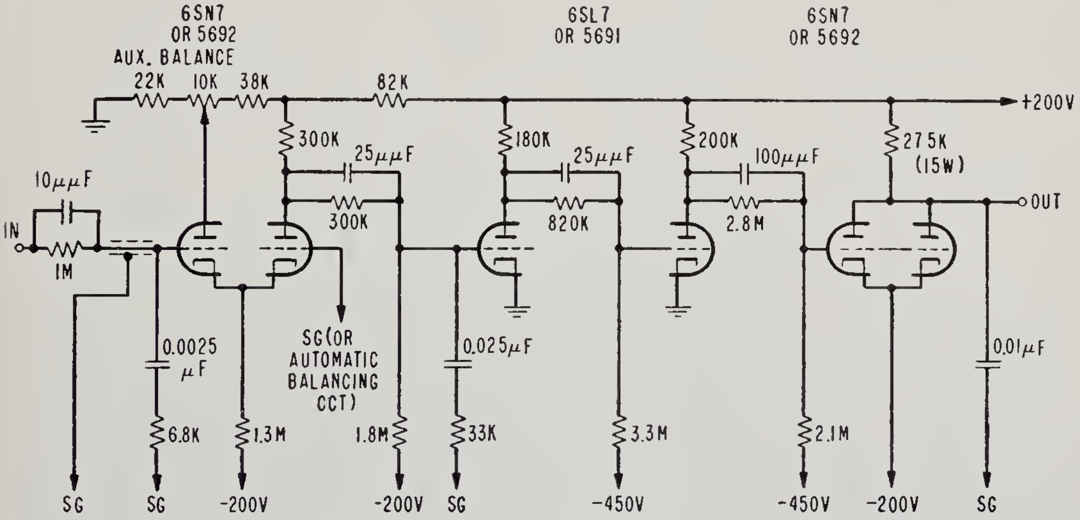

FIG. 5.34. Goodyear Aircraft Corporation computer amplifier.

of the series-compensated stage and the corresponding reference level. The output tube is a pentode. The input stage is operated at reduced filament voltage for improved stability and tube life[1] without any sacrifice in gain. The grid current in the input stage is minimized through reduced plate voltage and through the use of a medium-μ tube; the grid current in this stage will be less than 5×10^{-5} μa. In spite of the fact that no automatic balancing is used, the average long-time drift of the

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[1] Maintenance of the Electronic Analog Computer, *Report GER*-3046, Goodyear Aircraft Corporation, Akron, Ohio, July 15, 1949.

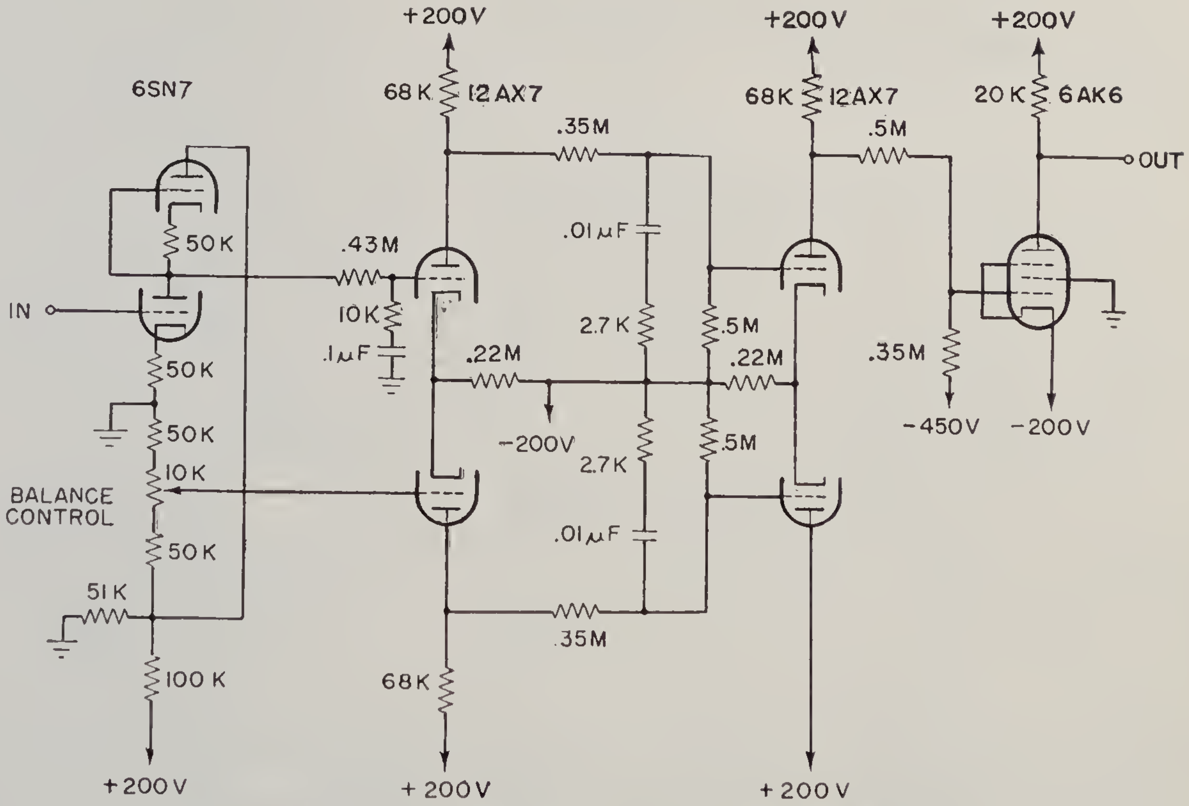


FIG. 5.35. Goodyear Aircraft Corporation computer amplifier.

amplifier shown in Fig. 5.35, used with the proper power supplies, is said to be less than 200 μv/hr (referred to the input), while the short-time drift is of the order of 50 μv in 5 min.

The effect of the high-frequency stabilization networks used in the amplifier circuits of Figs. 5.34 and 5.35 is essentially like that of the networks discussed in Sec. 5.4, although the arrangement is not exactly the same. The gain A of each Goodyear amplifier drops from its d-c value to unity at about 100 kc. When such an amplifier is used as a unity-gain phase inverter (Fig. 1.5*b*) with $R_1 = R_o = 1$ megohm, the over-all phase shift is less than 1 deg at 30 cps. The d-c amplifier circuits of Figs. 5.34 and 5.42, which are designed for automatic balancing (Sec. 5.6), have replaced the circuit of Fig. 5.35 in the Goodyear Electronic Analog Computer (Sec. 8.4).

5.6. IMPROVED D-C AMPLIFIER DESIGN: AUTOMATIC BALANCING CIRCUITS

Introduction. The development of improved d-c amplifiers and d-c analog computers has received a tremendous impetus through the invention of *automatic balancing circuits* which make it possible to obtain

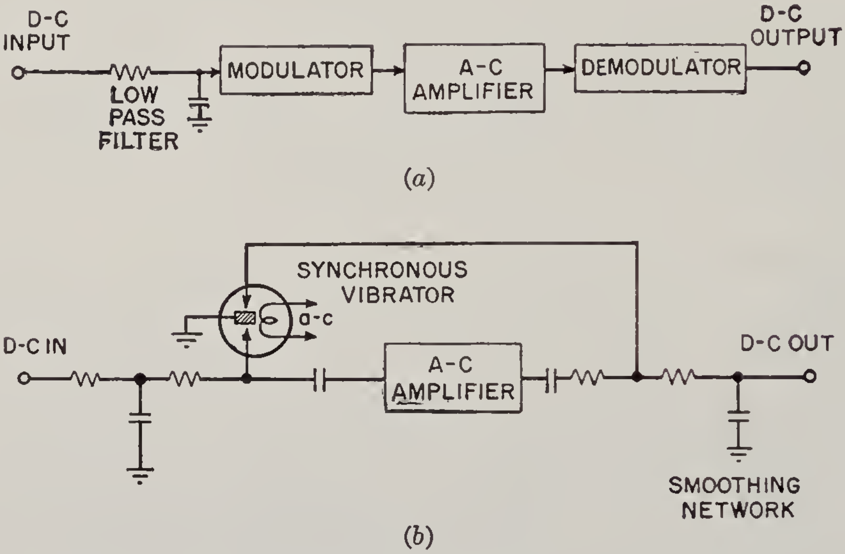
1. A reduction of d-c amplifier drift (see Sec. 5.1) by a factor of 3 to 20.
2. An increase of the d-c amplifier gain at low frequencies by a factor of 1,000 to 5,000, resulting in decreased distortion and improved accuracy of integration.
3. The elimination of the necessity for frequent manual balancing. Balancing controls may be eliminated entirely at the price of a very slight reduction in accuracy.

Such possibilities permit striking improvements in the design of d-c analog computers. This section will present a discussion of the principles of automatic balancing circuits as well as a number of practical examples.

Modulated-carrier-type Circuits for D-c Amplification. Amplifier circuits of the modulated-carrier type are frequently used instead of conventional d-c amplifiers in connection with various instrumentation applications.

Figure 5.36*a* shows the block diagram of such a circuit. The d-c input is used to modulate an a-c "carrier" voltage, so that the a-c amplitude is proportional to the d-c input voltage. The modulated carrier is amplified by means of a conventional a-c amplifier. The amplified alternating current is then demodulated to yield an amplified d-c output voltage proportional to the original d-c input voltage.

Figure 5.36*b* illustrates the use of a single *synchronous vibrator or chopper* for both modulation and demodulation. The d-c input voltage is interrupted periodically by the chopper relay, so that a modulated a-c voltage is obtained at the amplifier input. The amplifier output voltage is rectified synchronously by the same vibrator; the d-c output voltage is then smoothed by a smoothing network. It should be noted that such a modulated-carrier-type amplifier is *phase-sensitive;* if the sign of the input d-c voltage is reversed, the phase of the modulated a-c voltage changes by 180 deg, and the sign of the d-c output voltage will also reverse in the correct manner.



(*a*)

(*b*)

Fig. 5.36. Principle of the modulated-carrier-type d-c amplifier. In the circuit of (*b*), a single synchronous vibrator serves as both a phase-sensitive modulator and demodulator.

Modulated-carrier-type amplifiers permit d-c amplification with very substantial gain, without the use of carefully regulated power supplies. The use of synchronous vibrators in the manner shown in Fig. 5.36*b* permits one, moreover, to obtain d-c amplification with very low drift.

A synchronous vibrator is essentially an a-c-driven relay which can perform switching operations at carrier frequencies between 0 and 600 cps. Aside from possible contact and thermoelectric potentials, the main source of error voltages in synchronous vibrators is stray coupling of the driving a-c voltage into the signal circuit. This effect must be minimized by careful shielding and also by appropriate phasing of the driving voltage; it can also be reduced by grounding the adjustable center tap of a potentiometer connected across the relay coil. It may be useful to bring the coil terminals out through the top of the converter case, well away from the signal-carrying leads. Finally, it is possible to make the contact reed of a synchronous vibrator vibrate at twice the excitation frequency applied to the relay coil; such an arrangement will further reduce the effects of a-c pickup at the excitation frequency. It is also helpful to use an excitation frequency different from the a-c line frequency.

The value of possible noise and offset-voltage output of a synchronous vibrator depends on the impedance level at which the device is operated and may be less than 2 μv for a synchronous vibrator operated into the low-impedance primary of an input transformer and 2 to 5 μv for a synchronous vibrator operated into a vacuum-tube grid at an impedance level of the order of 1 megohm. It is possible to replace the synchronous vibrator by a cam-operated or rotating switch actuated periodically through a high-speed motor. Offset and noise voltages lower than 0.005 μv have been obtained with an input resistance of 5 ohms.[1] It is desirable to obtain vibrator excitation from a low-impedance source (step-down transformer).

The range of signal frequencies usable with modulated-carrier-type amplifiers is restricted to frequencies lower than the carrier frequency by about a factor of 10. If modulated-carrier-type amplifiers are to be employed in feedback circuits, the high-frequency components must be eliminated through the use of low-pass filters (indicated in Fig. 5.36). Otherwise, the high-frequency components of the signal and noise would not modulate the carrier properly but might be amplified in the a-c amplifier and give rise to spurious high-frequency feedback. Since synchronous vibrators cannot be operated consistently at carrier frequencies much higher than 400 cps, it has not been possible to employ modulated-carrier-type amplifiers using synchronous vibrators directly in high-gain operational amplifiers for computer applications. Higher carrier frequencies can be utilized by replacing the synchronous vibrators with diode[2] or transistor[3] switching circuits. The offset voltages obtainable with the best circuits of these types (of the order of 1 mv) are not too much lower than those of high-quality d-c amplifiers. Again, the design of stable operational amplifiers would require megacycle-high carrier frequencies with possible phase-shift and shielding problems. The direct application of modulated-carrier-type amplifiers in d-c analog computers is therefore not practical.

It is conceivably possible to utilize a modulated-carrier-type amplifier channel employing a synchronous vibrator to amplify the d-c and low-frequency components of the signal and to amplify the higher-frequency components in a parallel a-c amplifier channel. Such developments have been rendered obsolete by the invention of the automatic balancing circuits described below.

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[1] Liston *et al.*, *Rev. Sci. Instr.*, **17**: 194, 1946.

[2] See B. Chance, F. C. Williams, V. Hughes, D. Sayre, and E. F. MacNichol, *Waveforms*, Chap. 12, MIT Radiation Laboratory Series, Vol. 19, McGraw-Hill, New York, 1949, for a discussion of switch-type modulators.

[3] Bright, R. L., and A. P. Kruper, Transistor Choppers for Stable D-c Amplifiers, *Electronics*, April, 1955.

Automatic Balancing Circuits Based on Chopper Stabilization.[1] Figure 5.37*a* shows the block diagram of the basic automatic balancing circuit. A modulated-carrier-type amplifier using a synchronous vibrator is ingeniously combined with a conventional d-c amplifier in such a manner that the freedom from drift of the former and the superior high-frequency response of the latter are both realized. In addition, the circuit has a vastly increased forward gain at low frequencies; this feature is particularly useful for accurate integration (see Sec. 4.5). Circuits of this type can be adapted to almost any existing computer d-c amplifier.

The time constant R_IC_I of the low-pass filter or integrating network comprising the resistance R_I and the capacitance C_I in Fig. 5.37*a* is usually chosen so that

$$R_IC_I \geq \frac{1.5}{f_c} \tag{5.34}$$

where f_c is the carrier frequency to be used in the a-c amplifier (f_c will be between 60 and 500 cps).

The high-frequency components of the error voltage E are not affected by the automatic balancing circuit but are amplified by the d-c amplifier as in any operational-amplifier circuit. The high-frequency forward gain of the operational amplifier shown is thus simply

$$A_{HF} = A_1 \tag{5.35}$$

where $A_1 < 0$ is the gain of the d-c amplifier channel.

If the circuit shown in the solid lines in Fig. 5.37*a* is used, the low-frequency components of the error voltage E are also amplified by the d-c amplifier. Much more significantly, however, components of the error voltage E of sufficiently low frequencies to pass easily through the low-pass filter will be amplified by the modulated-carrier channel and the d-c amplifier in *cascade*. The low-frequency forward gain is thus seen to be

$$A_{LF} = A_1 + A_1A_2 < 0 \tag{5.36}$$

where $-A_2 < 0$ is the gain of the modulated-carrier channel.

The modulated-carrier channel precedes the d-c amplifier as far as the d-c and low-frequency components of the signal are concerned. Since this channel is not affected by voltage changes in the high-voltage and

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[1] Williams, A. J., R. E. Tarpley, and W. R. Clark, *Trans. AIEE*, **67**: 47, 1948; Goldberg, E. A., Stabilization of D-c Amplifiers, *RCA Rev.*, **11**: 296, 1950; McCool, W. A., Analysis of Zero Stabilization of Wide-Band D-c Amplifiers, *Report of NRL Progress*, February, 1951; Bradley, F. R., and R. McCoy, Driftless D-c Amplifier, *Electronics*, **25**: 122, 1952. To the best of the authors' knowledge, automatic balancing circuits were first used in the RCA TYPHOON computer. See also U.S. Patents No. 2,685,000 (A. W. Vance) and No. 2,684,999 (E. A. Goldberg and J. Lehmann), both assigned to Radio Corporation of America.

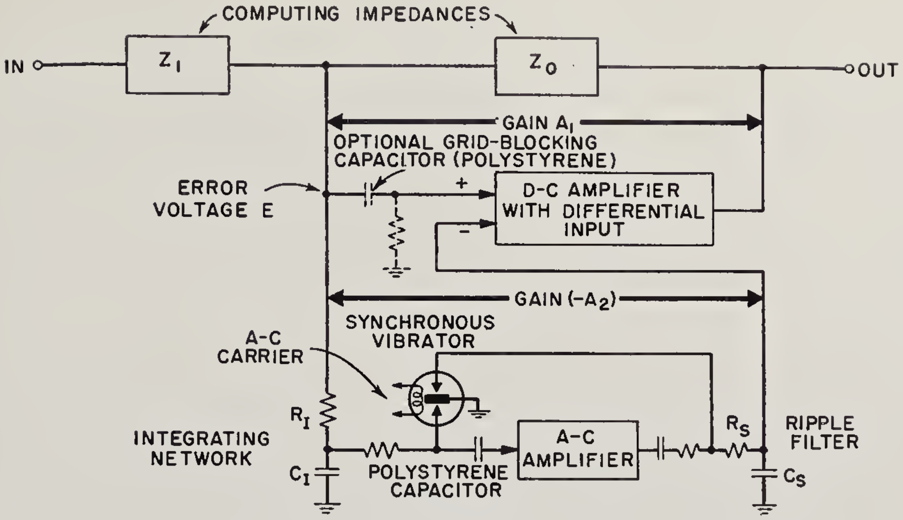

FIG. 5.37*a*. D-c amplifier with chopper stabilization.

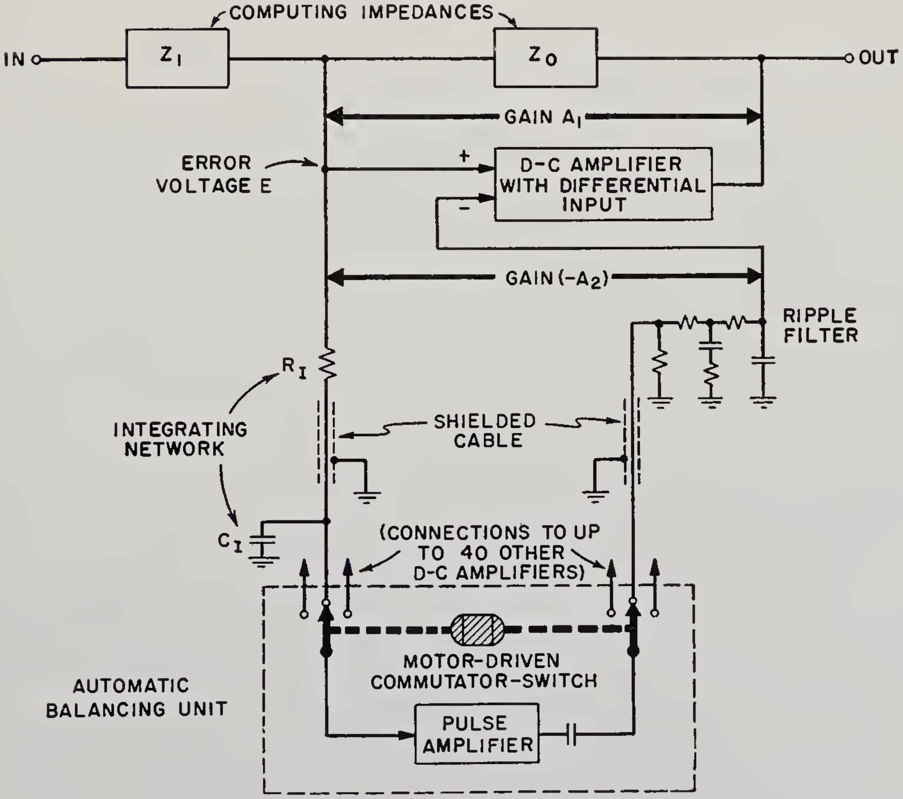

FIG. 5.37*b*. D-c amplifier with commutator stabilization.

filament power supplies, it acts essentially as an input stage having a relatively high gain A_2 and very low drift. Reference to Eq. (5.2) shows that *such an arrangement reduces the effects of drift in the d-c amplifier channel effectively by a factor equal to the gain A_2 of the modulated-carrier channel.* Although it will still pay to keep the drift in the d-c amplifier channel to a minimum by the techniques discussed in the previous section, the most significant d-c error voltages will now be *those acting at the input to the modulated-carrier amplifier channel* due to

1. Noise and drift from the synchronous vibrator, of the order of 2 to 20 μv.
2. Grid current of the input tube of the d-c amplifier; *e.g.*, a typical grid current of 10^{-4} μa through a resistance of 1 megohm will give rise to an error voltage of 100 μv. A portion of this error voltage will appear at the input of the modulated-carrier channel and will act like an input voltage.
3. D-c leakage from adjacent high-voltage points.
4. D-c ground currents.

Integral Control. The action of the automatic balancing circuit may be described in yet another manner. Any offset voltage due to drift in the d-c amplifiers, say, appearing at the output terminals of the operational amplifier will be fed back through the feedback networks and will tend to charge the capacitor C_I of the low-pass filter network at the input of the modulated-carrier channel. This "low-frequency error voltage" will be amplified by the modulated-carrier channel and fed back in such a manner as to reduce the effects of the original offset voltage. One may say that the low-pass filter comprising the resistor R_I and the capacitor C_I "remembers" the effects of past disturbances for a short time and seeks to annul their effects. This type of action is referred to as *integral control.*[1]

Use of a Grid-blocking Capacitor. *It is possible to eliminate the d-c offset voltage due to the constant component of the grid current in the d-c amplifier input tube through the use of a grid-blocking capacitor and separate grid-return resistor.* Such an arrangement is shown in dotted lines in the block diagram of Fig. 5.37*a*. With circuits of this type, the d-c component of the input signal cannot pass directly through the d-c amplifier channel but must pass first through the modulated-carrier channel; the d-c gain will be equal to A_1A_2. Higher frequency components of the input signal can still pass directly through the d-c amplifier channel by way of the grid-blocking capacitor.

The grid-blocking capacitor should be a high-quality-type capacitor having a minimum of leakage or absorption, such as a polyethylene or polystyrene capacitor.

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[1] Refer to James, Nichols, and Phillips, *op. cit.*, for a detailed discussion of integral control.

Automatic Balancing Circuits Based on Commutator Stabilization.[1] The alternative automatic balancing scheme illustrated by Fig. 5.37*b* utilizes a single stabilizing amplifier and a commutator-type switch to balance as many as 40 separate d-c amplifiers in periodic succession. The error voltage (summing-point voltage) E of the operational amplifier shown in Fig. 5.37*b* is sampled by a commutator about three times per second. Whenever the error voltage E is different from zero, the commutator will form a pulse which is amplified by the pulse amplifier shown. Another section of the commutator returns the amplified pulses to a ripple filter whose d-c output voltage is applied to the balancing grid of the d-c amplifier as in Fig. 5.37*a*. Other pairs of commutator sections serve other d-c amplifiers, so that the stabilizing pulse amplifier is utilized on a time-sharing basis. The action of this automatic balancing circuit on each operational amplifier served is in principle similar to that of individual automatic balancing circuits based on chopper stabilization; the d-c gain of the circuit shown in Fig. 5.37*b* is again given by Eq. (5.36).

Recovery of Automatic Balancing Circuits after Overloads. Whenever the absolute value of the output voltage of an operational amplifier exceeds a certain permissible level, the amplifier will "saturate" and can then no longer function as a linear device. In this condition, the feedback cannot act to reduce the "error voltage" at the amplifier input terminals.

With conventional d-c amplifier circuits, normal operating conditions can be reestablished quickly after an overload by simply returning the computer to the RESET condition (see Sec. 7.1). In amplifiers employing automatic balancing circuits, however, the filter and grid-blocking capacitors used in these circuits will tend to charge up while the amplifier is overloaded. When the computer is returned to the RESET condition following an overload, some time may pass until these capacitors are discharged and permit the amplifier output voltage to return to its normal operating range. Consequently, provision should be made for discharging the capacitors in question after an overload.

A device for this purpose might be in the form of a relay which shorts the guilty capacitors for a short time whenever the computer is placed in the RESET condition (see Sec. 7.1).

The recovery time will, in any case, be shorter with automatic balancing circuits employing a 400-cps carrier than with 60-cps carrier systems, since the filter and grid-blocking capacitors can have smaller values in this case. Commutator-type balancing circuits usually have longer recovery times than chopper-type circuits.

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[1] Ingerson, W. E., Drift Compensation in D-c Amplifiers, IRE National Convention paper, 1951.

Error-indicating Lights. Neon bulbs connected across the output terminals of each operational amplifier have long been found useful for indicating excessive output voltages and thus for eliminating computer runs during which one or more operational amplifiers were overloaded (Sec. 7.2).

In the case of d-c amplifiers employing automatic balancing circuits, it is possible to use neon bulbs for indicating excessive signal voltages in the output of each automatic balancing channel. Such circuits are indicated in the circuit diagrams of Figs. 5.39 through 5.42. Each circuit is arranged so that the neon "error-indicator light" will begin to glow whenever the absolute value of the voltage across the capacitor C_I exceeds a specified value of 1 to 5 mv.

Assuming that the automatic balancing channel is working properly, the error-indicator light will glow whenever the feedback circuit constituting the operational amplifier is not acting to reduce the low-frequency error voltage on the capacitor C_I. The error-indicator light will, thus, not simply indicate excessive output voltages in the operational amplifier in question. Output voltages substantially in excess of the threshold level of ordinary overload lights will, for instance, be tolerated if the amplifier is lightly loaded; this permits greater flexibility in the choice of scale factors and increased accuracy. Again, the new error-indicating system will indicate improper amplifier operation if an amplifier is loaded by too many potentiometers even though the output voltage may be relatively small. *Errors hitherto unnoticed can be eliminated in this manner, and more efficient utilization of the operational amplifiers is made possible.*

Finally, the error-indicating lights will tend to glow in certain cases of tube or component failures in the d-c amplifier channel. The new error indicators are seen to be more useful than ordinary overload lights; it may even be helpful to use both systems in conjunction so as to identify error sources more easily.

Comparison of the Two Types of Automatic Balancing Circuits. Initial costs as well as power and space requirements are appreciably less for a single 40-channel commutator and pulse amplifier than for 40 separate chopper-stabilization circuits. It may, however, be advisable to provide duplicate commutator-amplifier units. The stabilizer unit in use can be checked continuously by means of a test voltage applied to an extra commutator channel.

The best commutator brushes have an estimated life of between 4,000 and 9,000 hr. This is roughly equivalent to the operating life of high-quality synchronous vibrators. Aside from brush maintenance, which is no longer a serious problem, most development difficulties associated with commutator stabilization have been due to cross talk, hum pickup, and electrostatic charges (both electret and hysteresis effects) in the cables connecting the commutator to the individual d-c amplifiers. Cross talk may arise especially from the large pulses derived from overloaded d-c amplifiers but is minimized by physical separation of the offending connections.

The somewhat more expensive chopper-stabilization circuits offer higher usable gain as well as better frequency response and faster recovery times in each stabilizing channel. The drift of the best chopper-stabilized d-c amplifiers is thus less than that obtainable with commutator-stabilized amplifiers, especially since the use of grid-blocking capacitors does not appear practical with the latter.[1] The specifications of some actual circuits are presented in Sec. 5.7. The continuing development of photoelectric choppers (Sec. 5.8) may well result in the replacement of both synchronous vibrators and rotating commutators by all-electronic automatic balancing circuits in many future d-c analog computers.

5.7. EXAMPLES OF COMPUTER AMPLIFIERS EMPLOYING AUTOMATIC BALANCING CIRCUITS

Figures 5.38 to 5.42 show a number of practical d-c amplifier designs incorporating automatic balancing. The remarkable properties of high-quality computer amplifiers are expressed by the following representative specifications:

Gain of the d-c amplifier channel ($|A_1|$), 30,000 to 150,000
Gain of the automatic balancing channel (A_2), 2000 to 3000
Total d-c gain, 25×10^6 to 300×10^6
RMS noise, referred to input, 2 to 10 mv
Grid current, less than 10^{-4} μa
Average drift over an eight-hour period, referred to input, less than 20 to 200 μv (chopper stabilization); or less than 100 to 500 μv (commutator stabilization)
Frequency response of phase inverter (Fig. 1.5*b*) with $R_o = R_1 = 1$ megohm: flat within 1 db between d-c and 2 to 5 kc, with a resonant peak of 1 to 4 db beyond that frequency; at 100 cps, the amplitude error is less than 0.04 to 0.1 per cent, and the phase shift is less than 0.05 to 0.1 deg (all at 1 volt amplitude)
Linear output range, at least plus or minus 100 volts into rated load; decreases above 100 cps, depending on tube types

Note carefully that *the range of output voltages permitting linear operation decreases with operating frequency* because of the decreasing loop gain.

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[1] Note, however, that this difference in drift may well be negligible compared to the d-c offset voltages due to ground currents in actual computer installations.

Reputable manufacturers supply curves showing output voltage ranges above 100 cps, and show amplitude and frequency-response curves for large as well as small amplitudes.

Offset voltages can be further reduced by careful adjustment of auxiliary manual balance controls, through the use of cathode-follower input stages (Fig. 5.39) or electrometer-type input tubes in the d-c amplifier channel, or through the use of a grid-blocking capacitor (Fig. 5.37*a*). Chopper-stabilization circuits should comprise some means for protecting the vibrator contacts from excessive currents, particularly when an automatic balancing channel is overloaded.

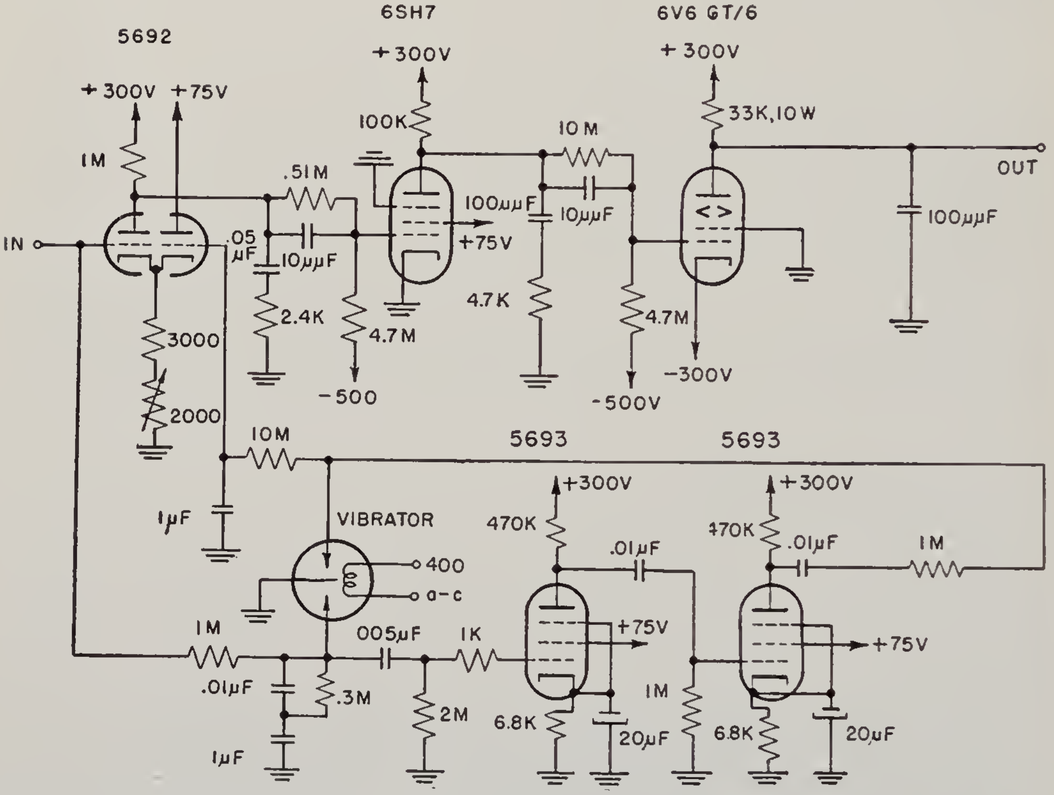

Fig. 5.38. RCA amplifier with 400-cps chopper stabilization. D-c channel gain, about 50,000; stabilizer gain, about 2,000; net d-c gain, about 100×10^6; net gain at 5 cps, about 300,000; grid current, less than 10^{-4} µa; long-term drift, about 50 µv (RCA Research Laboratories, Princeton, N.J.).

The pioneer RCA circuit[1] of Fig. 5.38 is a laboratory model featuring 400-cps chopper stabilization. Figure 5.39 shows the automatic balancing and error-indicating circuits used by the RAND Corporation[2] in connection with a modified Reeves[3] d-c amplifier. An earlier version of the

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[1] Goldberg, *op. cit.*

[2] Gunning, W. F., and A. S. Mengel, *Report RM* 236, RAND Corporation, Santa Monica, Calif., with unpublished circuit corrections.

[3] See Bradley, *op. cit.*, for the original Reeves circuit.

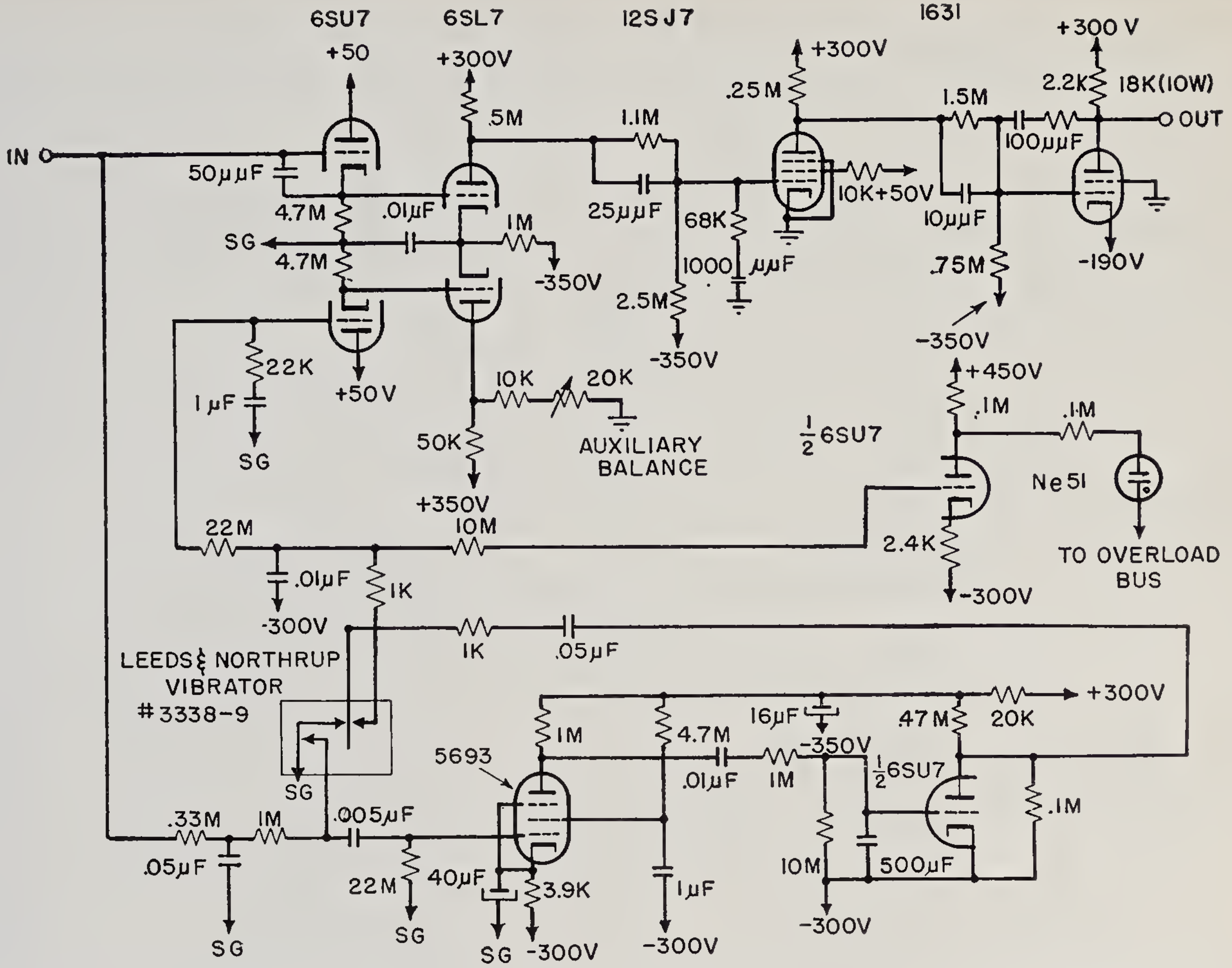

FIG. 5.39. High-quality d-c amplifier developed by the RAND Corporation, Santa Monica, Calif. The input circuits must be well shielded. Note the separate signal ground strap (SG) used to avoid circulating currents in the ground connections. Net d-c gain, about 30 × 10^6; long-term drift, 50 to 100 μv.

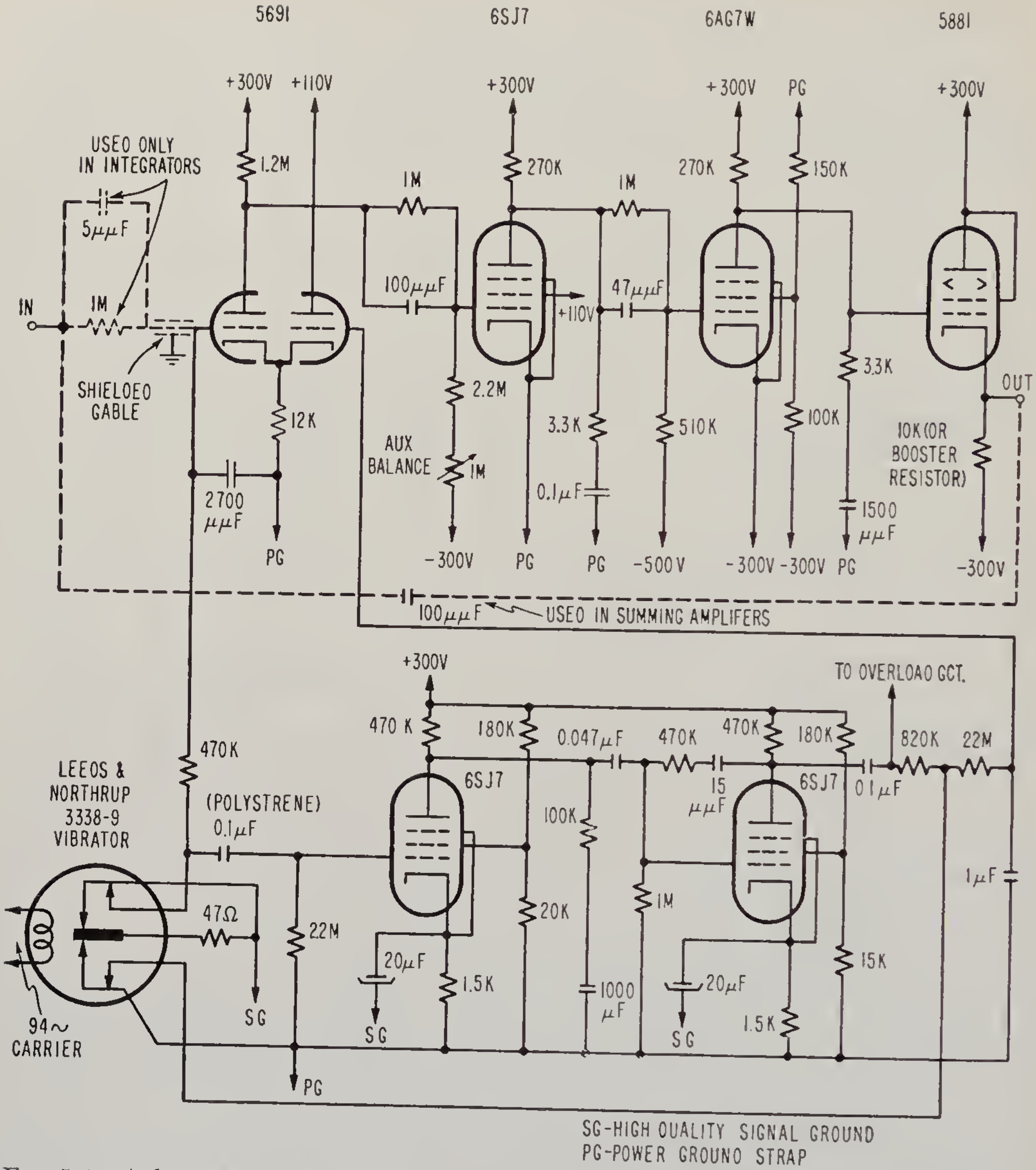

FIG. 5.40. A d-c channel gain in excess of 100,000 and a stabilizer gain of 3,000 combine to give this precision computer amplifier a net d-c gain greater than 300 × 10^6 with a specified 8-hr drift voltage of less than 100 μv. Note the powerful output stage; the output range is at least plus and minus 100 volts up to 30 ma with booster resistor (Electronic Associates, Inc.).

RAND automatic balancing circuit included a polystyrene grid-blocking capacitor of 1 μf with a grid-leak resistor of 10 megohms. In order to avoid the necessity of discharging the grid-blocking capacitor after overloads, it was decided to use the low-plate-voltage cathode-follower input stage shown in Fig. 5.39; the low grid current (10^{-6} μa) of this circuit makes the use of a grid-blocking capacitor practically unnecessary. The vibrator-contact arrangement shown results in a peak-detection scheme which is said to reduce the demodulator-ripple filtering problem. The second stage of the a-c amplifier has a purposely low plate voltage, so that excessive error voltages due to overloads will tend to be limited and cannot damage the demodulator contacts of the 60-cps vibrator. The error-indicating light appears at the output of a d-c amplifier stage following the synchronous rectifier. All error-indicating lights in the RAND computer are returned to a common bus; a warning buzzer or relay system can thus be actuated by an error indication in any amplifier. A switch (not shown in Fig. 5.39) is provided to disconnect the automatic balancing circuit from the d-c amplifier after severe overloads.

The circuit of the elaborate and accurate computer amplifier used in the PACE computer[1] (Fig. 5.40) is similar to the RCA circuit of Fig. 5.38, with the addition of a powerful cathode-follower output stage. 94-cps chopper excitation is used to minimize the effects of 60-cps hum pickup in the a-c amplifier. The high-quality multielement vibrator used is claimed to be capable of long life without special limiting circuits. Coaxial cables and terminals insulate input and output terminals with metal (Sec. 5.9). The 100-μv drift specified may be further reduced through the use of an (optional) grid-blocking capacitor in the manner of Fig. 5.37*a*.

The REAC[2] dual d-c amplifiers shown in Fig. 5.41 are particularly instructive examples of precision computer components designed for commercial production. The circuits use relatively few parts. Two chopper-stabilized d-c amplifiers are neatly assembled on an ethoxyline printed-circuit board and share several dual-triode tubes requiring similar d-c reference voltages for their filament supplies. Diode demodulation in each stabilizer channel permits the use of a single modulating vibrator; the latter has an excellent service life, since it is protected by nonlinear resistors and is not subjected to stabilizer output voltages. Demodulator drift is not serious because of the preceding high-gain a-c amplifier.

An outstanding feature is the series-balanced ("cascode") output stage (see also Sec. 5.3), which has lower quiescent-current requirements (7 ma) than conventional output stages of comparable power. Booster resistors (Sec. 5.9) cannot be used with this output circuit, nor are they usually required; for special applications provision is made for connecting external vacuum tubes in parallel with the output stage.

In the newer REACs, the printed circuits and the associated connectors prevent surface leakage into amplifier input terminals by surrounding them with grounded printed-circuit lands (see also Sec. 5.9).

The commutator-stabilized Goodyear Aircraft Corporation amplifier circuit shown in Fig. 5.42 is capable of excellent performance even without its stabilizing circuit. Three cascaded differential-amplifier stages permit very high gain with low drift; the input stage is operated at low plate voltage to minimize grid current. The long-term drift is less than 10 mv *without* automatic balancing.

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[1] Registered trade mark of Electronic Associates, Inc., Long Branch, N.J.

[2] Registered trade mark of Reeves Instrument Corporation, New York City.

5.8. AUTOMATIC BALANCING CIRCUITS EMPLOYING PHOTOELECTRIC CHOPPERS

The maintenance problems often associated with electromechanical automatic balancing circuits (Sec. 5.6) have stimulated the development of *photoelectric choppers.* A photoelectric chopper consists of a photocell (phototube or photoconductor) whose impedance is periodically reduced by light pulses from a gas-discharge tube or pulsed incandescent lamp; the photocell may be connected like an electromechanical chopper to yield an a-c component whose amplitude is proportional to an applied d-c voltage.[1]

The development of satisfactory photoelectric choppers is quite difficult, since ambient-temperature changes, spurious photoelectric effects, and mechanical vibrations affecting the relative orientation of photocell and light source all tend to produce offset voltages. The use of photoconductor cells with specially prepared contacts appears to be most promising,[2] even though the frequency response of such devices is limited because of hysteresis effects.

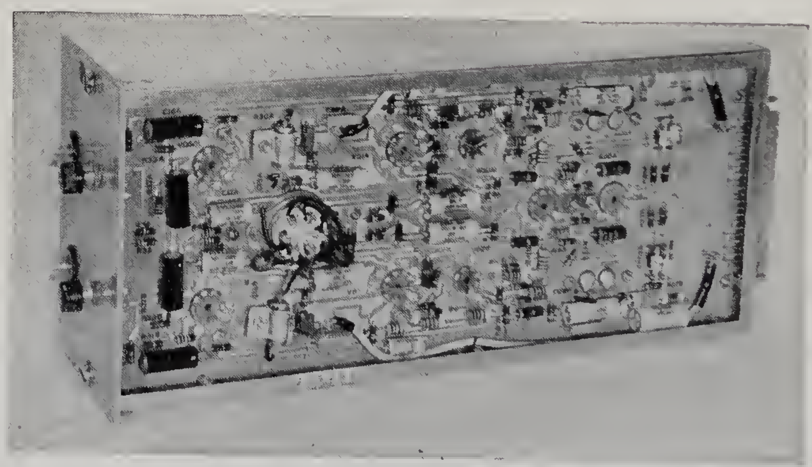

FIG. 5.41*a*. For legend see opposite page.

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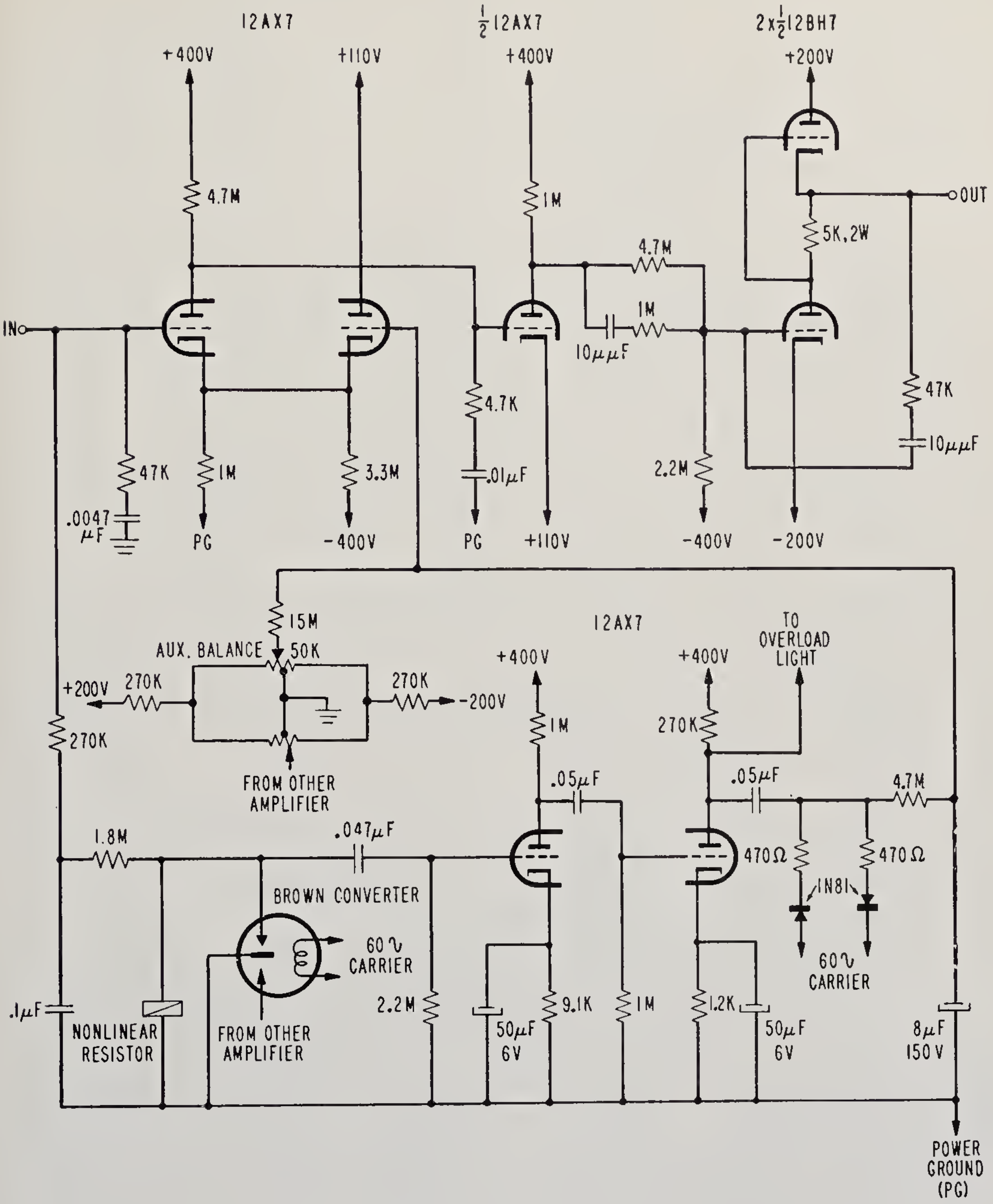
[1] Schwartz, J., and R. Solomonoff, Photoelectric Chopper for Guided Missiles, *Electronics*, November, 1954.

[2] Sternberg, S., Electronic Zero Stabilization of D-c Amplifiers, *TYPHOON Symposium III*, Oct. 13, 1953.



(*b*)

Fig. 5.41. Two computer amplifiers constructed on a printed-circuit board share several tubes and a common synchronous vibrator in the remarkable production design shown in Fig. 5.41*a*. D-c channel gain, greater than 60,000; stabilizer gain, about 1,000; net d-c gain, about 60×10^6; grid current, less than 10^{-4} μa; long-term drift (24 hr), less than 250 μv; output range, plus and minus 100 volts at 20 ma (400-series REAC dual d-c amplifier, Reeves Instrument Corporation). The similar 200-series REAC amplifier circuit of Fig. 5.41*b* has about one-half the gain and current output of the 400-series amplifier; the 200-series circuit is particularly instructive because of its simplicity.

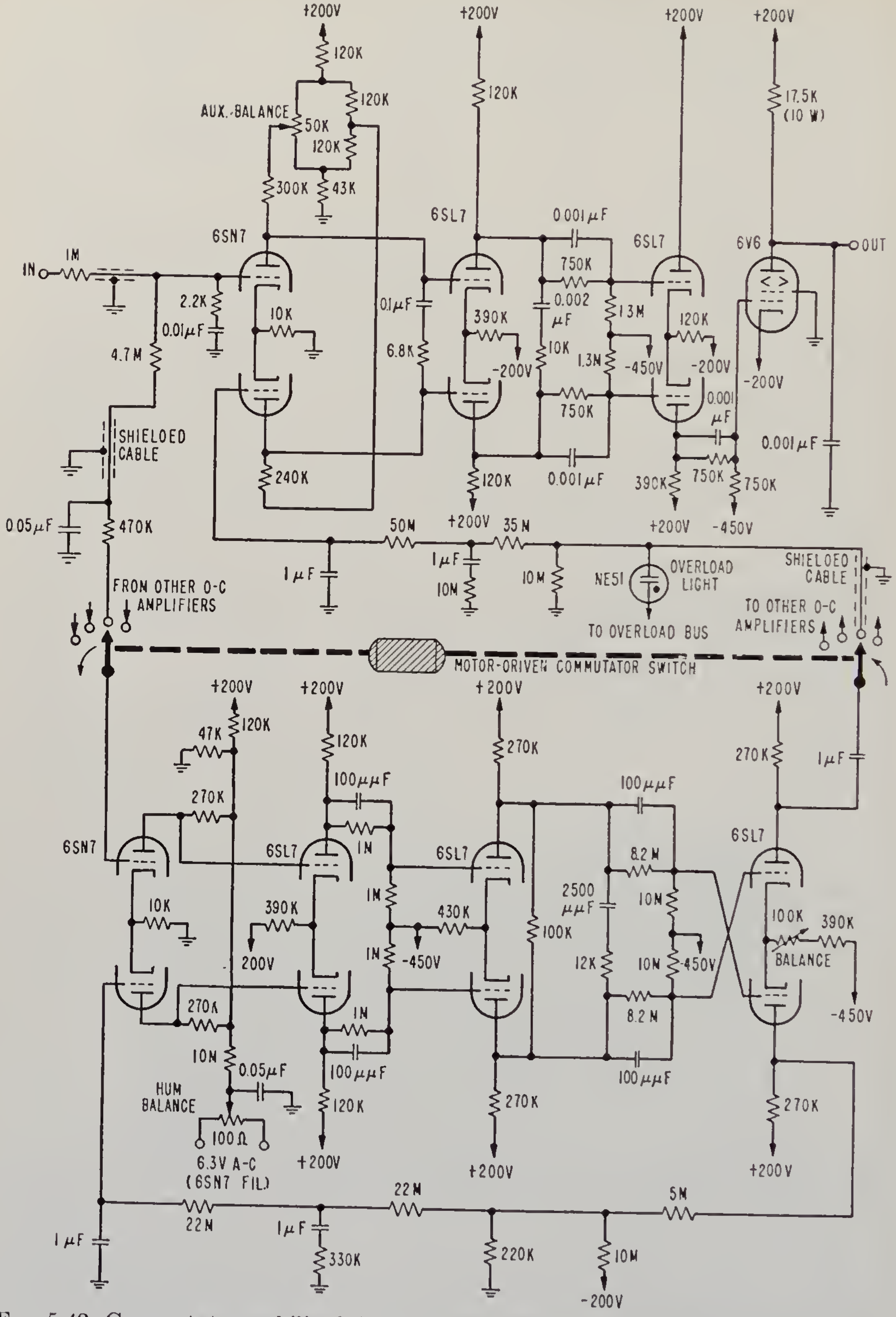


FIG. 5.42. Commutator-stabilized d-c amplifier used in the Goodyear Electronic Differential Analyzer (GEDA). D-c channel gain, 150,000; stabilizer gain, 500; net d-c gain, at least 30 × 10^6; grid current, less than 5 × 10^{-5} μa; long-term offset voltage, less than 500 μv (includes drift and leakage); output range, plus and minus 100 volts at 5 ma (15 ma with booster resistor; Goodyear Aircraft Corporation).

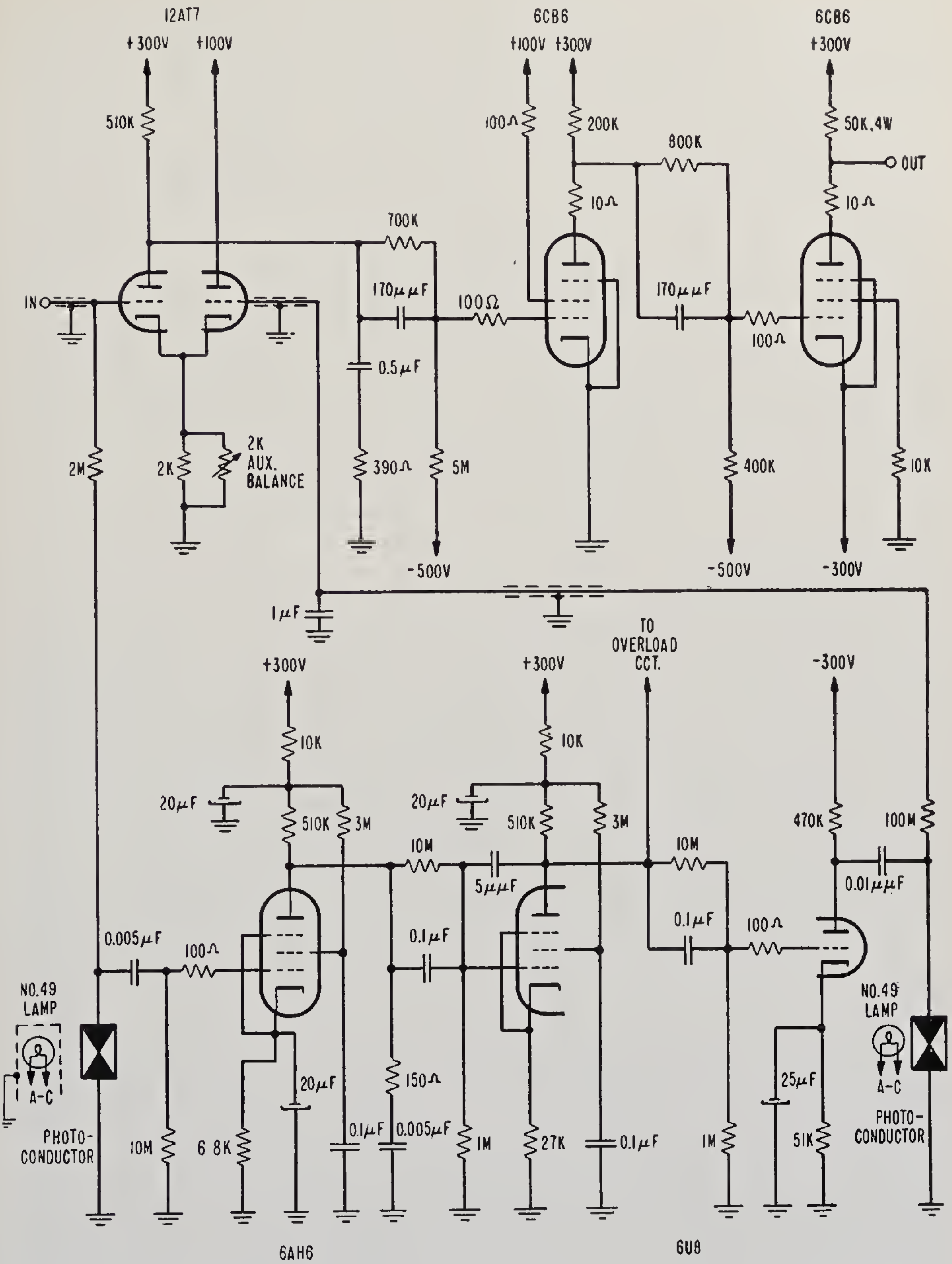


FIG. 5.43. An experimental amplifier circuit employing photoelectric choppers (RCA Laboratories Division).

In an experimental high-quality d-c amplifier designed by the RCA Laboratories Division under a contract with the United States Air Force (Fig. 5.43), two crystalline-cadmium-sulfide photoconductors are connected essentially like the electromechanical choppers in Fig. 5.37*a*. Pulsed light decreases the impedance of each photoconductor periodically in a ratio exceeding 3 to 1; this finite impedance ratio requires a slightly higher a-c amplifier gain than that used with electromechanical choppers. The Type 49 flashlight bulbs used have sufficiently low thermal capacity for pulsed operation with 1.5 volt (RMS) half-wave rectified sinusoidal pulses at repetition rates up to 60 cps. The *long-term* drift of the experimental d-c amplifier is less than 1 mv; this figure may be capable of further improvement, perhaps through the use of temperature compensation and/or bridge-balanced circuits. The life expectancy of the photoconductors is practically indefinite.

5.9. HINTS FOR THE CONSTRUCTION OF D-C AMPLIFIERS. CHOICE OF COMPONENTS

In most respects, the physical construction of d-c amplifiers is not essentially different from that of other electronic devices. The construction of all computer components should emphasize ruggedness; the chassis wiring can be done either with uninsulated solid wire and/or with insulated stranded wire. All a-c filament leads should be tightly twisted pairs. It is advisable, although not absolutely necessary, to keep all leads short and to shield the amplifier input leads. In order to avoid microphonism, particularly in the input stages of high-gain amplifiers, it will be wise to shock-mount the tube assemblies in these stages by means of small rubber grommets placed in the screw holes. Relays associated with d-c amplifiers for control purposes often tend to shake the entire chassis and must, therefore, also be shock-mounted.

The vacuum-tube operating voltages in each stage of a d-c amplifier are affected by resistors in preceding stages as well as in the stage in question. For this reason, the resistances of all resistors carrying direct current may have to be specified to within plus or minus 1 per cent.

In order to minimize d-c amplifier drift, all resistors carrying direct current must be *stable: i.e.*, they must maintain their resistance values within specified tolerances over reasonable ranges of current, humidity, and ambient temperature. In order to avoid excessive temperature changes, all resistors used in computer amplifiers should have about twice the power rating actually required; resistors which must dissipate more than 1 or 2 watts may be mounted on top of the amplifier chassis for better heat dissipation.

High-quality metalized-type resistors are recommended for low-cost computer amplifiers. *In high-precision computers, all critical resistors (in the power-supply regulators and servomechanisms as well as in the d-c amplifiers) should be special precision types of metalized, carbon-film, or wire-wound resistors.* Wire-wound resistors will not exhibit the required temperature stability unless special alloys are used for the windings.

Generally speaking, it is usually less expensive to use high-quality resistors in all doubtful positions than to try to compute the drift due to the resistance variations of cheaper resistors.

Adjustable resistors and balancing potentiometers must also be chosen with a view to stability and freedom from microphonism.

Ground Connections. In order to minimize d-c offset voltages,[1] hum pickup, and feedback due to circulating ground currents, high-quality computer amplifiers employ *separate ground straps* for signal ground (in particular, chopper ground) and power ground; these ground connections may be returned to the grounded rack assembly at a suitable point.

Integrating Capacitors, Absorption, and Leakage. It has been shown in Chap. 4 that *high leakage resistance* is a desirable property in integrating capacitors. Although the effects of capacitor leakage could be compensated by means of regenerative circuits, the leakage resistance of poor capacitors tends to change with humidity, so that accurate compensation may be difficult. The effects of *dielectric hysteresis* (soak-in) are somewhat like those of leakage but change with frequency.

Both capacitor leakage and hysteresis are minimized through the use of the proper types of capacitors. Integrating capacitors in high-accuracy computers should use either polystyrene or similar plastics as the dielectric and have leakage resistances in excess of 5×10^5 megohms/μf. Mica or ceramic capacitors may be used as integrating capacitors in low-cost computers and particularly in repetitive computers; ceramic capacitors are recommended for use in the high-frequency stabilization networks of d-c amplifiers.

In all parallel-feedback-type integrators, care should be taken to confine the leakage path between the plates of each integrating capacitor to the dielectric proper. Circuit points respectively connected to the d-c amplifier input and output terminals should *always* be separated by grounded metal parts. Coaxial input leads and terminals are recommended. Such construction practice will eliminate leakage paths over the surface of insulating materials. The analysis of Sec. 4.5 shows that leakage to ground is essentially harmless; "insulation with metal" is, therefore, the best possible insulation.

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[1] Improper ground connections in automatic balancing circuits can cause d-c offset voltages as high as 1 to 10 mv in large computer installations. In an unpublished lecture, C. M. Edwards (Bendix-Detroit) has suggested *separate grounding of all coefficient-setting potentiometers* as well as a *separate chopper ground bus* to reduce such offset voltages to below 100 μv.

"Insulation with metal" will also prevent d-c leakage between d-c amplifier input terminals and adjacent high-voltage points (see also Sec. 8.3).

Amplifier Loading and the Use of Booster Resistors. For a given d-c amplifier, the range of output voltages not to be exceeded for proper linear operation will be specified for a given minimum load resistance. If the use of smaller load resistances is necessary (*e.g.*, if several potentiometers in parallel are to be connected to an amplifier output), the need for additional operational amplifiers in parallel with the first can sometimes be avoided through the use of *booster resistors.* A booster resistor is simply a resistor connected in parallel with the output plate-load resistor of the amplifier output stage in circuits like those shown in Figs. 5.33 to 5.35. Such a resistor can be introduced into the circuit by means of a simple switch or plug-in arrangement. A booster resistor will decrease the amplifier output impedance at a slight expense in terms of amplifier forward gain and d-c power drain.

5.10. TRANSISTOR D-C AMPLIFIERS

The inherent drift characteristics, low output voltage ranges, and low input impedance of transistor d-c amplifiers do not make such circuits attractive for use in general-purpose analog computers. Transistor d-c amplifiers may be useful in special applications, where the advantages of miniaturization, high reliability, and low power requirements outweigh those of extreme accuracy. An experimental circuit described by Ettinger[1] has a gain of 25,000 with an output range of plus and minus 5 volts at 3 ma; the long-term drift is of the order of 30 mv but may be reduced by a special temperature-compensation circuit. The only power supplies required are two 8-volt batteries. The drift could be improved through bridge-balanced circuits similar to those used with vacuum-tube d-c amplifiers and through the use of simple temperature-controlled ovens. The input impedance of transistor amplifiers may be increased through special feedback circuits.[2] Finally, a transistor chopper[3] and a transistor a-c amplifier may be used in the automatic-balancing channel of Fig. 5.37*a*; the resulting offset voltage should be of the order of 1 mv.

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[1] Ettinger, G. M., Transistor Amplifiers for Analog Computers, *Electronics*, July, 1955.

[2] Schenkerman, S., Feedback Simplifies Transistor Amplifiers, *Electronics*, November, 1954.

[3] Bright, *op. cit.*

CHAPTER 6

MULTIPLIERS AND FUNCTION GENERATORS

6.1. INTRODUCTION AND SURVEY

The designer of operational amplifiers or computing networks can draw freely on the well-established theory of linear circuits. Such knowledge is not sufficient for the design of multipliers or function generators, which must necessarily involve nonlinear operations. Each multiplier or function generator may be regarded as a transducer whose gain (transfer function) is dependent upon one or more of the applied voltages.

The most frequently employed types of multipliers and function generators for d-c analog computers are the *servo-driven potentiometers* briefly introduced in Sec. 1.3. Servo multipliers and function generators are discussed in more detail in Secs. 6.8 to 6.11. Note that a single computer servomechanism can multiply each of several machine variables by a function of the servo input voltage. The design of such devices is highly developed and permits accuracies better than 0.1 per cent of full output scale. Nevertheless, the use of electromechanical computing elements necessarily limits the frequency ranges of some machine variables to frequencies less than 3 to 40 cps; dynamic errors will depend radically on the amplitudes of the high-frequency components contained in the signals in question. Such restrictions impose limitations on the computer time scale and on the speed of computation.

A number of *all-electronic multipliers* are discussed in Secs. 6.2 to 6.5. The electronic multiplying circuits most frequently employed in d-c analog computers are the time-division multipliers (Sec. 6.4) and the quarter-square diode multipliers (Sec. 6.5). Accuracies better than 0.2 per cent at frequencies as high as 100 to 1,000 cps are possible with such circuits, but circuit complexity increases if better accuracy is required.

Electronic function generators are described in Secs. 6.6 and 6.7. The most frequently used types are diode function generators which permit accuracies between 0.2 and 1 per cent at frequencies up to several kilocycles.

Sections 6.12 and 6.13 deal, respectively, with division and with the generation of functions of more than one variable.

6.2. MISCELLANEOUS DEVICES PERMITTING MULTIPLICATION OF TWO MACHINE VARIABLES

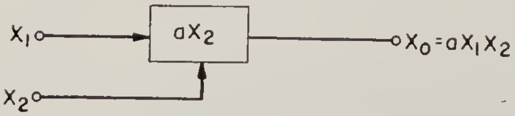


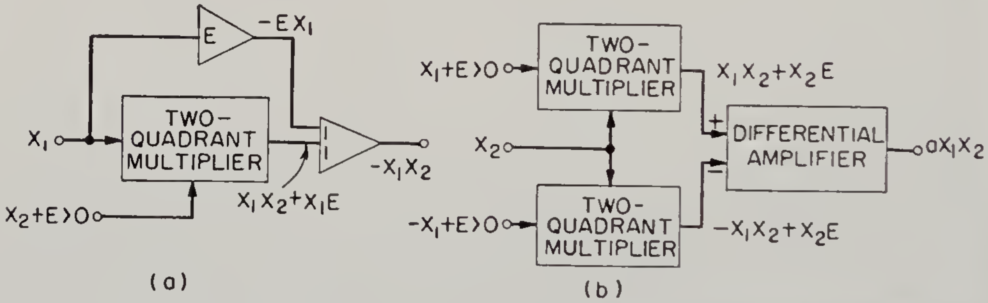
FIG. 6.1. General block diagram of a multiplier for multiplying two d-c voltages.

Devices capable of multiplying two machine variables may be represented by a block diagram like the one shown in Fig. 6.1. The device will have two input voltages, X_1 and X_2, corresponding to the two machine variables to be multiplied. The output voltage X_o will be proportional to the product of the two input voltages X_1 and X_2 so that

$$X_o = aX_1X_2 \tag{6.1}$$

where a is a positive or negative constant.

Multiplication of Positive and Negative Quantities. The ideal multiplier would accept voltages of both signs for either X_1 or X_2, and its output voltage X_o would have the correct sign associated with the product X_1X_2. Such a multiplier is said to be capable of *four-quadrant operation.* Other types are capable of *two-quadrant operation:* here the input voltage X_1 may be either positive or negative, and the output voltage X_o will change sign with the input voltage X_1. The input voltage X_2, however, can be only either positive or negative throughout the computation. Such multipliers are called *two-quadrant multipliers.* A third class of multipliers is capable of accepting only voltages of one sign for either X_1 or X_2. Such multipliers might be called *one-quadrant multipliers.*



(a) (b)

FIG. 6.2. Adaptation of two-quadrant multipliers to four-quadrant multiplication.

Generally speaking, it is usually possible to build four-quadrant multipliers by combining simpler multiplying devices which are not capable of four-quadrant operation. Figure 6.2*a* shows how a two-quadrant multiplier can be made to serve for four-quadrant multiplication. The multiplier is of a type which will accept positive and negative voltages X_1 but only values of the second input voltage, V, say, which are positive and greater than some value V_0. X_1 is to be multiplied by a machine variable

X_2 which can assume both positive and negative values. A constant positive voltage E, sufficiently large to make the sum of X_2 and E greater than V_0 for all values of X_2, is added to the voltage X_2. The voltage

$$V = X_2 + E \tag{6.2}$$

is used as an input to the multiplier, which will compute the product

$$X_1V = X_1X_2 + X_1E \tag{6.3}$$

Figure 6.2*a* shows how the product X_1E may be subtracted from the multiplier output voltage in order to obtain the correct product X_1X_2. The arrangement of Fig. 6.2*b* utilizes two two-quadrant multipliers. This circuit may be more complex than that of Fig. 6.2*a*, but some errors may cancel because of the symmetrical push-pull arrangement. The scheme of Fig. 6.2*a* can be applied to both X_1 and X_2 (Fig. 6.15*b*), or both schemes of Fig. 6.2 can be combined to yield four-quadrant operation with one-quadrant multipliers (see also Sec. 6.3).

Voltage-sensitive Variable-gain Devices Used for Multiplication. The majority of devices used for the multiplication of two variables may be described by a block diagram such as the one shown in Fig. 6.3*a*. The gain or transfer function $K = X_o/X_1$ of the device in question with respect to an input voltage X_1 is variable and will increase or decrease with the value of a second input voltage X_2. Figures 6.3*b* and *c* show two different specific examples of devices which can be described by the block diagram of Fig. 6.3*a*. Figure 6.3*b* shows the servo-positioned potentiometer already discussed in Sec. 1.3. The potentiometer gain or transfer function X_o/X_1 depends on the "gain-determining" voltage X_2 which controls the position of the potentiometer wiper. The output voltage X_o is, accordingly, a function of both input voltages X_1 and X_2 and may be made equal to the product X_1X_2 of the two input voltages. Figure 6.3*c* shows a variable-gain amplifier consisting of a multigrid vacuum tube with an input voltage X_1 applied to one control grid and a second input voltage X_2 applied to

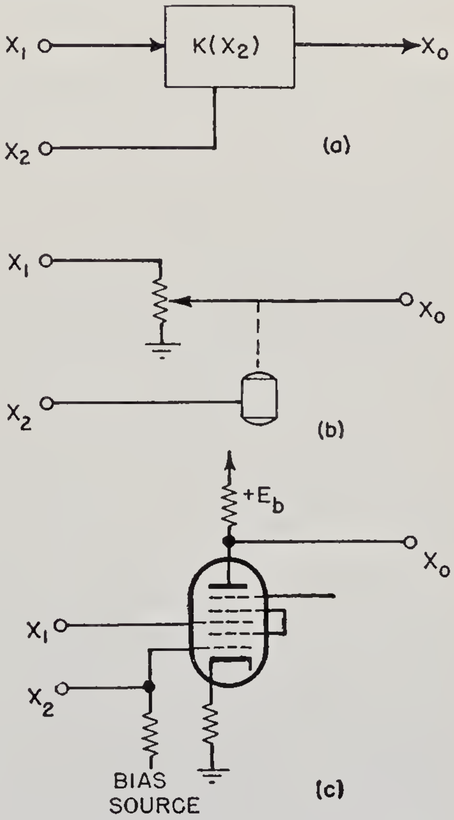


FIG. 6.3. Voltage-sensitive variable-gain devices used for multiplication of two d-c voltages.

the second control grid. The output voltage X_o may be made roughly proportional to the product of these two input voltages.

There are several fundamental differences between the circuits shown in Figs. 6.3*b* and 6.3*c*, although the principle of operation as shown by the block diagram is the same in either case. The servo-driven potentiometer is an electromechanical device, whereas the variable-gain amplifier is a purely electronic device. Again, the output voltage of the potentiometer multiplier will be quite accurately proportional to the first input voltage X_1 unless there is excessive heating of the potentiometer winding. In the variable-gain amplifier, the output voltage will depend on a function of the input voltage X_1 which is only roughly linear; the output voltage is dependent on the tube characteristics.

Another difference between the two multiplying devices is in their respective abilities to multiply both positive and negative machine variables. The simple potentiometer of Fig. 6.3*b* is a two-quadrant multiplier. The variable-gain amplifier shown in Fig. 6.3*c* is a two-quadrant multiplier, since the input voltage X_1 may be either positive or negative if the bias levels are chosen correctly. It is seen, however, that a constant voltage depending on the plate supply voltage is added to the desired output voltage aX_1X_2. This zero-offset voltage may be subtracted out by one of the methods shown in Fig. 6.2. The variable-gain amplifier shown will, however, multiply with reasonable accuracy only for a very small range of operating conditions, and these operating conditions will be difficult to reproduce.

Multiplier Circuits Based on Dual Modulation of an A-c Carrier.[1] The d-c component derived from a suitably detected and averaged (filtered) sinusoidal or square-wave carrier can be made proportional to the product of two d-c input voltages X_1 and X_2. Such multipliers may be based on (1) simultaneous *amplitude and pulse-width modulation* of a square wave whose (running) average value is then proportional to the desired product (Sec. 6.4), (2) simultaneous *amplitude and phase modulation* of a square-wave carrier which is then detected by a phase-sensitive detector[2] to yield aX_1X_2,[3,4] (3) simultaneous *amplitude and frequency modulation* (*AM-FM multipliers*, Sec. 6.3). The modulator and demodulator (detector) circuits used in the best multipliers of these types are essentially carrier-controlled *switches,*[1] which are either *on* or *off*. The output of such multipliers will, therefore, depend less critically on the characteristics of tubes and rectifiers than, for instance, that of the variable-gain d-c amplifier of Fig. 6.3*c*.

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[1] Detailed discussions of many practical modulator and demodulator circuits are presented in B. Chance, F. C. Williams, V. Hughes, D. Sayre, and E. F. MacNichol, *Waveforms*, Chaps. 11 and 14, MIT Radiation Laboratory Series, Vol. 19, McGraw-Hill, New York, 1949.

[2] Chance, B., *et al.*, *op. cit.*

[3] Craig, L. J., The Magnetic Amplifier as an Analog Computer Component, *Proc. IRE*, **41**: 1477, 1953.

[4] Fuller, H. W., and M. A. Meyer, Two New Analog Multipliers, *Rev. Sci. Instr.*, **25**: 1666, 1954.

By contrast, the multiplier circuit of Fig. 6.4 depends on the characteristics of nonlinear circuit elements, such as vacuum tubes or rectifiers, to "mix" the input d-c voltage with an input a-c "carrier" voltage. Suppose the sum of a d-c voltage

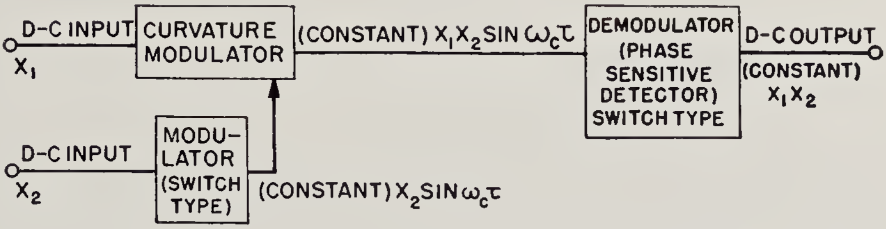

FIG. 6.4. Block diagram of a multiplier based on amplitude modulation.

$$V_1 = X_1 \tag{6.4}$$

and an a-c voltage

$$V_2 = X_2 \sin \omega_c\tau \tag{6.5}$$

is applied to a nonlinear device; ω_c is the carrier circular frequency. For the sake of simplicity, consider a square-law modulator with the output voltage

$$\begin{aligned} k(V_1 + V_2)^2 &= k(X_1^2 + X_2^2 \sin^2 \omega_c\tau + 2X_1X_2 \sin \omega_c\tau) \\ &= k\left(X_1^2 + \frac{1}{2} X_2^2\right) - \frac{k}{2} X_2^2 \cos 2\omega_c\tau + 2kX_1X_2 \sin \omega_c\tau \end{aligned} \tag{6.6}$$

where k is a constant. The last term on the right is seen to be proportional to the product of the two input voltages (6.4) and (6.5) and constitutes the desired modulated a-c output voltage. The d-c and second-harmonic terms in Eq. (6.6) can be eliminated from the output voltage by means of blocking capacitors and push-pull circuits (so-called balanced modulators). The product is detected by a phase-sensitive demodulator whose d-c output voltage changes sign whenever the carrier phase is reversed. Multipliers of this type are frequently used in special-purpose computers requiring only rough accuracy (1 to 7 per cent), such as correlators (Sec. 3.12). The utilization of *saturable transformer modulators* (*magnetic amplifiers*) in the block diagram of Fig. 6.4 might deserve further investigation.[2] A possible method for improving the accuracy by means of feedback is discussed in Sec. 6.3.

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[1] Chance *et al.*, *op. cit.*.

[2] See also Craig, *op. cit.*

All-electronic modulation-type multipliers are capable of good frequency response if the carrier frequency is chosen sufficiently high to permit filtering without phase shift at signal frequencies (see also Sec. 6.4).

Electromechanical Vibration Multipliers. The output voltage of a strain-gauge pickup driven at a constant audio frequency by means of a loudspeaker is proportional to the product of the loudspeaker excitation voltage and the d-c voltage applied across the strain-gauge bridge.[1,2] The device is thus an electromechanical modulator.[3] For carrier frequencies in excess of 2 kc, the frequency response of such a multiplier is satisfactory for most computer applications, and accuracies better than 0.2 per cent of full scale have been obtained. The construction of *strain-gauge multipliers* does, however, require considerable precision, so that such devices may be expensive. In a d-c analog computer, it will be necessary to rectify the a-c output voltage, and some drift may result from this operation.

Dynamometer-type Multipliers. An electrodynamometer consists essentially of a moving coil, somewhat like those used in meter movements, which can rotate in the field of an electromagnet as shown in Fig. 6.5. If the direct current in the coil is i_1 and the direct current in the electromagnet is i_2, the resulting torque on the moving coil will be proportional to the product $i_1 i_2$ of the two currents i_1 and i_2.

If two electrodynamometers are arranged with their moving coils rigidly connected to a common shaft as in Fig. 6.5, the resulting torque will be zero if and only if

$$i_1 i_2 = -i_3 i_4 \tag{6.7}$$

where i_1, i_2, and i_3, i_4 are the currents in the first and second electrodynamometers, respectively. This balance condition may be maintained by using an electromechanical or photoelectric pickoff to convert any angular displacement of the dynamometer shaft into an error voltage; the latter is then used to drive a high-gain current amplifier whose output is equal to i_3 (see Fig. 6.5). If the polarity of the error voltage is chosen correctly, such a feedback arrangement will tend to minimize the shaft displacement and thus the shaft torque at least for slow variations of the currents i_1 and i_2. As a result, one has very nearly

$$i_3 = -\frac{i_1 i_2}{i_4} \tag{6.8}$$

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[1] Hall, A. C., A Generalized Analogue Computer for Flight Simulation, *Trans. AIEE*, **69**: 308, 1950.

[2] Soroka, *op. cit.*, p. 70.

[3] A similar modulator using a condenser microphone as a vibration pickup was used by the RCA Research Laboratories, Princeton, N.J., in the feedback-stabilized circuit of Fig. 6.7*a* (RCA *vibration multiplier*, Project TYPHOON).

for a certain range of working frequencies for i_1 and i_2. The relation (6.8) can be made the basis of a compact d-c voltage multiplier if current amplifiers, which produce currents proportional to d-c input voltages, are available. If the current i_4 is made proportional to a third input voltage, simultaneous division and multiplication are possible. Accuracies better than 2 per cent should be feasible. The frequency response of the electrodynamometer multiplier is limited to perhaps 100 cps by the effects of the inertia of the moving coils and of coil inductance; a detailed analysis of these effects must be made in each case.

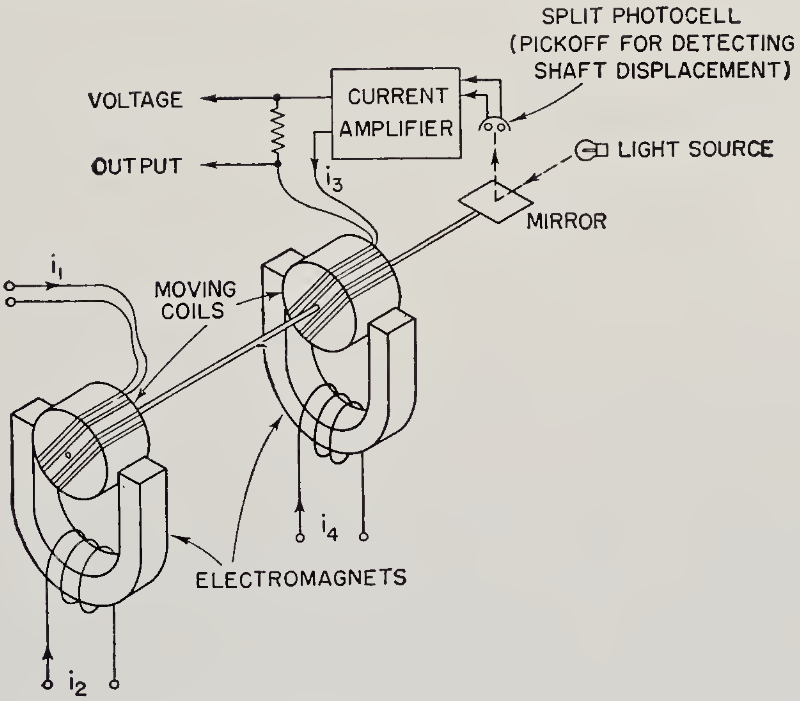

Fig. 6.5. Principle of the electrodynamometer multiplier. The essentially similar Weston Inductronic Product Resolver uses an inductive error pickup operating at 200 kc instead of the photoelectric pickup shown here.

At least two multiplying devices operating on the electrodynamometer principle have been built for control applications.[1] The accuracy of an electrodynamometer multiplier can be better than 0.5 per cent at signal frequencies up to about 20 cps.

The Crossed-fields Electron-beam Multiplier.[2] The principle of the crossed-fields electron-beam multiplier is somewhat similar to that of the electrodynamometer multiplier. Unlike the latter, the new multiplier does not contain any moving parts, so that its high-frequency response is excellent.

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[1] Automatic Current Balance, *Descriptive Sheet* S144A, Specialties, Inc., Syosset, Long Island, N.Y., December, 1950; Gilbert, R. W., The Inductronic Product Resolver, *Report* 356, Weston Electrical Instrument Corporation, Sept. 14, 1951; see also Greenwood *et al.*, *op. cit.*, p. 57.

[2] MacNee, A. B., An Electronic Differential Analyzer, *Proc. IRE*, **37**: 1315, 1949.

Figure 6.6 illustrates the principle of the crossed-fields multiplier. An ordinary cathode-ray tube having electrostatic deflection plates and a short-persistence screen is fitted with an axial coil wound around the tube in the vicinity of the second pair of deflection plates.

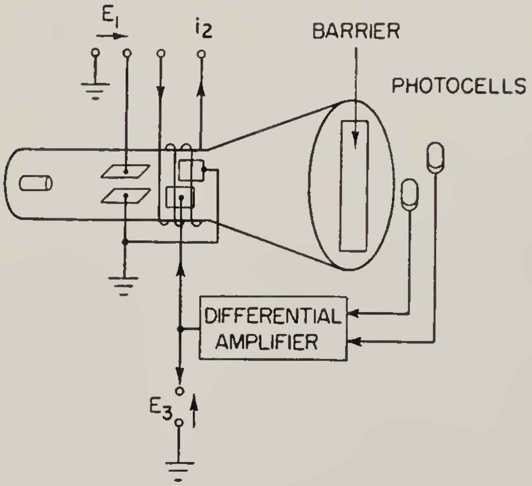

FIG. 6.6. Principle of the crossed-fields electron-beam multiplier (A. B. MacNee, *Proc. IRE*, **37**: 1315, 1949).

Let E_1 and E_3 be the voltages across the first and second pairs of deflection plates, respectively, and let i_2 be the current in the axial coil. The first pair of deflection plates shown will impart a vertical velocity proportional to E_1 to the electron beam. As a result, the beam experiences a *horizontal* magnetic deflection force proportional to E_1i_2 as the electrons pass through the coil.

Any actual deflection of the electron beam will be detected by a pickoff consisting of a pair of high-gain photoelectric cells arranged on either side of a vertical barrier in front of the cathode-ray tube screen.

The difference between the photocell output voltages is amplified by a differential amplifier. The amplifier output voltage E_3 is applied to the second pair of deflection plates in such a manner as to counteract the original deflection. The plates will exert a *horizontal* force proportional to the voltage E_3 on the electron beam. If the gain of the feedback loop consisting of the cathode-ray tube, the photocells, and the amplifier is large, the horizontal deflection of the electron beam will be kept very small. The horizontal forces acting on the beam must then balance each other, so that at least approximately

$$E_3 = (\text{constant})\ E_1i_2 \tag{6.9}$$

If E_1 and i_2 are made proportional to two input voltages X_1 and X_2, the output voltage will be proportional to the product X_1X_2. Equation (6.9) compares favorably with the experimentally determined performance equation

$$E_3 = (\text{constant})\ (E_1i_2 + 0.009E_1^2 + 0.015i_2 + 0.0005) \tag{6.10}$$

of a typical unit. It is believed that the error terms could be reduced by a factor of 10 through better magnetic shielding and through careful positioning of the photocells and the barrier. The use of two units in a push-pull circuit and of automatic balancing devices (see Sec. 5.6) might result in further improvements.

The crossed-fields electron-beam multiplier is a useful and fairly simple device. Perhaps its greatest advantage is its good high-frequency response, which makes this multiplier especially useful for repetitive computers. The frequency response is flat (within 1 per cent) and the phase shift negligible up to *at least* 20,000 cps for E_1 and 3,000 cps for i_2; the latter figure is capable of improvement through suitable equalization of the current amplifier driving the axial coil.

Finally, if the connections in Fig. 6.6 are changed so that the first pair of deflection plates receives its deflection voltage E_1 from the amplifier whereas an input voltage E_3 is applied across the second pair, then

$$E_1 = (\text{constant}) \frac{E_3}{i_2} \qquad (i_2 \neq 0) \tag{6.11}$$

is the output voltage; the crossed-fields electron-beam multiplier may be used for *division* in this manner.

6.3. EXAMPLES OF MULTIPLYING DEVICES USING FEEDBACK TO LINEARIZE THE GAIN CHARACTERISTICS

Frequently, the gain K of a voltage-sensitive variable-gain device can be made *accurately* proportional to the ratio X_2/Y of two voltages (machine variables) X_2 and Y through the use of a *feedback circuit.* Figure 6.7*a* shows such an arrangement. The gain K of the variable-gain device shown in the solid lines *must be independent of its input voltage* (Y in this case). The gain K does vary monotonically but not necessarily linearly with the gain-setting voltage V_o obtained at the output of a high-gain differential amplifier. The latter is seen to amplify the difference between the input voltage X_2 and the voltage KY. Since K varies monotonically with V_o, it is possible to write

$$K = K(V_o) = A(X_2 - KY) \qquad (Y \neq 0) \tag{6.12}$$

where A is a positive quantity proportional to the amplifier gain but in general depending on the value of V_o. It follows that

$$K = \frac{AX_2}{1 + AY} \tag{6.13}$$

If the value of A is made sufficiently large by increasing the amplifier gain, Eq. (6.13) will reduce to

$$K \approx \frac{X_2}{Y} \tag{6.14}$$

The feedback will thus tend to set the gain K automatically near the value X_2/Y. The accuracy with which Eq. (6.14) approximates the correct expression (6.13) will depend on the value of the amplifier gain.[1]

Feedback Multipliers. Assume, now, that it is possible to obtain a second variable-gain device whose gain will vary with the voltage V_o exactly like the gain of the first variable-gain device.

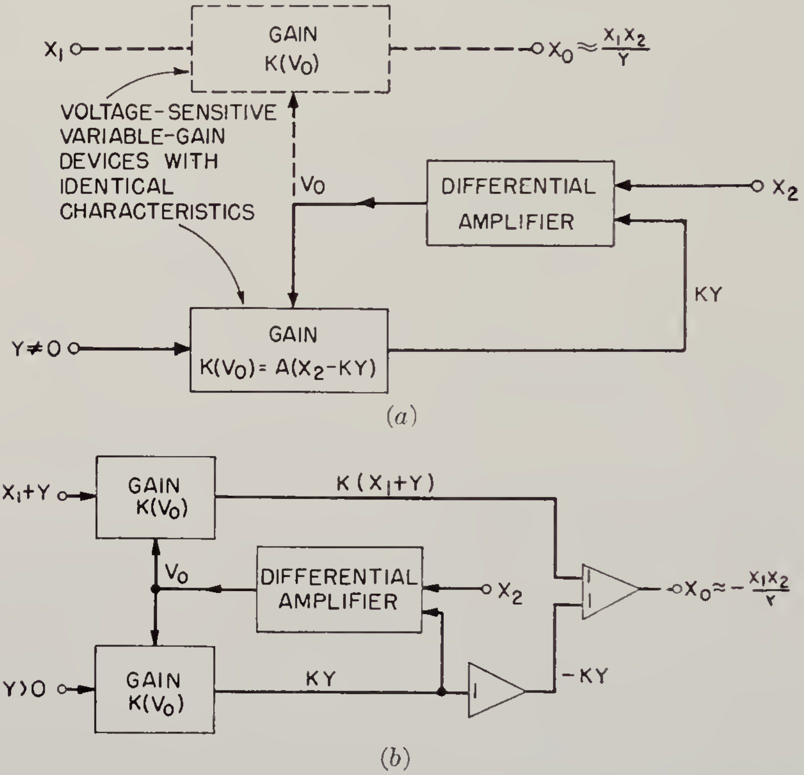

Fig. 6.7. Feedback circuits used for setting the gain of a variable-gain device accurately to a value determined by the input voltages X_2 and Y. Note that X_1, X_2, Y, and X_o are measured in machine units.

If, then, the output voltage V_o of the differential amplifier shown in Fig. 6.7*a* is fed to the second variable-gain device, the latter can be used to multiply any input voltage X_1 by $K \approx X_2/Y$. The dotted lines in Fig. 6.7*a* indicate how the multiplication

$$X_o = X_1 \frac{X_2}{Y} \tag{6.15}$$

may be performed in this manner. It is seen that simultaneous multiplication and division (see Sec. 6.12) is possible by this method. One may perform several multiplications by X_2/Y by providing additional variable-gain devices with identical characteristics and with gains determined by the voltage V_o.

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[1] A uniformly high gain can be obtained only for a limited range of signal frequencies in each case, and the entire circuit will have to be carefully designed for high-frequency stability in a manner similar to that indicated in Sec. 5.4.

The schemes of Fig. 6.2 may be combined with the block diagram of Fig. 6.7*a* to yield four-quadrant multiplication even if the variable-gain devices used permit only positive values of input voltage and/or gain. Figure 6.7*b* shows one such modification of the block diagram of Fig. 6.7*a* (see also Figs. 6.12*b* and 6.15*b*).

The reader will find it interesting to make an analysis of the effects of unbalance voltages due to drift at different points of the circuit in the manner of Sec. 5.1.

Design Problems. Aside from certain difficulties involved in attaining stable feedback action throughout the desired range of operating frequencies, *the chief problems in designing multipliers based on the block diagram of Fig.* 6.7*a are in obtaining*

(1) *voltage-sensitive variable-gain devices whose gain is independent of the input voltage* (X_1 or Y in Fig. 6.7*a*)

(2) *two or more voltage-sensitive variable-gain devices whose gain vs. gain-setting-voltage characteristics are accurately similar*

Practically all simple voltage-sensitive circuit elements, such as thyrite resistors, crystals, or even vacuum tubes, do not satisfy either of the conditions given above; their gain may be dependent on the input voltage, and their characteristics cannot be reproduced with sufficient accuracy. One may overcome these deficiencies (1) by using one and the same variable-gain device to attenuate both Y and X_1 in Fig. 6.7 on a time-sharing or frequency-sharing basis; (2) by using vacuum tubes only as switches (on-off devices). The circuit gain will then be independent of the tube characteristics to a first-order approximation (Sec. 6.4).

Servomultipliers. The operation of the ordinary *multiplying servomechanisms* (which have been shown only as block-diagram symbols in the previous sections) depends on the principle of Fig. 6.7. In Fig. 6.8, the variable-gain devices in this case are potentiometers positioned by a servomotor which is actuated by the output voltage V_o of a differential amplifier. The latter may take the form of a servo amplifier or of a sensitive relay. Servo-driven potentiometers satisfy the design requirements given above quite well. The use of electromechanical devices will, however, limit the range of operating frequencies for the variable X_2. The design and performance of computer servomechanisms is discussed in detail in Secs. 6.8 to 6.11.

Heat-transfer Multipliers.[1] Remarkably small and simple feedback-stabilized multipliers are based on transducers of the type shown in Fig.

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[1] Savet, P. H., Analog Computing by Heat Transfer, *Tele-Tech and Electronic Industries*, February, 1954; Davidson, G. M., Thermal Elements Simplify Computing Circuits, *Elec. Mfg.*, 1954; Davidson, G. M., W. Djinis, and P. H. Savet, Subminiature Thermal Control Element, *Electronic Equipment*, June, 1955. The Arma Corporation (Brooklyn and Garden City, New York), which developed these circuits, has also used the heat-transfer lag in similar computing elements to differentiate and integrate the RMS values of a-c currents.

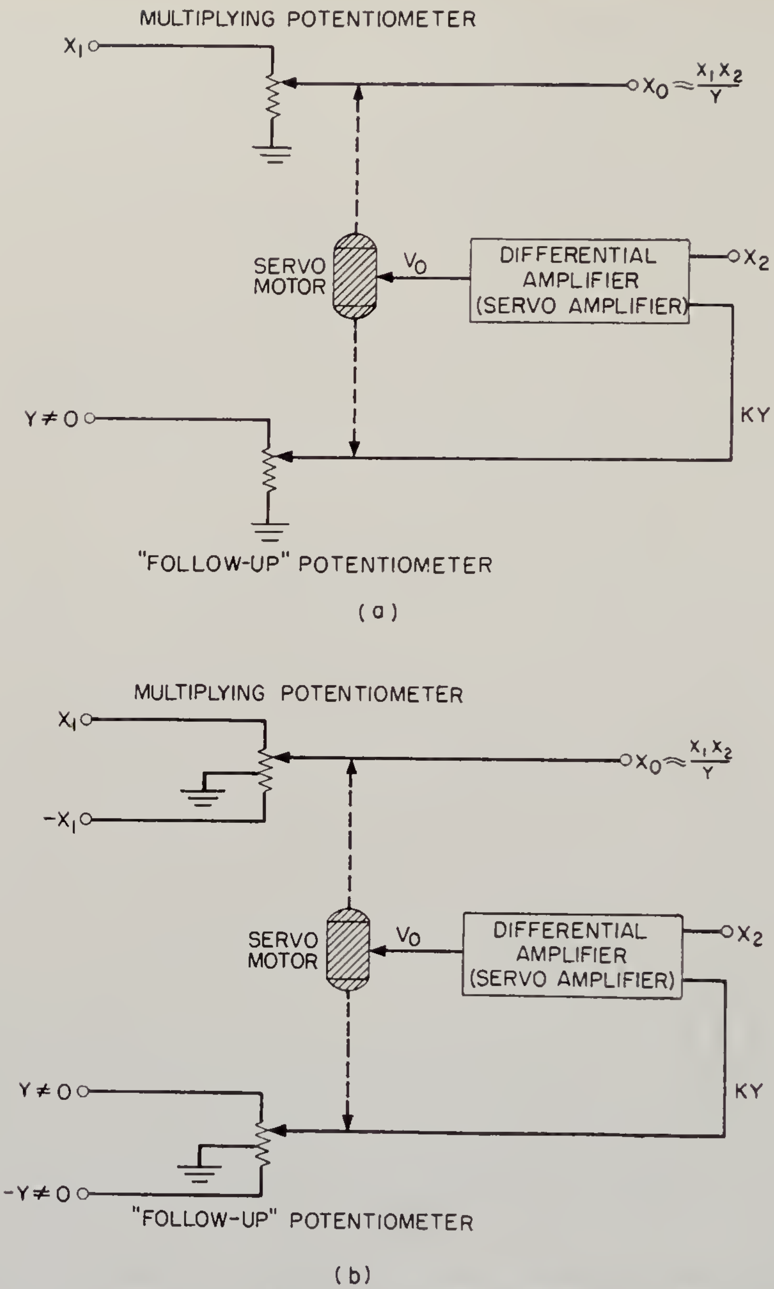
MULTIPLYING POTENTIOMETER
X_1
$X_0 \approx \frac{X_1 X_2}{Y}$
SERVO MOTOR
V_0
DIFFERENTIAL AMPLIFIER (SERVO AMPLIFIER)
X_2
KY
$Y \neq 0$
"FOLLOW-UP" POTENTIOMETER
(a)
MULTIPLYING POTENTIOMETER
X_1
$-X_1$
$X_0 \approx \frac{X_1 X_2}{Y}$
SERVO MOTOR
V_0
DIFFERENTIAL AMPLIFIER (SERVO AMPLIFIER)
X_2
KY
$Y \neq 0$
$-Y \neq 0$
"FOLLOW-UP" POTENTIOMETER
(b)

FIG. 6.8. Servomultipliers connected for two-quadrant operation (*a*) and four-quadrant operation (*b*).

6.9*a*. The gain-changing voltage V_o is added to the heater voltage E; the resulting temperature change varies the resistance of the sensing element and thus the gain K. Figure 6.9*b* shows a feedback-stabilized push-pull multiplier circuit based on this principle; the circuit was originally used to operate on a-c voltages.

Each of the transducers encircled in broken lines in Fig. 6.9*b* comprises a bifilar heating element (No. 40 teflon-insulated nichrome wire) and two sensing elements (No. 40 teflon-insulated nickel wire) wound together on a pyrex rod. Two or more such units may be mounted in a miniature-tube envelope; no vacuum is required.

Accuracies better than 1 per cent of full scale have been obtained with carefully matched gain-setting and output-sensing elements. The frequency response is restricted to about 10 cps by the inherent heat-transfer lag, and the output-voltage range may be limited by self-heating of the sensing elements. Such multipliers are useful mainly in special-purpose computers used as control-system components.

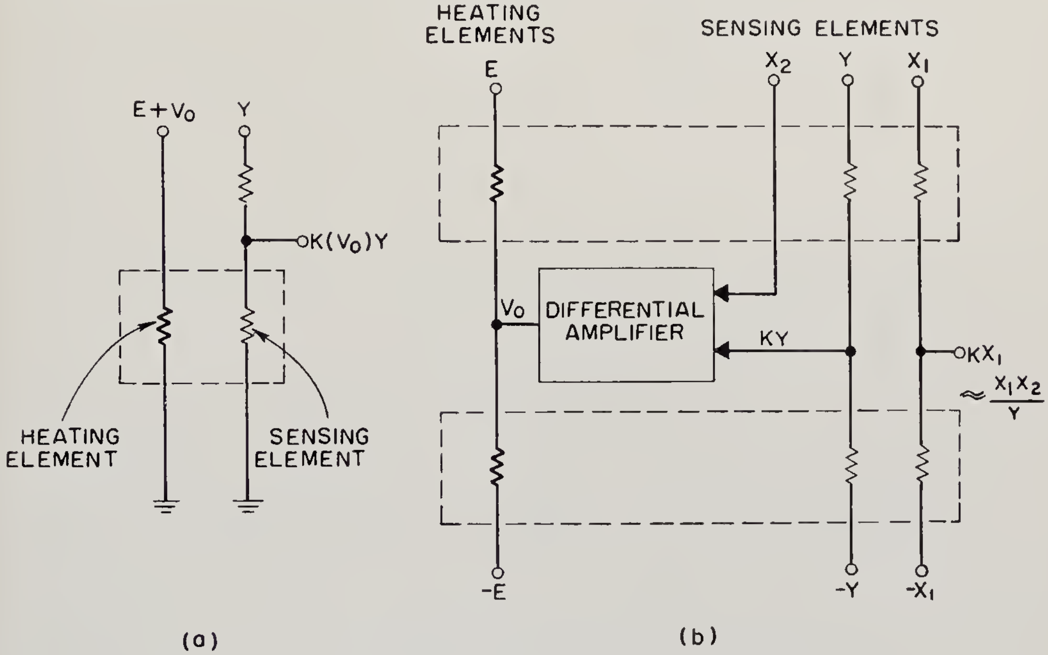

FIG. 6.9. Principle of the heat-transfer multiplier.

The Step Multiplier. The step multiplier, developed by the RCA Research Laboratories,[1] is a very accurate computing element useful for multiplications by functions of machine variables as well as for multiplications by machine variables. The step multiplier is not a pure analog-computing element since it uses a digital counter to regulate the gain of a variable-gain amplifier.

Figure 6.10 illustrates the principle of the step multiplier.[2] One may vary the transfer function of the operational amplifier shown by placing different combinations of resistors in parallel by means of a set of relays. If the relays are initially open and are closed successively as time progresses, the gain K of the operational amplifier will increase progressively as the input resistance is effectively decreased. If different combinations of relays are closed (or opened) in succession with equal time intervals, the various resistors can be chosen so as to permit multiplication of the input voltage by a stepwise-varying monotonic function of the time. A somewhat similar device, utilizing one large stepping relay instead of a multiplicity of simple relays, has actually been used to generate functions of the time.[1]

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[1] Princeton, N.J.

[2] Goldberg, E. A., Step Multiplier in Guided Missile Computer, *Electronics*, **24**: 120, 1951.

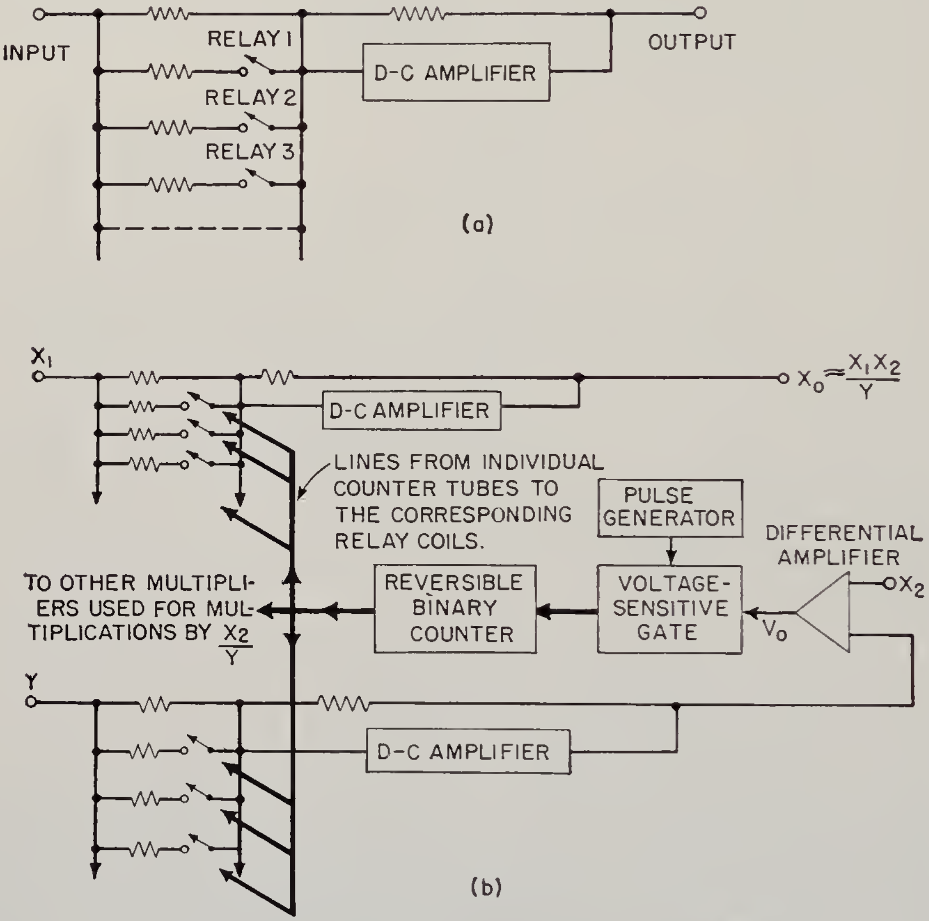
INPUT
RELAY 1
RELAY 2
RELAY 3
D-C AMPLIFIER
OUTPUT
(a)
X_1
$X_0 \approx \frac{X_1 X_2}{Y}$
D-C AMPLIFIER
LINES FROM INDIVIDUAL COUNTER TUBES TO THE CORRESPONDING RELAY COILS.
PULSE GENERATOR
DIFFERENTIAL AMPLIFIER
X_2
TO OTHER MULTIPLIERS USED FOR MULTIPLICATIONS BY $\frac{X_2}{Y}$
REVERSIBLE BINARY COUNTER
VOLTAGE-SENSITIVE GATE
V_0
Y
D-C AMPLIFIER
(b)

FIG. 6.10. The step multiplier (simplified block diagrams).

If different combinations of the relays are opened or closed in a manner dependent on the value of a *voltage* V_o, this device becomes a voltage-sensitive variable-gain amplifier whose gain K varies step by step as the voltage V_o changes. This principle has been made the basis of a feedback multiplier of the type illustrated in Fig. 6.7*a*.

Figure 6.10*b* shows a simplified block diagram of a step multiplier. The relays are opened and closed by corresponding stages of a reversible binary counter.[1] The reversible counter counts pulses generated by a pulse generator and is controlled (gated) by the gain-determining output voltage V_o of the differential amplifier shown. If the voltage V_o is positive, the electronic gate applies the pulses to the counter so that it will count up. Under these conditions, the counter stages will open and close relays so as to increase the gain K of the two identical variable-gain devices until it becomes equal to the correct value X_2/Y. The gate will then close. Whenever the gain K is less than X_2/Y, the output voltage V_o of the differential amplifier will become positive, and the electronic gate will cause the counter to count up and thus increase the multiplier gain K. In this manner, the latter can be stabilized at the desired value X_2/Y. The stepwise variation of the multiplier output voltage may be smoothed by means of a smoothing filter.

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[1] Hagelbarger, D. W., C. E. Howe, and R. M. Howe, Investigation of the Utility of an Electronic Analog Computer in Engineering Problems, *UMM*-28, Apr. 1, 1949, Aeronautical Research Center, University of Michigan, Willow Run.

The accuracy of the step multiplier can be improved by increasing the number of relays used. With 14 counter stages, the tubes of the binary counter could energize the relays in the sequence: (none); (1); (2); (1,2); (3); (1,3); (2,3); (1,2,3); (4); etc.; the 28 possible relay positions permit $2^{14} = 16{,}384$ distinct values of the multiplier gain or a theoretical accuracy of better than 0.01 per cent of full scale. The gain can be made to increase or decrease in equal steps if the resistance values of the resistors associated with the different relays are chosen judiciously. The actual accuracy of the step multiplier is limited to about 0.01 per cent by d-c amplifier drift and by the accuracy of the gain-setting resistors.

The frequency range of the gain-setting variable X_2 in step multipliers is severely limited by the frequency response of the relays. Not only must the relays open or close within the time required for the smallest gain changes under consideration, but the relays in two or more variable-gain devices must "track" in order to yield accurately similar gain-vs.-voltage characteristics. The maximum counting rate usable with high-speed relays is of the order of 1,500 counts per second, corresponding to a maximum voltage rate of change

$$\frac{dX_2}{d\tau} \leq \frac{1{,}500 V_{\max}}{2^n} \tag{6.16}$$

where $V_{\max}$ is the voltage range and n is the number of relays used. Generally speaking, step multipliers can be about as fast as servomechanisms of equal accuracy. The accuracy can be improved at the expense of frequency response.

The step multiplier may be used as an accurate function generator if the resistors associated with the various relays are chosen correctly. Although one counter and gate circuit may, in any case, be used for several multiplications by the same machine variable or by functions of this machine variable, the relative complexity of the step-multiplier circuits (over 50 tubes may be required for a single 14-relay multiplier) will restrict the usefulness of such devices to applications where high accuracy is of paramount importance.

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[1] For a detailed discussion of the circuits used, see *Project TYPHOON Progress Report* 2, Contract N6oni196, RCA Laboratories Division, Princeton, N.J.

Time-sharing Schemes. The very serious difficulty in obtaining two or more *identical* attenuator elements for use in feedback multipliers can sometimes be resolved by using a single attenuator on a time-sharing basis. As an example, Fig. 6.11 shows a *commutated-potentiometer servo* of the type developed at the RCA Research Laboratories.[1] The circuit shown is essentially the same as the servomultiplier of Fig. 6.8*b*. *A single potentiometer is switched back and forth between the multiplying and gain-adjusting sections of the circuit.* This is accomplished by means of three commutator-type switches rotated at 3,600 rpm or by means of synchronous vibrators. The potentiometer output voltage for each switch position is "remembered" by an *RC* smoothing network which filters out the commutator ripple. The use of a single potentiometer in a computer servomechanism assures perfect "tracking" of the multiplying and follow-up sections of the circuit; no expensive accurately linear wire-wound potentiometers are required. A fairly low-cost potentiometer having good resolution, such as a metalized or carbon type, may be used. The frequency response of the servomechanism is not affected by the new arrangement and still constitutes a limit to the usefulness of the multiplier.

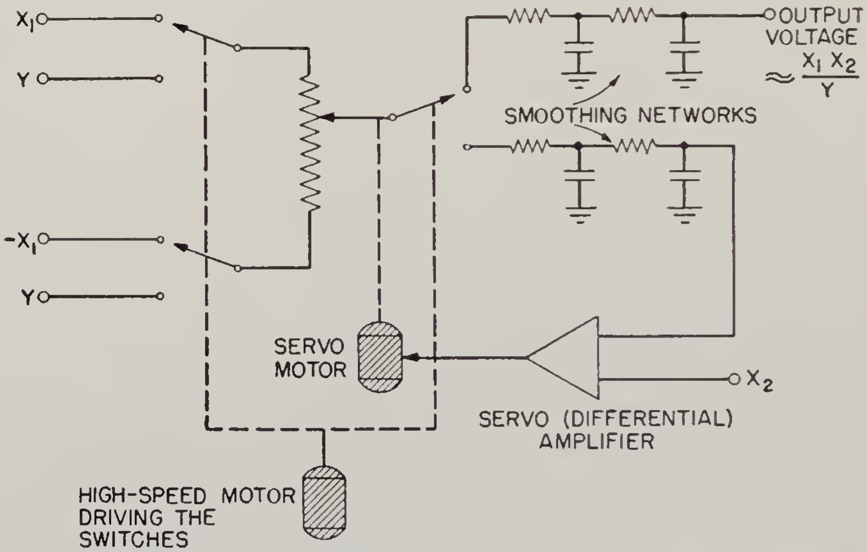
X_1
Y
$-X_1$
Y
OUTPUT VOLTAGE $\approx \frac{X_1 X_2}{Y}$
SMOOTHING NETWORKS
SERVO MOTOR
X_2
SERVO (DIFFERENTIAL) AMPLIFIER
HIGH-SPEED MOTOR DRIVING THE SWITCHES

FIG. 6.11. Commutated-potentiometer servo. The motor-driven switches may be replaced by vibrating relays. The load resistances in the multiplier and follow-up branches must be equal for good accuracy.

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[1] Princeton, N.J. (TYPHOON Project).

Frequency-sharing Schemes and AM-FM Multipliers. With voltage-sensitive attenuators whose gain is independent of frequency over a sufficiently wide range, it is again possible to use a single attenuator in a feedback-multiplier arrangement. Figure 6.12*a* shows a possible arrangement. The input machine variable X_1 and the reference voltage Y are simply added and applied to the voltage-sensitive attenuator, which may be a simple variable-gain amplifier or a modulator-demodulator arrangement (see Sec. 6.2). The reference voltage Y is the amplitude of an a-c voltage whose frequency is above the desired range of signal frequencies to be used with the multiplier. The resulting high-frequency component of the attenuator output voltage is filtered by a band-pass filter, rectified, and used for adjusting the attenuator gain. The product output voltage, on the other hand, is freed of the high-frequency components by means of a low-pass filter. It is seen that the attenuator transfer function can be made accurately proportional to X_2/Y in this manner; arrangements of this type have permitted accuracies better than 0.2 per cent of full scale.[1]

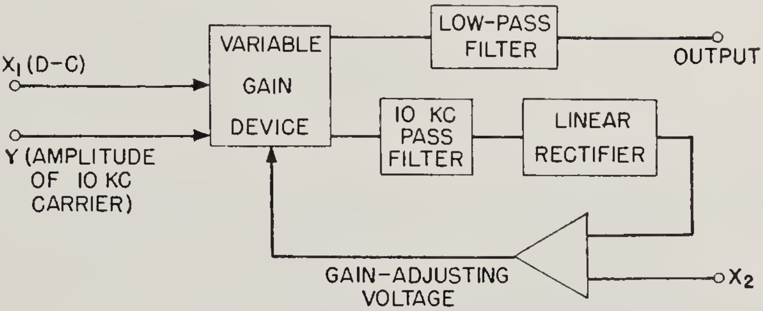


FIG. 6.12*a*. Frequency-sharing feedback multiplier.

Figure 6.12*b* shows the block diagram of an AM-FM multiplier[2] based on the feedback scheme of Fig. 6.7*b*. The output voltage of each FM discriminator (Foster-Seely discriminator) is proportional to the carrier amplitude ($X_1 + Y$ or Y) and increases with the carrier frequency; the latter is determined by the gain-setting voltage V_o applied to a frequency modulator.[3] The frequency deviation, and thus each discriminator output voltage, changes sign together with V_o. The amplitude-modulator input voltages, which *cannot* change the sign of the corresponding discriminator output voltages, are always positive (see also Fig. 6.2*a*). Note that the amplitude modulation as well as the frequency modulation is linearized by feedback.

[1] Chance, Williams, Hughes, Sayre, and MacNichol, *op. cit.*, Sec. 19.5.

[2] McCool, W. A., An AM-FM Electronic Analog Multiplier, *Proc. IRE*, **41**: 1470, 1953; see also R. Price, An FM-AM Multiplier of High Accuracy and Wide Range, *RLE Report* 213, Oct. 4, 1951, Massachusetts Institute of Technology, Cambridge, Mass.; Somerville, M. J., An Electronic Multiplier, *Elec. Eng.*, February, 1952.

[3] Refer to McCool, *op. cit.*, for the complete circuit.

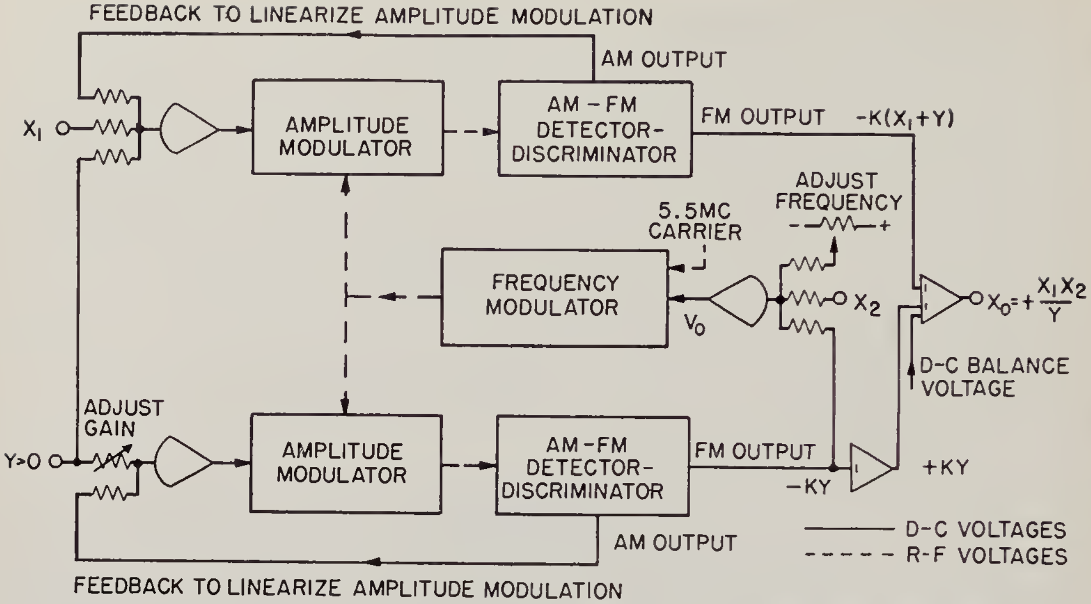


FIG. 6.12*b*. Block diagram of an AM-FM multiplier. The two discriminators used should be adjusted so that their characteristics are as closely identical as possible; temperature-compensated tuning circuits are recommended. The GAIN, FREQUENCY, and BALANCE controls are adjusted for zero output with $X_1 = 0$, $X_2 = 0$, and $X_1 = X_2 = 0$, respectively.

AM-FM multipliers using chopper-stabilized d-c amplifiers and carefully matched discriminators permit accuracies better than 0.3 per cent of full scale (200 volts); the high carrier frequencies used insure good high-frequency response (flat within 1 db, less than 15-deg phase shift up to 1 per cent of the carrier frequency).

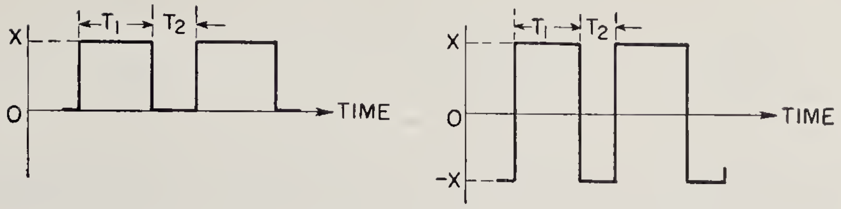
6.4. MULTIPLIER CIRCUITS BASED ON SIMULTANEOUS PULSE-AMPLITUDE MODULATION AND PULSE-WIDTH MODULATION

Introduction. The operation of most commercially available electronic multipliers is based on simultaneous pulse-amplitude modulation and pulse-width modulation of a square-wave carrier. Such multipliers are capable of high static accuracy (better than 0.05 per cent of full scale) and fair frequency response (usable to above 2 kc). Each complete multiplier requires between four and nine tubes in addition to three or four d-c amplifiers.

The average (smoothed) values of the square-wave pulse trains pictured in Figs. 6.13*a* and *b* are respectively given by

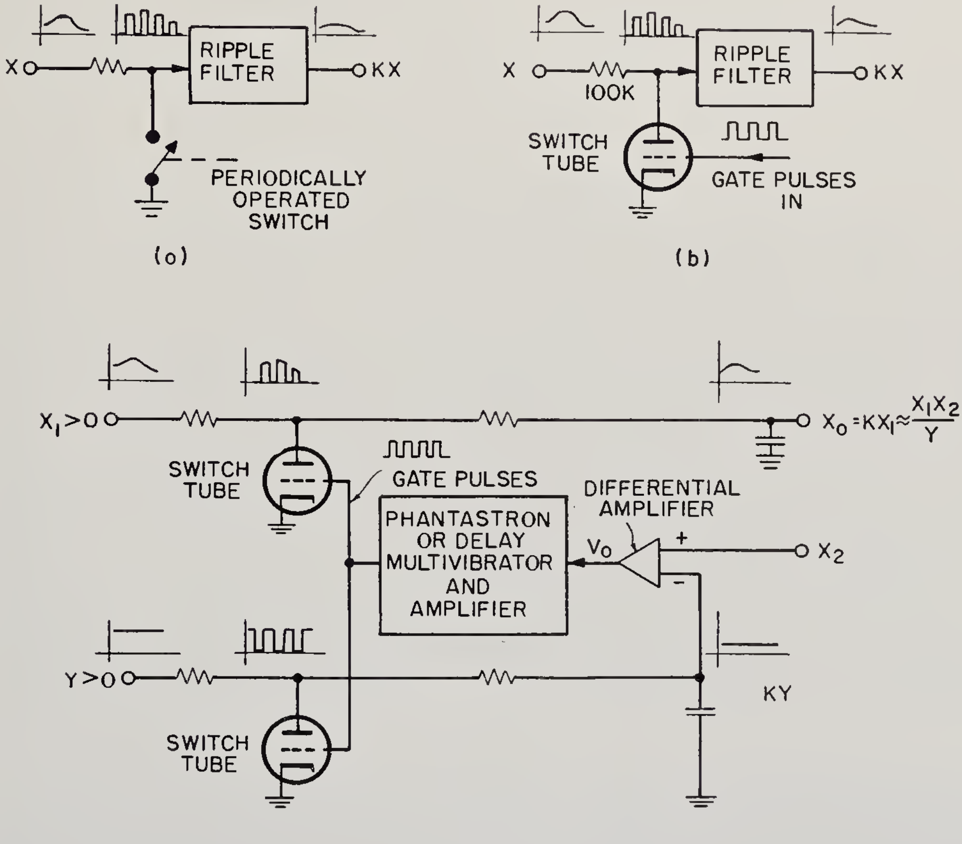
$$X\frac{T_1}{T_1 + T_2} = XnT_1 \qquad \text{and} \qquad X\frac{T_1 - T_2}{T_1 + T_2}$$

where X is the pulse amplitude, T_1 is the width of the positive pulse, and $n = 1/(T_1 + T_2)$ is the pulse repetition rate. The circuits to be



(a) (b)

FIG. 6.13. The average value (d-c component) of the square-wave voltages shown is $X\frac{T_1}{T_1+T_2}$ in Fig. 6.13*a* and $X\frac{T_1-T_2}{T_1+T_2}$ in Fig. 6.13*b*.



(a) (b)

(c)

FIG. 6.14(*a*), (*b*). Principle of the pulsed attenuator. The gain K is determined by the ratio of the pulse width T_1 to the repetition period $T_1 + T_2$ of the periodically switched waveform. (*c*) Two simple pulsed attenuators arranged as a one-quadrant feedback multiplier.

described utilize these relations to multiply a voltage X_1 controlling the pulse amplitude X by a second voltage X_2 which controls either $T_1/(T_1 + T_2)$ or $(T_1 - T_2)/(T_1 + T_2)$.

Pulsed Attenuators. Figure 6.14*a* illustrates the principle of a *pulsed attenuator.* The switch shown is operated at a repetition rate n large compared to the highest signal frequency and chops the input signal into approximately square pulses of amplitude X and width T_1. The resulting pulse-amplitude-pulse-width modulated square-wave voltage is averaged (smoothed) by a low-pass filter which yields the output voltage

$$X_o = aXnT_1 = KX \tag{6.17}$$

If $K = anT_1$ is made to depend on a second input voltage by means of suitable timing circuits, the pulsed attenuator becomes a voltage-sensitive variable-gain device.

Triggered Pulsed-attenuator Multipliers.[1] The simple pulsed-attenuator circuit of Fig. 6.14*b* incorporates an electronic switch which is actuated by amplified gate pulses from a phantastron or delay multivibrator[2] triggered at a constant repetition rate n. The latter circuits make the duration T_1 of each negative gate-pulse excursion roughly proportional to a gain-setting voltage, so that the relation (6.17) results in multiplication with an accuracy of the order of 1 per cent.

More accurate pulse-width modulation, and thus more accurate multiplication, is made possible by the feedback principle indicated in Fig. 6.7. Figure 6.14*c* shows two pulsed attenuators arranged as a simple (one-quadrant) feedback multiplier. The "on" period T_1 for both attenuators is determined by the gain-setting input voltage V_o to the phantastron or delay multivibrator. As a result of feedback through a differential amplifier and one of the pulsed attenuators, the gain $K(V_o) = anT_1$ of each attenuator will become practically equal to X_2/Y, just as in Fig. 6.7*a*. Since the switch tubes in Fig. 6.14 require a positive plate voltage to operate, the basic circuits shown restrict the range of input voltages and permit only one-quadrant multiplication. Four-quadrant operation can be obtained in the manner discussed in Sec. 6.1 (see also Fig. 6.15*b*).

Operation of a Self-excited Time-division Multiplier[3]: The Relay Multiplier. The operation of a self-excited time-division multiplier is well illustrated by that of the simple four-quadrant relay-multiplier cir-

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[1] Early circuits of this type were developed at Cornell University, Ithaca, N.Y., under the direction of H. S. Sack (*NDRC Report* 14-435 and Cornell thesis of A. C. Beer and H. W. Boehmer). See also I. A. Greenwood, J. V. Holdam, and D. MacRae, *Electronic Instruments,* MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948; and L. G. Walters, A Study of the Series-motor Relay Servomechanism, *Ph.D. thesis,* University of California, Los Angeles, 1951.

[2] In such circuits, each trigger pulse starts a square pulse which is integrated until the absolute value of the integral equals a predetermined fraction of a d-c input voltage. At this point the pulse is terminated by regenerative action. For a detailed discussion, see Chance *et al., op. cit.,* Chap. 5.

[3] Goldberg, E. A., A High-accuracy Time-division Multiplier, *RCA Rev.,* **23**: 265, September, 1952; Morrill, C. D., and R. V. Baum, A Stabilized Electronic Multiplier, *Trans. PGEC, IRE,* December, 1952; Korn, G. A., and T. M. Korn, Relay Time-division Multiplier, *Rev. Sci. Instr.,* **25**: 977, October, 1954. The term *time-division multiplier* is often applied to all types of multiplier circuits based on pulse-amplitude-pulse-width modulation, including those operating at constant repetition rates.

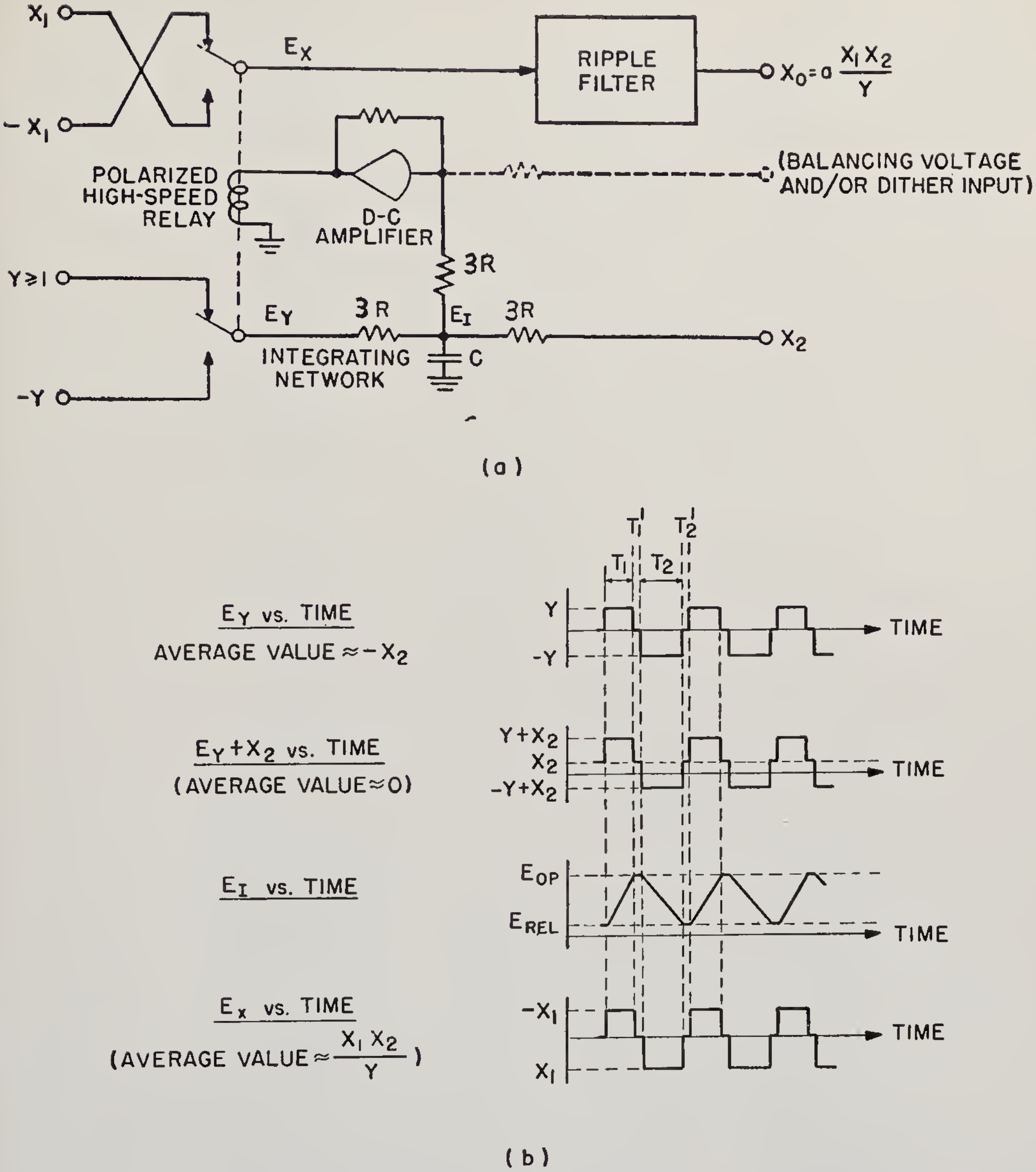

Fig. 6.15*a*, *b*. This simple relay time-division multiplier circuit illustrates how double-throw switching operations yield four-quadrant multiplication. For faster switching and extended high-frequency response, the relay is replaced by a pair of double-throw electronic switches and the associated gate generator (Fig. 6.15*c* and *d*). The principle of operation remains the same.

cuit shown in Fig. 6.15*a*. The positive voltage $E_Y = Y + X_2$ charges the integrating capacitor at the rate $(Y + X_2)/RC$ until the voltage E_I reaches a value E_{OP} sufficient to energize the relay.[1] The integrating

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[1] Since a relay operates at a specified value of coil *current* rather than coil voltage, the voltages E_{OP} and E_{REL} depend slightly on the rates at which E_I increases or decreases, and thus on the voltages X_2 and Y. This effect can be neglected if the relay operating and release delays and switching times are negligible compared to T_1 and T_2.

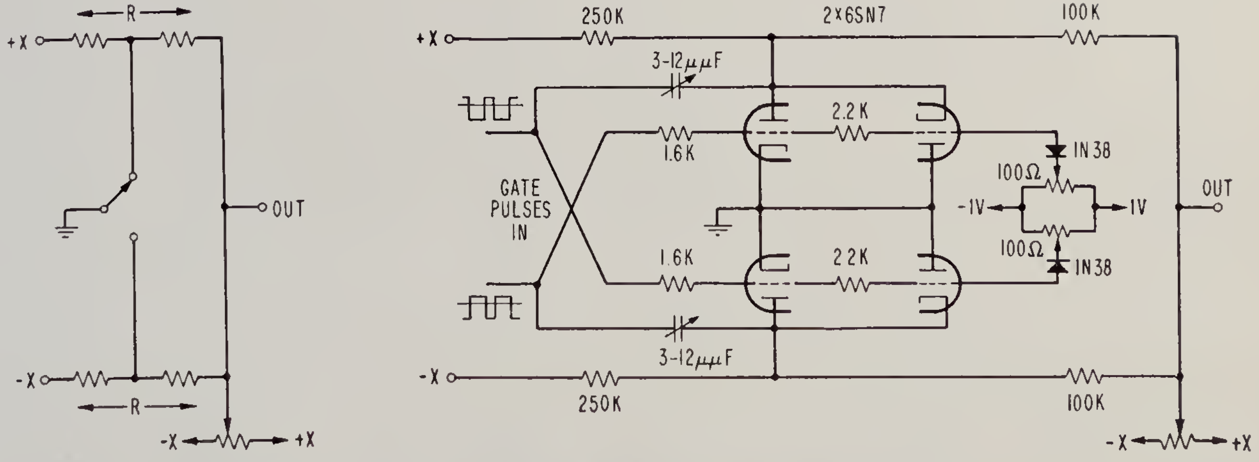

FIG. 6.15*c*. This double-throw shunt switch (voltage switch) alternately grounds the X input and the $-X$ input. Both the input voltage X and the gain K may be either positive or negative. The negative gate-pulse excursion is -50 volts. Crystal-diode clamping circuits permit individual adjustments of the positive gate-pulse excursion for each switch tube. Two such switches are needed for an electronic multiplier (Reeves Instrument Corporation).

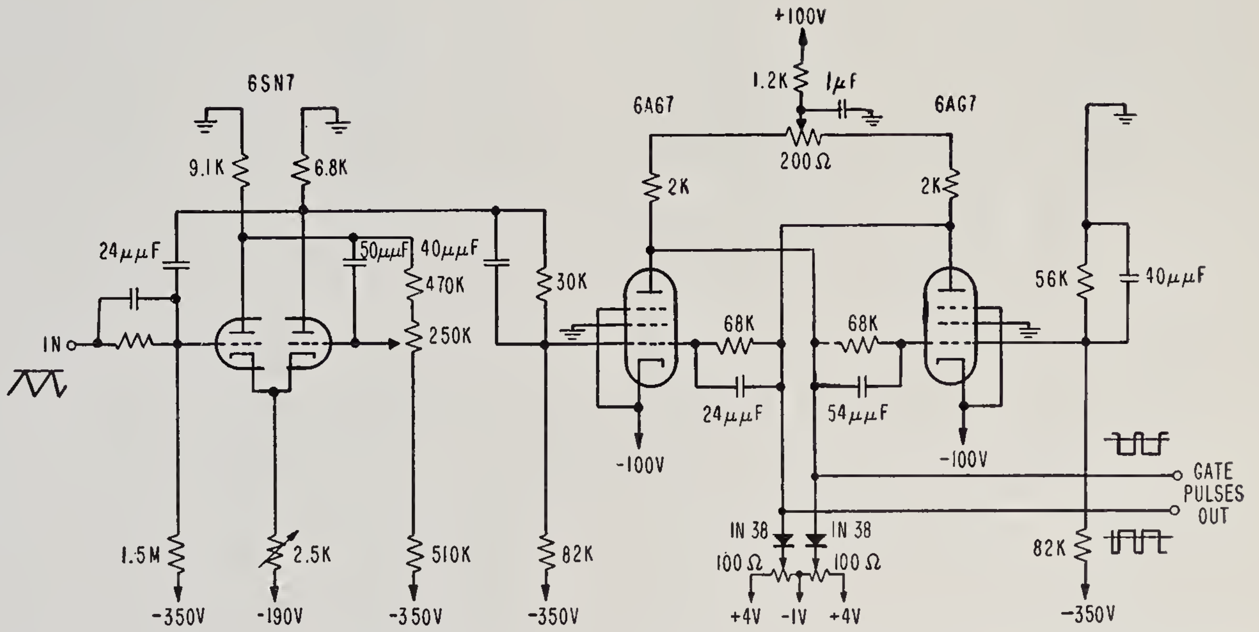

Fig. 6.15*d*. Gate-generator circuit for the electronic switch of Fig. 6.15*c*. The first bistable multivibrator (Schmitt trigger circuit) triggers the second one whenever the voltage E_I (Figs. 6.15*a* and *b*) reaches the values E_{OP} and E_{REL}. The second bistable multivibrator yields steep-sided gate pulses. The repetition rate is of the order of 5 kc (Reeves Instrument Corporation).

network will then integrate $E_Y = -Y + X_2$, so that E_I decreases at the rate $(Y - X_2)/RC$. When E_I has decreased to the value E_{REL} the relay releases, and the entire process is repeated.

For sufficiently slow repetition rates, the relay operating and release delays as well as the switching times T_1' and T_2' are negligible, so that the relay is deenergized and energized for the respective time intervals

$$T_1 = \frac{RC(E_{OP} - E_{REL})}{Y + X_2} \qquad T_2 = \frac{RC(E_{OP} - E_{REL})}{Y - X_2} \tag{6.18}$$

and the repetition rate is

$$n = \frac{1}{T_1 + T_2} = \frac{1}{2RC(E_{OP} - E_{REL})} \frac{Y^2 - X_2^2}{Y} \tag{6.19}$$

If the X_1 contacts of the relay move in exact synchronism with the Y contacts (identical pulsed attenuators), then the output voltage X_o of the ripple filter is proportional to the desired product; specifically

$$X_o = -a \frac{T_1 - T_2}{T_1 + T_2} X_1 = a \frac{X_1 X_2}{Y} \tag{6.20}$$

where a is the low-frequency transfer function of the ripple filter.

The simple relay multiplier requires no electronics other than an ordinary computer d-c amplifier. A laboratory model, limited by the switching speeds of available relays[1] to repetition rates below 60 cps, yielded a static accuracy of better than 2 per cent of full scale (plus and minus 80 volts), with a frequency response comparable to that of a 60-cps computer servomechanism. One might improve this performance considerably by using faster relays and by substituting a comparator or bistable multivibrator (circuits 6.7 and 6.8 of Table 6) for the simple d-c amplifier indicated in Fig. 6.15*a*. It is also possible to use a polarized DPDT relay with a neutral center position. Finally, one may obtain triggered pulsed-attenuator action by introducing a sinusoidal "dither" voltage into the auxiliary input terminal indicated in Fig. 6.15*a*; this should result in faster relay operation as well as in a constant repetition rate.

Electronic Time-division Multipliers. Much faster switching and thus higher repetition rates and improved multiplier high-frequency response become possible if the relay in Fig. 6.15*a* is replaced by a pair of double-throw electronic switches driven by a common gate generator (Figs. 6.15*c* and 6.15*d*). The principle of operation remains the same: the gate-generating bistable multivibrators (Fig. 6.15*d*) "flip" and "flop" whenever the voltage E_I reaches the limiting values E_{OP} and E_{REL}, so

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[1] Siemens & Halske type T rls 66 a T Bv 3602/54, polarized by a 1N34 crystal diode.

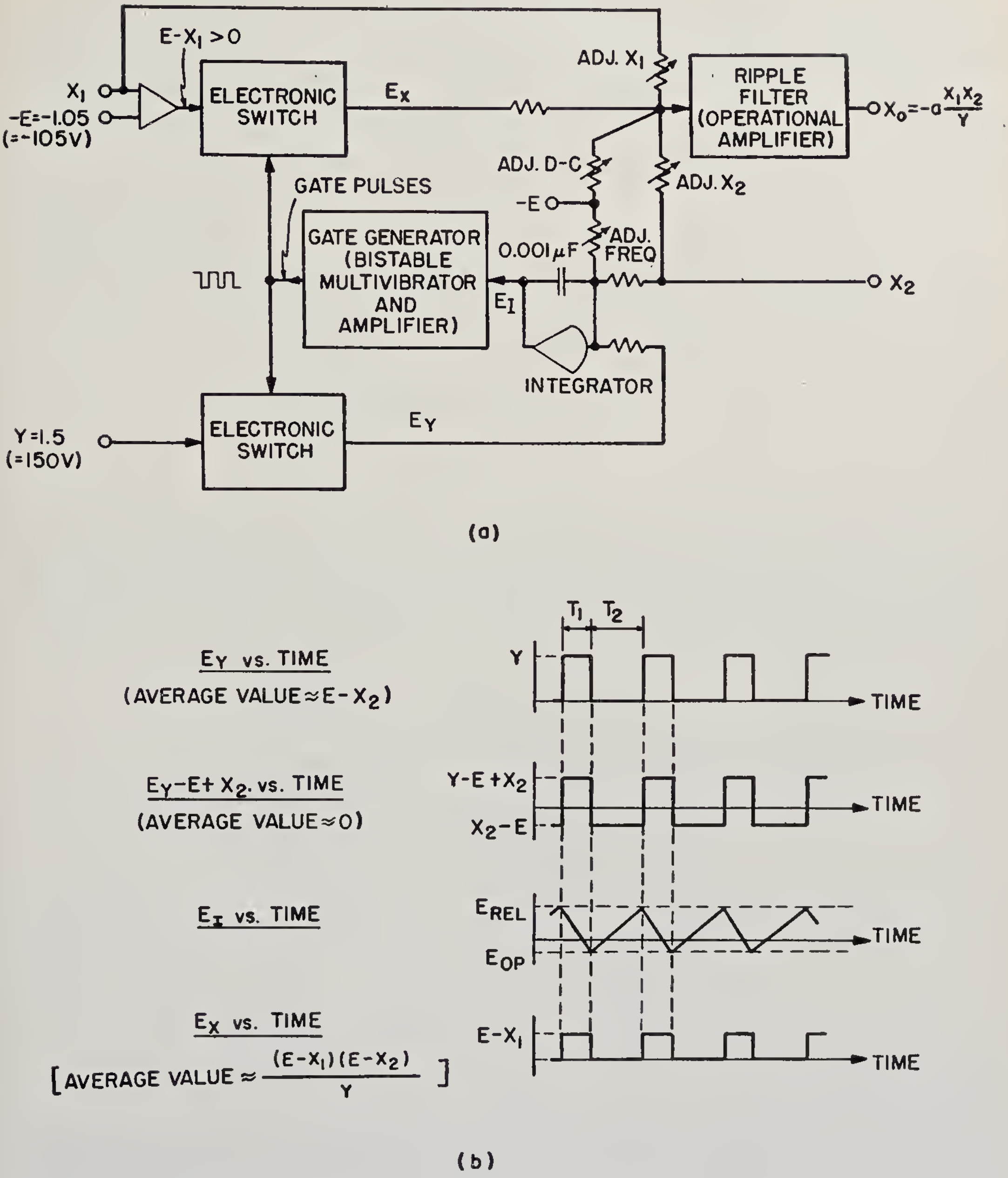

Fig. 6.16*a*, *b*. Block diagram and idealized waveforms for a time-division multiplier employing single-throw electronic switches which can operate only for positive values of both input voltage and gain (Fig. 6.16*c*). Four-quadrant operation is obtained through addition of a positive voltage E to both $-X_1$ and X_2 and subtraction of the unwanted product terms $-X_1E/Y$, $-X_2E/Y$, and E^2/Y from the output voltage of the electronic switch (see also Fig. 6.2*a*).

that each electronic switch (Fig. 6.15*c*) periodically reverses the sign of its output voltage, just like the relay in Fig. 6.15*a*.

Figure 6.16 illustrates the operation of an electronic time-division multiplier utilizing somewhat simpler (single-throw) electronic switches; four-quadrant operation is obtained in the manner of Fig. 6.2*a*.

Precision Switching Circuits. Whereas the electronic switch circuit of Fig. 6.15*c* permits individual adjustments of the switch-tube resistances, the precision switching schemes shown in Fig. 6.17 minimize the effects of switch-tube plate resistances by means of inverse feedback. Both circuits are suitable for operation at repetition rates of at least 20 kc; some matching of tubes for interelectrode capacitances may be necessary.

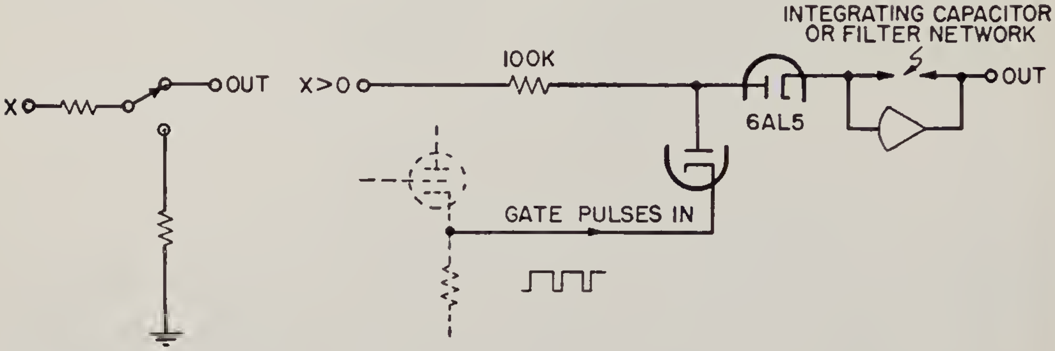

Fig. 6.16*c*. A single-throw electronic-switch circuit based on combined shunt and series switching. Both the input voltage X and the gain K of the pulsed attenuator must be positive. Two such switches are needed for the multiplier of Fig. 6.16*a* (Donner Scientific Company).

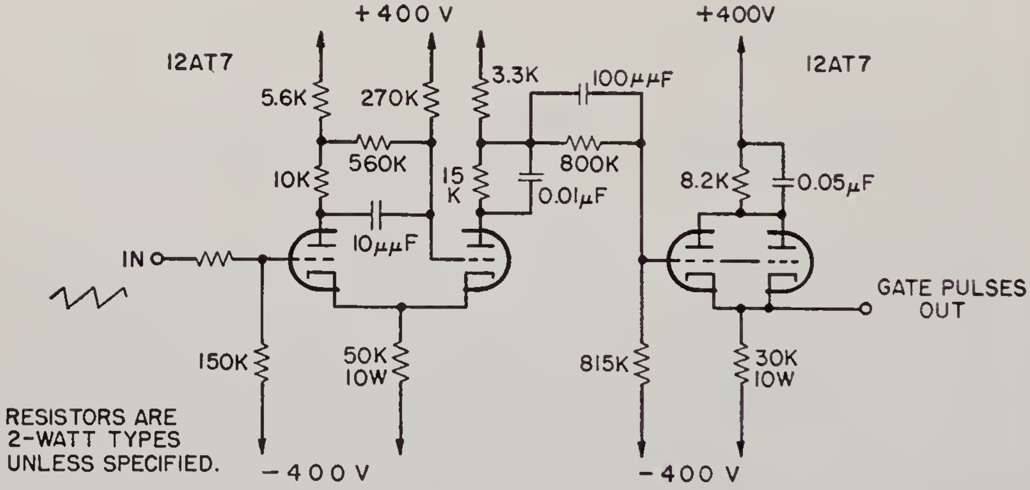

Fig. 6.16*d*. This simple gate generator for the electronic switch of Fig. 6.16*c* comprises a bistable multivibrator and a cathode-follower output stage. The output voltage "flops" to its most negative value (and thus opens the switch) when the input voltage decreases below the value E_{OP}; the output voltage "flips" back when the input voltage increases again to the level E_{REL} (Donner Scientific Company).

All resistors indicated are wire-wound precision units. In the circuit of Fig. 6.17*b*, one may eliminate two wire-wound resistors carrying high-frequency current by grounding the plate of V_1 and using the circuit as a single-throw switch in the manner of Fig. 6.16. In an earlier version of the circuit of Fig. 6.17*b*, the high impedance of a pentode used to drive the switches helped to remove the relatively small a-c components appearing on the switch-tube cathodes from the input d-c amplifier and its wire-wound feedback resistors.[1] In the case of the triode circuit shown, a similar effect is obtained as an incidental benefit of the dynamic balancing circuit described below.

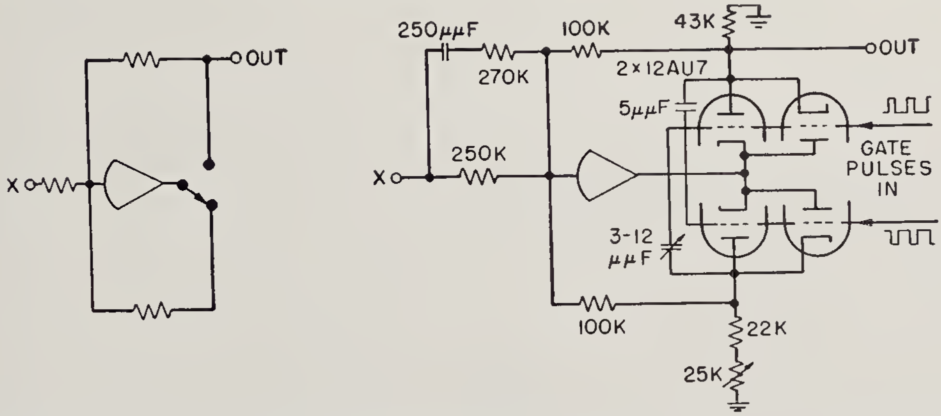

FIG. 6.17*a*. Goodyear's precision single-throw switch uses shunt feedback to make the output voltage largely independent of the switch-tube characteristics. A high-gain commutator-stabilized d-c amplifier is used. The gain K is necessarily negative, but the input voltage X may be positive or negative. One may improve the accuracy further by restricting X to positive values, so that two of the switching triode sections can be omitted (Goodyear Aircraft Corporation).

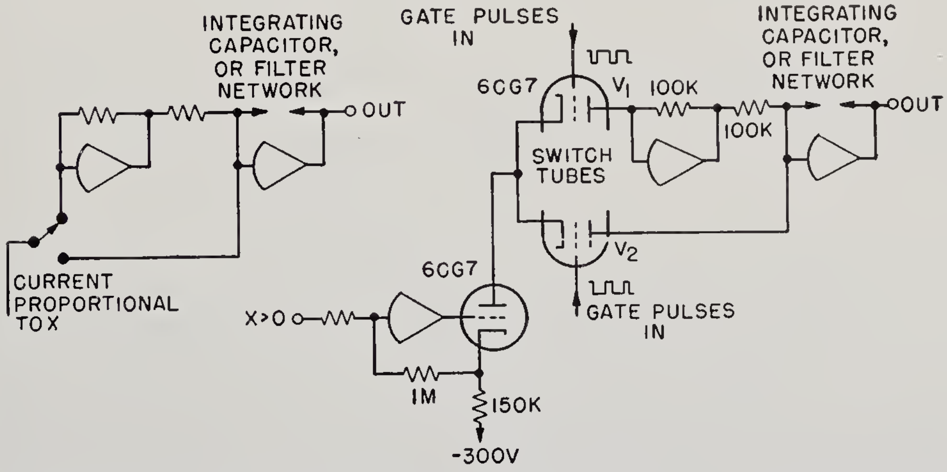
INTEGRATING CAPACITOR, OR FILTER NETWORK
OUT
CURRENT PROPORTIONAL TO X
GATE PULSES IN
6CG7
V1
100K
100K
INTEGRATING CAPACITOR, OR FILTER NETWORK
OUT
SWITCH TUBES
V2
GATE PULSES IN
6CG7
X>0
1M
150K
-300V

FIG. 6.17*b*. This precision double-throw series switch (current switch) uses a chopper-stabilized high-gain d-c amplifier with current feedback as a current generator (high-impedance source), so that the effect of the switch-tube plate resistances on the output voltage is very small. The gate voltages vary between −50 volts and −65 volts. The gain K can take positive or negative values, but the input voltage X must be positive (RCA Laboratories Division).

The theoretical performance of the feedback schemes indicated in Figs. 6.17*a* and 6.17*b* will be slightly impaired by the by-passing action of stray capacitances at the ripple frequency. Thus, in Fig. 6.17*b*, the capacitance between the switch-tube cathodes and ground shunts the high impedance of the current generator, so that any difference in the switch-tube plate resistances will cause a small unbalance of the output voltage. For switching frequencies below 10 kc, one can reduce the resulting error below 0.1 per cent by keeping stray capacitances as small as possible, by matching switch tubes, by adjusting the 100K resistors shown in Fig. 6.17*b*, or by adjusting the gate-pulse levels in the manner of Fig. 6.15*b*. The latter purpose is accomplished automatically by the *RCA dynamic balancing circuit* shown in Fig. 6.17*c*. An a-c amplifier senses the square-wave unbalance voltage on the switch-tube cathode and sets both positive and negative gate-pulse excursions on each switch tube so as to minimize the unbalance.

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[1] Goldberg, E. A., *op. cit.*

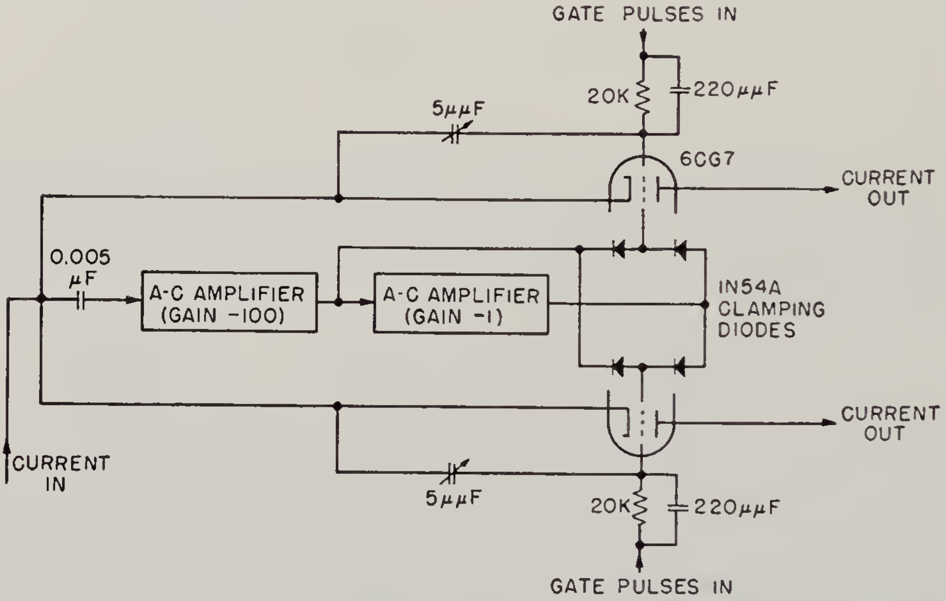

FIG. 6.17*c*. The RCA dynamic balancing circuit may be used with the electronic switch of Fig. 6.17*b* to reduce any remaining effects of switch-tube unbalance in the ratio of 100 to 1. Only two dual tubes are required (RCA Laboratories Division).

The electronic switching circuits shown in Figs. 6.15*c*, 6.16*c*, and 6.17 may be used in triggered pulsed-attenuator multipliers as well as in self-excited time-division multipliers.

"Ganging" of Multipliers. Both triggered and self-excited pulsed-attenuator multipliers are capable of multiplying not just one but several input voltages, X_1, X_1', X_1'', . . . , say, by the voltage X_2. It is merely necessary to provide as many synchronously driven "slave" switches and ripple filters as there are multiplicands X_1, X_1', X_1'', ... If more than three electronic switches are to be driven by a single gate generator (more than two multiplications by X_2), one may either use a power amplifier to amplify the gate pulses, or each electronic switch may have its own gate-generating multivibrator triggered by pulses from the master gate generator.

Ripple Filtering and Multiplier Frequency Response. If, as is often the case, the multiplier output voltage is to be integrated, then little or no filtering of the pulsed-attenuator ripple is required. If, on the other hand, the multiplier feeds nonlinear devices, and, in particular, other similar multipliers, then one must reduce the RMS ripple voltage below about 0.1 per cent of full scale to avoid serious errors due to ripple rectification. The ripple filter may comprise a low-pass resistance-capacitance network (integrating network) followed by a tuned filter (inductance-capacitance filter, parallel-T filter, or an operational amplifier similar to circuit 2.2, Table 2) tuned to the mean repetition rate. Since the two pulsed attenuators should be identical, the impedance presented by the output amplifier and/or ripple filter to the X_1 switch should be equal to that of the amplifier, integrating circuit, or ripple filter connected to the Y switch.[1]

The repetition rate of most self-excited time-division multipliers varies through a ratio of approximately 2:1; triggered pulsed-attenuator multipliers have a constant repetition rate and permit slightly better filtering than time-division multipliers.

The frequency response of a pulsed-attenuator multiplier is essentially determined by that of its ripple filter. For an RMS ripple output of 0.1 per cent of full scale, the frequency response can be made flat within 0.5 db, and the phase shift kept below 3 deg, up to a frequency of the order of 1 per cent of the lowest repetition rate.

Other Design Considerations. The accuracy of triggered or self-excited pulsed-attenuator multipliers depends critically on d-c amplifier drift and on the stability of the various summing and feedback resistances. The corresponding requirements may be met through the use of automatic balancing circuits and stable resistors (Chap. 5). A remaining critical factor is the *switching time, i.e.*, the sum of the rise and decay times of the switching waveforms. During the rise and decay of each gate pulse, switch-tube characteristics can affect the output voltages, and thus the matching, of the individual pulsed attenuators operating on Y, X_1, X_1', etc. *The resulting errors decrease with the ratio of switching time to repetition period.*[2]

It is seen that the multiplier design will involve a compromise between static accuracy (best for low repetition rates), frequency response (best for high repetition rates and high ripple voltage), and ripple noise.

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[1] See also Morrill and Baum, *op. cit.*

[2] See Korn and Korn, *op. cit.*, for a quantitative analysis of this effect in the special case of a relay time-division multiplier.

Representative Circuits. Accuracy. The essentially similar Berkeley[1] Model 1056 and Donner[2] Model 33B electronic multipliers employ the block diagram of Fig. 6.16*a* with the electronic switch circuit of Fig. 6.16*c* and unstabilized d-c amplifiers. In this low-cost design, the overall accuracy (0.5 per cent of full scale or 200 volts) is determined as much by d-c amplifier drift as by the switching waveforms. The designers were, therefore, able to utilize a simple gate-generator circuit (Fig. 6.16*d*) and high repetition rates (50 to 100 kc) to obtain remarkably good high-frequency response (flat within 0.5 db to 700 cps; less than 2.5-deg phase shift at 100 cps).

The electronic switching circuits shown in Figs. 6.15*c* and *d* are those used in the Reeves Instrument Corporation EM-101 Model 0 electronic multiplier; the precision switching circuits of Figs. 6.17*a* and *b* are respectively employed in Goodyear Aircraft Corporation and RCA multipliers. Sharp gate pulses may be obtained by means of relatively well-known radar techniques.[3] The Goodyear time-division multiplier has an especially elaborate and accurate timing circuit involving the usual bistable multivibrator (Schmitt trigger circuit), a pulse-differentiating network, and a blocking oscillator driving individual gate-generating multivibrators and output amplifiers for each electronic switch. The electronic multiplier used in the Electronic Associates PACE computer (Sec. 8.4) is not a self-excited time-division multiplier but uses a phantastron triggered at 10 kc to actuate single-throw versions of the electronic switch shown in Fig. 6.17*b*. The Reeves, RCA, Goodyear, and Electronic Associates multipliers all have automatically balanced d-c amplifiers (Sec. 5.6).

The static accuracy of such high-quality multipliers is better than 0.1 per cent of full scale. The frequency response depends on individual requirements for ripple filtering; commercially available ripple filters may be modified to permit the user to exchange ripple filtering for frequency response. A typical combination involves 0.2 volts RMS ripple, with a response flat within 0.2 per cent and less than 3-deg phase shift up to 100 cps.

6.5. LOGARITHMIC MULTIPLIERS, QUARTER-SQUARE MULTIPLIERS, AND RELATED CIRCUITS

Logarithmic Multipliers. The nonlinear devices needed for multiplication may take the form of function generators whose output voltages are proportional to the square, logarithm, etc., of the respective input voltages. Thus

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[1] Berkeley Division, Beckman Instruments, Inc., Richmond, Calif.

[2] Donner Scientific Company, Berkeley, Calif.

[3] Chance *et al., op. cit.*

$$X_1X_2 = a^{(\log_a X_1 + \log_a X_2)} = \text{antilog}_a(\log_a X_1 + \log_a X_2) \qquad (6.21)$$

Figure 6.18*a* shows the block diagram of a multiplier based on this well-known relation. It is easy to see that circuits based on the properties of the logarithmic function may be used to obtain *division*, *powers*, and *roots* as well as multiplication. The direct application of the circuit of Fig. 6.18*a* is restricted to positive values of X_1 and X_2 (one-quadrant operation). Accuracies better than 1.5 per cent have been obtained with diode function generators (Sec. 6.7).

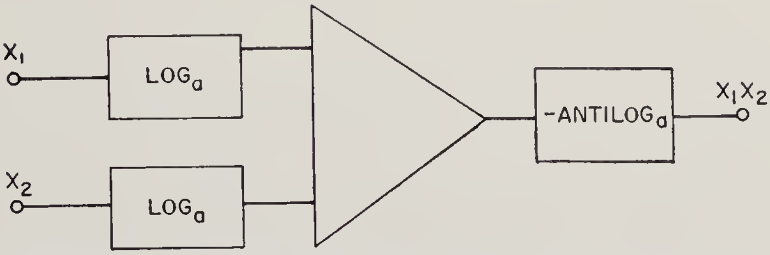

FIG. 6.18*a*. Logarithmic multiplier. Diode function generators (Sec. 6.7) may be used in this circuit; the antilogarithm (exponential function) may be generated as the inverse function of the logarithm in the manner of Sec. 6.12.

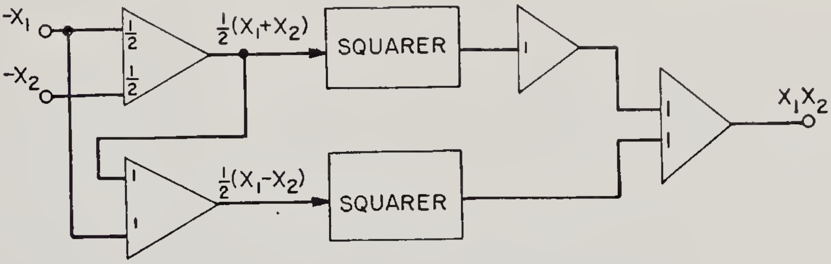

FIG. 6.18*b*. Quarter-square multiplier.

Quarter-square Multipliers. The *quarter-square multiplier* shown in Fig. 6.18*b* is based on the relation

$$X_1X_2 = \tfrac{1}{4}[(X_1 + X_2)^2 - (X_1 - X_2)^2] \qquad (6.22)$$

The use of cathode-ray function generators (Sec. 6.6)[1] or diode function generators[2] (Sec. 6.7) in the circuit of Fig. 6.18*b* permits four-quadrant operation with excellent frequency response (up to more than 100 kc), so that repetitive operation (Secs. 1.1 and 8.5) is possible. The *percent-*

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[1] Miller, J. A., A. S. Soltes, and R. E. Scott, Wide-band Analog Function Multiplier, *Electronics*, February, 1955.

[2] Giser, S., *An All-Electronic High-speed Multiplier*, Massachusetts Institute of Technology Instrumentation Laboratory *Report* R-67, November, 1953.

age error of a quarter-square multiplier may be quite large for small values of the difference (6.22).

The *Reeves Diode Electronic Multiplier* (see also Sec. 8.4)[1] utilizes the relation

$$X_1X_2 = \frac{a}{4}\left\{\left[X_1 + X_2 + \frac{(X_1 + X_2)^2}{a}\right] + \left[X_1 - X_2 - \frac{(X_1 - X_2)^2}{a}\right] - 2X_1\right\} \tag{6.23}$$

rather than the relation (6.22); the functions $U + U^2/a$ and $U - U^2/a$ are monotonic for $|U| \leq 1$ if $a \geq 2$, and are thus somewhat easier to set up on a diode function generator than the function U^2. The frequency response of this multiplier is 3 db down at 1,000 cps, with a phase shift of 1 deg at 100 cps. When the diode squaring circuits are properly adjusted, the d-c output error is less than 0.25 volt.

The Philbrick[2] Amplitude-selection Multiplier. The ingenious *amplitude-selection multiplier* used in Philbrick analog computers (see also Sec. 8.5) is based on the circuit shown in Fig. 6.19*a*. X and Y are machine variables, and V is a 2-mc triangular-wave voltage (carrier voltage) whose half-amplitude V_0 exceeds the maximum value of $|X| + |Y|$. V may be generated by integration of a square wave (multivibrator output); the sum $V + X$ is obtained from a simple summing network. The greater of the two voltages $V + X$ and Y, or

$$U = \max\,(V + X,\ Y) \tag{6.24}$$

appears as the output voltage of a two-diode selection circuit (Sec. 6.7) and is integrated to yield the output voltage E. Figure 6.19*b* shows a graph of U vs. the time; E is proportional to the area under this curve, so that

$$E = a(V_0 + X - Y)^2 + bY \tag{6.25}$$

where a and b are constants determined by the carrier frequency and the circuit resistances.

The actual multiplier circuit shown in Fig. 6.19*c* generates four voltages of the form (6.25), *viz.*

$$\left.\begin{aligned} E_1 &= a(V_0 + X_1 + X_2)^2 - bX_2 \qquad & E_2 &= a(V_0 - X_1 - X_2)^2 + bX_2 \\ E_3 &= a(V_0 + X_1 - X_2)^2 + bX_2 \qquad & E_4 &= a(V_0 - X_1 + X_2)^2 - bX_2 \end{aligned}\right\} \tag{6.26}$$

The summing amplifiers shown yield the desired product

$$X_o = 2k(E_1 + E_2 - E_3 - E_4) = 16kaX_1X_2 \tag{6.27}$$

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[1] Reeves Instrument Corporation, New York City.

[2] G. A. Philbrick Researches, Inc., Boston, Mass.

with a claimed accuracy better than 1 per cent of full scale at frequencies up to 50 kc. This four-quadrant multiplier circuit is, thus, suitable for repetitive computers.

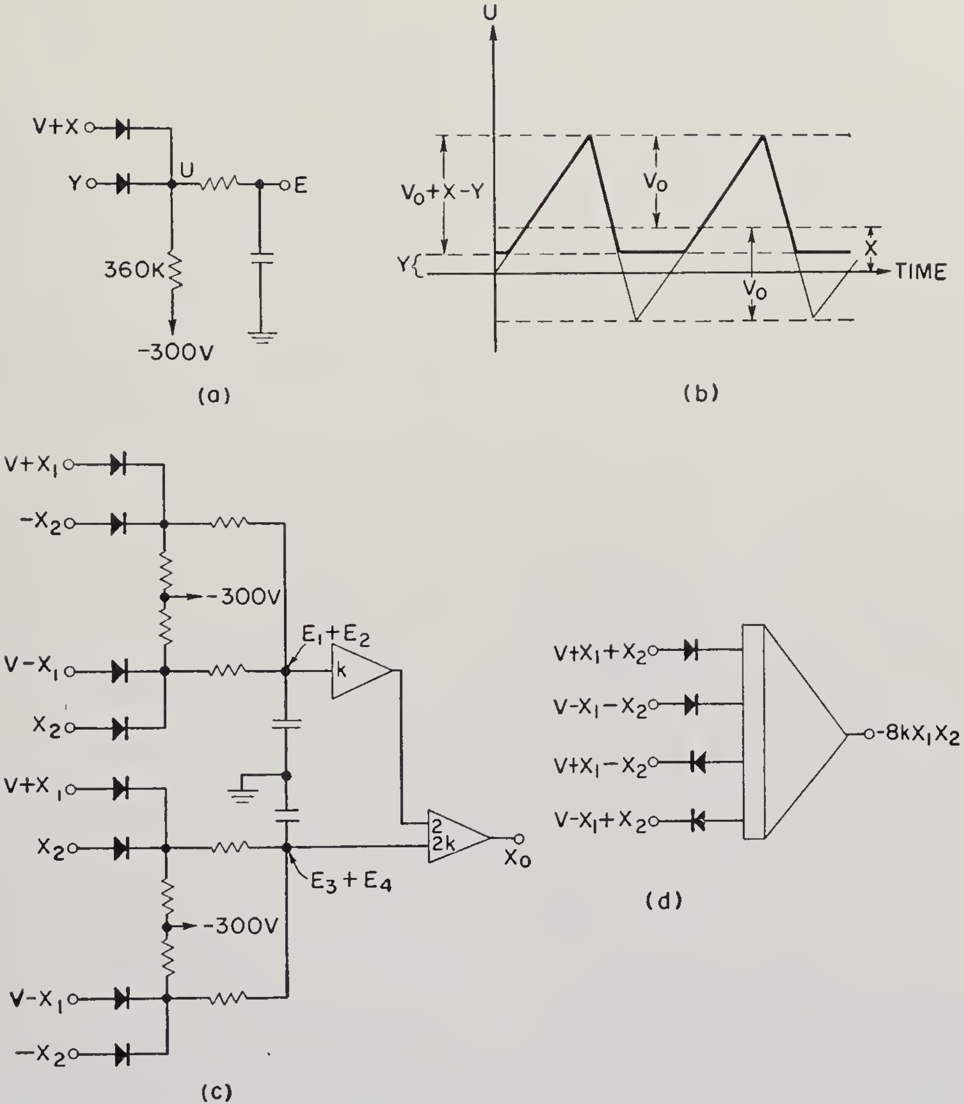

FIG. 6.19. Multiplication by amplitude selection. In the simplified circuit of Fig. 6.19*d*, the amplitude-selection circuits are simple rectifiers; the circuit computes

$$k(V_0 + X_1 + X_2)^2 + k(V_0 - X_1 - X_2)^2 - k(V_0 - X_1 + X_2)^2 - k(V_0 + X_1 - X_2)^2 = 8kX_1X_2$$

The Philbrick multiplier circuit shown in Fig. 6.19*c* is designed so that no operational amplifier has to operate at the carrier frequency. In a "slow" d-c analog computer, the latter requirement could be eliminated through reduction of the carrier frequency below 1 kc, so that the simpler circuit of Fig. 6.19*d* might become practical. This circuit could be set up on existing d-c analog computers. The use of chopper-stabilized summing amplifiers (Sec. 5.6), improved selection circuits (Table 6), and a precision carrier supply might well make the accuracy of the relatively simple amplitude-selection multiplier comparable to that, say, of a diode electronic multiplier.

6.6. FUNCTION GENERATORS

Many relations to be set up on d-c analog computers involve the generation of a voltage as a function of time or as a function of another voltage. In other cases, the multiplication of a voltage by such a function is required. The computing elements capable of performing these operations are called *function generators.*

Like other computing elements, function generators are characterized by their accuracy, flexibility, reliability, frequency response, and cost. The first class of function generators described in the following (Secs. 6.6 and 6.7) are all-electrical devices. The relatively good high-frequency response of such *electronic function generators* makes them suitable for computer applications requiring fast time scales and, in particular, for repetitive computers. *Electromechanical function generators* involving relays (Sec. 6.7) or servomechanisms (Secs. 6.8 to 6.11) are capable of somewhat greater accuracy than electronic function generators. The frequency response of electromechanical function generators is, of course, limited by the inertia of mechanical components.

Generation of Functions of the Time. Many functions of the time can be generated in a d-c analog computer as solutions of simple differential equations. Table 4 shows a number of computer setups suitable for generating frequently used functions. Many other functions can be generated in a similar fashion. One may, furthermore, use the machine variables T, T^2, $\log_e T$, etc., obtained in the manner of Table 4 as input voltages of function generators to produce more complicated functions of the form $F(T)$, $F(T^2)$, $F(\log_e T)$, etc.

One may also generate functions of the machine time τ by driving function potentiometers, switches, and resolvers (Sec. 6.9) with a synchronous motor connected either to the 60-cps a-c line or to the power output stage of a special stable timing oscillator of the tuning-fork type. The latter arrangement is almost mandatory for good accuracy, since the 60-cps line frequency is capable of instantaneous variations as large as 0.7 per cent.

Use of Nonlinear Circuit Elements. The voltage-current characteristics of certain nonlinear circuit elements such as tubes, rectifiers, and thyrite resistors can be used in many different circuit arrangements to establish functional relationships between input and output voltages.

Only a few nonlinear circuit elements have characteristics which can be reproduced with an accuracy better than 1 per cent. For this reason, it is again preferable to use nonlinear elements such as vacuum or crystal diodes as *switching elements* rather than as continuously variable nonlinear resistance elements. Function generators based on this principle are discussed in detail in Sec. 6.7.

Cathode-ray Function Generators: Photoformers. Figure 6.20*a* shows the block diagram of the basic cathode-ray function generator or *photoformer.*[1] The input voltage X_1 is applied between the horizontal deflection plates of a cathode-ray tube through a suitable d-c amplifier. The voltage X_o between the vertical deflection plates is taken to be the output voltage; this voltage is made to vary as a function

$$X_o = f(X_1)$$

of the input voltage X_1 by a feedback arrangement which forces the electron beam to follow the boundary of an opaque mask placed over the lower portion of the cathode-ray screen.

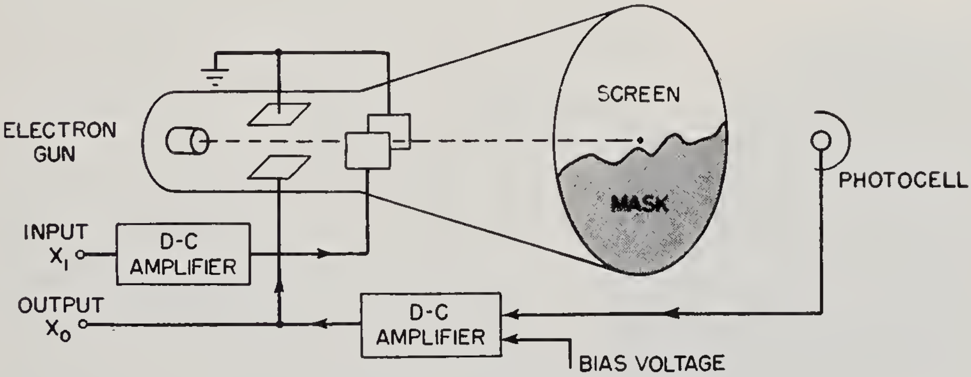

FIG. 6.20*a*. Block diagram of a photoformer.

The bias voltage indicated in Fig. 6.20*a* tends to force the electron beam upward away from the mask. As soon as the spot on the cathode-ray tube screen emerges from behind the mask, however, a photocell mounted in front of the screen applies an error voltage across the input terminals of the vertical deflection d-c amplifier. This error voltage is phased so that the beam is forced downward toward the edge of the mask. *The feedback loop comprising the cathode-ray tube, the photocell, and the vertical deflection amplifier will, then, tend to keep the spot on the screen traveling just below the edge of the mask as the input voltage X_1 (horizontal deflection voltage) changes.*

Since the deflections of the electron beam produced by the deflection voltages X_1 and X_o are approximately proportional to these respective voltages, *the output voltage X_o is forced to vary as a function*

$$X_o = f(X_1)$$

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[1] Mynall, D. J., Electronic Analog Computing, *Elec. Eng.*, August, 1947; Sunstein, D. E., Photoelectric Waveform Generator, *Electronics*, **22**: 100, 1949.

of the input voltage X_1*; the function* $f(X_1)$ *is determined by the shape of the mask boundary.* If the equation of the latter in rectangular cartesian coordinates ξ and η with origin in the center of the cathode-ray screen is

$$\eta = F(\xi)$$

then

$$f(X_1) = \frac{1}{k_\eta} F(k_\xi X_1) \tag{6.28}$$

where k_ξ and k_η are the horizontal and vertical deflection sensitivities

$$k_\xi = \frac{\partial \xi}{\partial X_1} \qquad k_\eta = \frac{\partial \eta}{\partial X_o} \tag{6.29}$$

respectively. It is, then, possible to obtain a mask for the generation of the function $f(X_o)$ by cutting the corresponding curve

$$\eta = F(\xi) = k_\eta f\left(\frac{1}{k_\xi}\xi\right) \tag{6.30}$$

out of a piece of opaque cardboard or plastic.

Functions may be generated in this manner with an accuracy of the order of 1 per cent for input voltages containing frequency components in excess of 100,000 cps if suitable amplifiers are used.

Better accuracy might be possible if Eq. (6.30) is not relied on to yield the shape of the mask for the desired function. *An adjustable photoformer mask like that shown in Fig. 6.20b could be adjusted until the measured value of the output voltage* X_o *is equal to the correct value* $f(X_1)$ *for each value of the input voltage* X_1. Errors due to changes in the deflection sensitivities k_ξ and k_η with the applied voltages as well as errors due to the finite gain of the feedback loop could be eliminated in this manner. Accurate masks can also be prepared through point-by-point photography of the cathode-ray tube screen with known deflection voltages.

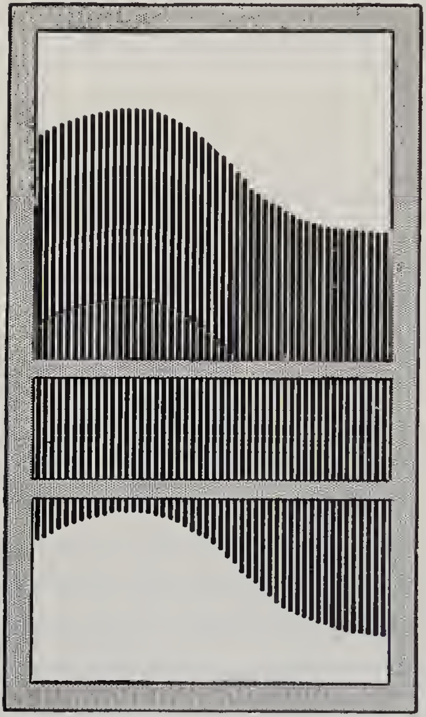

FIG. 6.20*b*. An adjustable photoformer mask consisting of a set of metal shims clamped in a suitable frame (A. B. MacNee, A High-speed Product Integrator, *MIT-RLE Report* 136, Aug. 17, 1949).

The actual circuit of a photoformer (essentially as developed at the Massachusetts Institute of Technology[1]) is shown in Fig. 6.20*c*. The use of a multiplier photocell in connection with a very simple push-pull deflection amplifier results in ample feedback gain. The P-11 cathode-ray tube screen used could be replaced by a P-5 screen for high-frequency applications. The maximum error (distortion) observed with this circuit was about 2.5 per cent (with a fixed mask).

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[1] Cambridge, Mass.

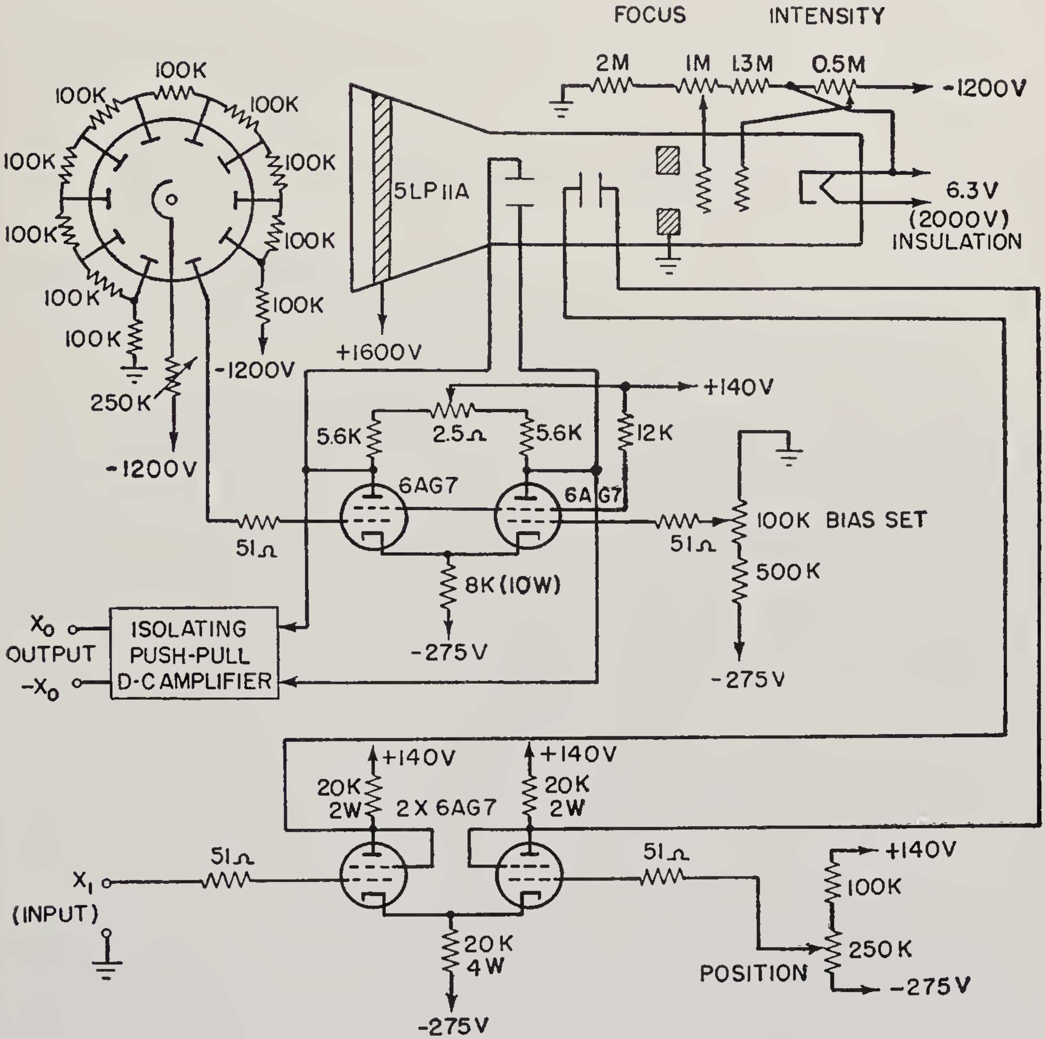

Fig. 6.20c. Practical photoformer circuit. Both positive and negative output may be obtained. (Based on the circuit used in the MIT repetitive computer described in Sec. 8.5.)

The all-electronic photoformers are, for all practical purposes, free from frequency-response limitations other than those imposed by associated computer amplifiers and cathode-ray tube phosphors. Photoformers are very simple devices, and their reliability and accuracy are being improved.

Other Cathode-ray Function Generators. The *monoformer* developed by the Philco Corporation[1] operates essentially like the photoformer, but the external mask and photocell are replaced by a mask controlling the secondary emission and a collector electrode inside the cathode-ray tube envelope. Since the monoformer mask is not easily replaceable, a monoformer is capable of generating a single function only; the accuracy is comparable to that of a photoformer.

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[1] Philadelphia, Pa.

The Raytheon Manufacturing Company has developed another type of cathode-ray function generator[1] in which a flat electron beam is narrowed down by a mask in accordance with a given function as the beam is deflected by an input voltage. The output voltage is obtained as a function of the beam current. The accuracy is better than 1 per cent.

Special cathode-ray function generators are useful at frequencies in excess of 20 kc and may be useful for quarter-square or logarithmic multipliers (Sec. 6.5),[2] as function generators in special-purpose computers, and as relatively accurate square-law modulators.

Noise Generators (see also Sec. 3.9). Voltages representing noise in systems simulated by d-c analog computers may be introduced as functions of time by means of function generators set up to simulate typical noises. Various types of noise may also be introduced by means of commercially available noise generators using noisy diodes, photocells, radioactive sources, or thyratrons in magnetic fields, with or without output filters[3]; or recorded noise may be used.

Generation of Sinusoidal Test Voltages. The generation of sinusoidal voltages of the time or of other machine variables is particularly important in connection with the solution of problems involving Fourier analysis or synthesis, problems involving frequency-response tests of automatic control systems, and problems involving coordinate transformations (see Sec. 6.11).

The possibility of generating sinusoidal functions of the machine time by implicit computation has been mentioned; the use of special sine and cosine function potentiometers is discussed in Sec. 6.11. Sinusoidal functions of the time may also be generated by means of *electronic oscillators* of the phase-shift type,[4] although the design of oscillators stable with respect to amplitude and frequency presents somewhat of a problem, especially if the frequencies required are below 0.5 cps. The correct phasing of electronic oscillators may raise still another problem.

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[1] Tube type QK 329, Raytheon Manufacturing Co., Waltham, Mass.; see also A. S. Soltes, A Wide-band Square-law Computing Amplifier, *Trans. PGEC, IRE*, June, 1954.

[2] Miller, J. A., *et al., op. cit.*

[3] Random Noise Generator for Simulation Studies, *Report GER*-6436, Goodyear Aircraft Corporation, Akron, Ohio, Dec. 13, 1954; see also GEDA Simulation Study of a Relay Servomechanism, *Report GER*-4799, Goodyear Aircraft Corporation, Akron, Ohio, May 9, 1952.

[4] See, for instance, F. E. Terman, *Radio Engineers' Handbook*, McGraw-Hill, New York, 1943.

Rotatable transformers, such as synchros and resolvers, will modulate an a-c carrier voltage with sinusoidal functions of the shaft displacement. Together with suitable phase-sensitive demodulators or detectors, such devices may be used as d-c function generators; they are somewhat less subject to wear than function potentiometers.

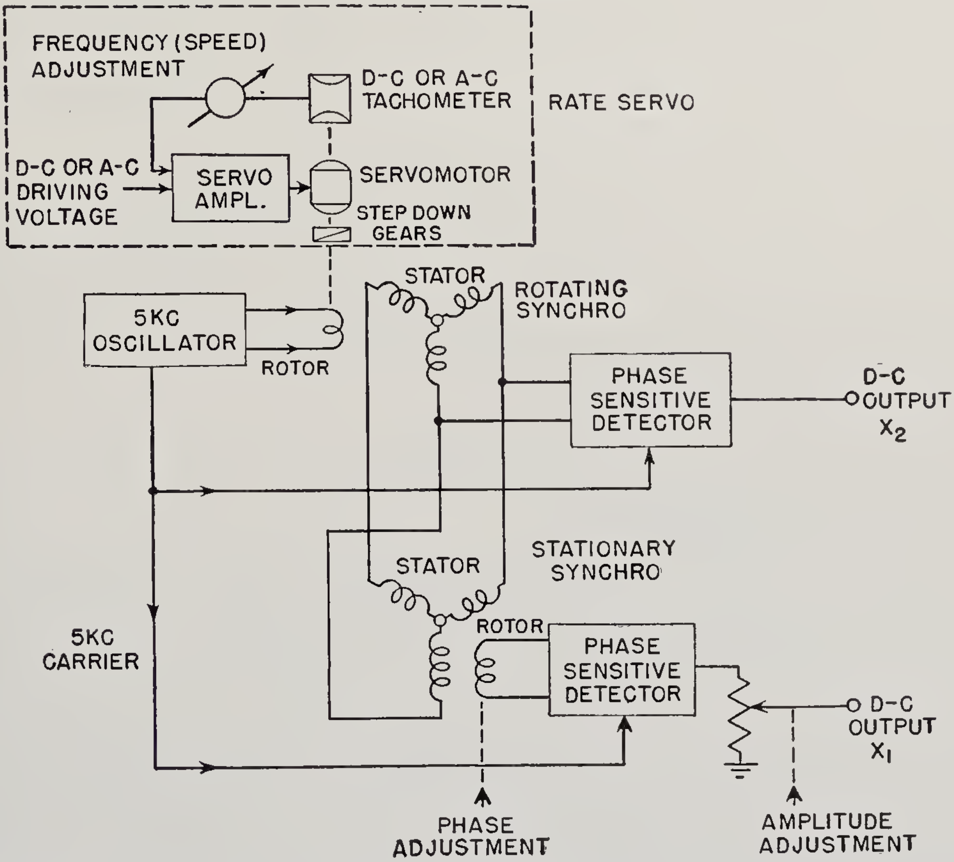

FIG. 6.21. Block diagram of a low-frequency oscillator used for testing automatic control circuits.

The input a-c carrier may be modulated with another d-c voltage, so that multiplication is possible; several applications of this type are discussed in Sec. 6.11. D-c offset voltages originating in the modulators and demodulators present a serious difficulty. The d-c frequency response is limited, as with potentiometers.

Figure 6.21 shows the block diagram of a flexible and convenient low-frequency oscillator using synchros to generate sinusoidal d-c voltages of the time for testing the frequency response of simulated or actual automatic control mechanisms.[1] A rate servo with adjustable tachometer feedback is used to rotate a synchro transmitter at any desired rate between 0.001 and 20 cps. The speed of rotation can be calibrated conveniently by stroboscopic reference to the 60-cps line frequency through the use of a simple neon bulb and a small stroboscopic disk on the high-speed shaft.

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[1] Korn, G. A., and T. M. Korn, Modern Servomechanism Testers, *Elec. Eng.*, September, 1950.

A 5,000-cps carrier voltage is used. The three-phase output of the synchro transmitter is impressed on the stator winding of a second synchro. The rotor winding of this second synchro is held stationary and affords a particularly convenient means for adjusting the phase of the d-c output voltage X_1 obtained through the use of a phase-sensitive detector circuit.[1] A calibrated potentiometer attenuator permits one to adjust the amplitude of the output voltage X_1.

A second sinusoidal d-c output voltage X_2 is obtained by demodulating the stator voltage of the first synchro directly. This voltage is useful as a *reference voltage. This voltage may, for instance, be used as an input voltage for a real or simulated automatic control system whose output phase and amplitude can be measured by comparison with the calibrated phase and amplitude, respectively, of the voltage* X_1.

6.7. GENERATION OF FUNCTIONS BY MEANS OF DIODES AND OTHER SWITCHING DEVICES

Properties of High-vacuum Diode Circuits.[2] Figure 6.22 illustrates the essential properties of a high-vacuum diode circuit. *An ideal diode* (solid-line characteristic in Fig. 6.22) *may be regarded as a voltage-sensitive on-off switch. The circuit is closed if the diode plate is positive with respect to the cathode* ($E > E_C$), *and the circuit is open if the plate is negative with respect to the cathode* ($E < E_C$). Actual high-vacuum diodes have a finite incremental resistance (diode plate resistance, 30 to 2,000 ohms) when they are ON, and the thermionic effect generates a small positive plate current even in the OFF condition (broken-line characteristic in Fig. 6.22). The finite resistance of the bias source indicated in Fig. 6.22 may also have to be taken into account in the circuit design. Note that the bias voltage E_C can be negative as well as positive, so that the *breakpoint* of the characteristic may be moved at will.

Diode Function Generators: Basic Limiter Circuits. Diode function generators utilize the transfer characteristics (output voltage vs. input voltage) of *resistive networks containing biased diode switches.* These characteristics will be linear between *breakpoints* corresponding to input voltage levels which cause a diode switch to conduct or to cut off. The

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[1] Refer to Chance *et al., op. cit.,* Chap. 14, for a discussion of phase-sensitive detectors.

[2] Chance *et al., op. cit.*

slopes of the line segments as well as the breakpoint voltages can be adjusted by suitable choices of resistances and bias values; such circuits permit approximation of many functions by *polygonal functions.*

The widely used basic *limiter circuits* shown together with their transfer characteristics in Fig. 6.23 involve only one or two breakpoints; at each breakpoint, the slope (gain) changes from a finite value to zero. The diode circuits of Fig. 6.24 illustrate various practical methods of providing bias voltages. Shunt limiters are easily biased (Figs. 6.24*c* and *d*), but the load impedance r_L, as well as the limiting resistance R_1, must be large for good limiting action with reasonable attenuation. The resulting high impedance level also tends to decrease switching speeds. Series-diode circuits do not have these disadvantages. Feedback limiters are particularly useful: they will prevent amplifier overloads, even though the gain over the sloping portion of the limiter characteristic can be made very high.

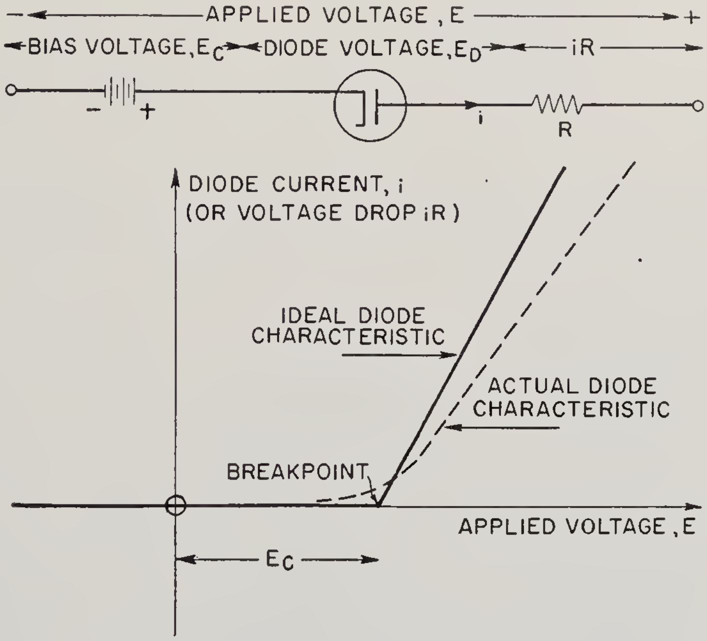


FIG. 6.22. Characteristics of a high-vacuum diode circuit operated within its voltage ratings. Note that E_C can be negative as well as positive.

Generation of More General Types of Functions. If a resistance is placed in parallel with the diode in Figs. 6.23*a*, *b*, and *c*, or in series with the diode in Figs. 6.23*d*, *e*, and *f*, the horizontal portions of the corresponding characteristics are replaced by segments of positive slope. The resulting three-segment polygonal functions serve as useful approximations for many given functions. The multisegment diode function-generator circuits shown in Fig. 6.25 are based on a generalization of the same principle. Note that these circuits can generate only monotonic

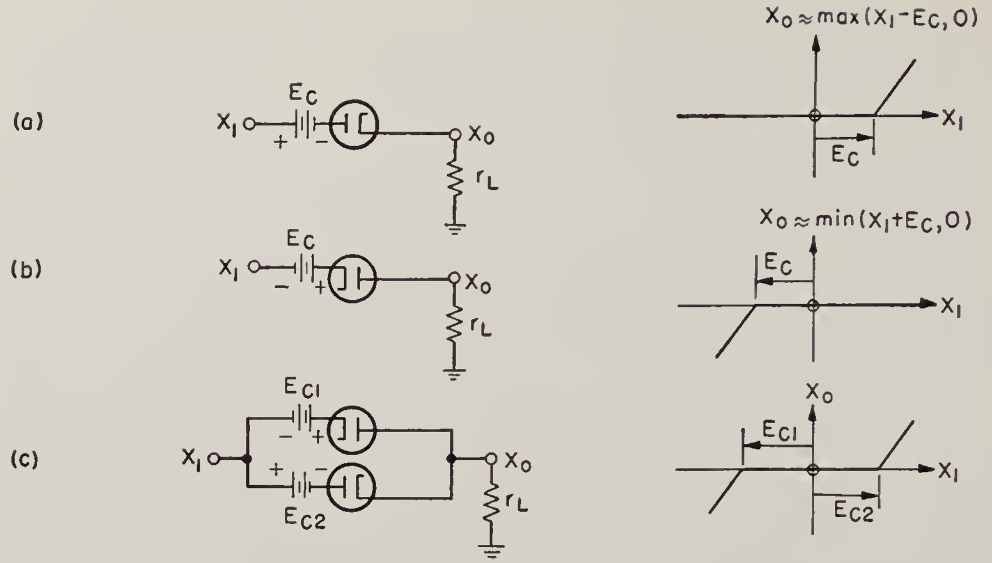


FIG. 6.23*a, b, c*. Series-diode circuits. (*a*) and (*b*) are series limiters, and (*c*) is a simple dead-space circuit (see also Sec. 3.3 and Table 6). r_L is frequently an input resistor of an operational amplifier.

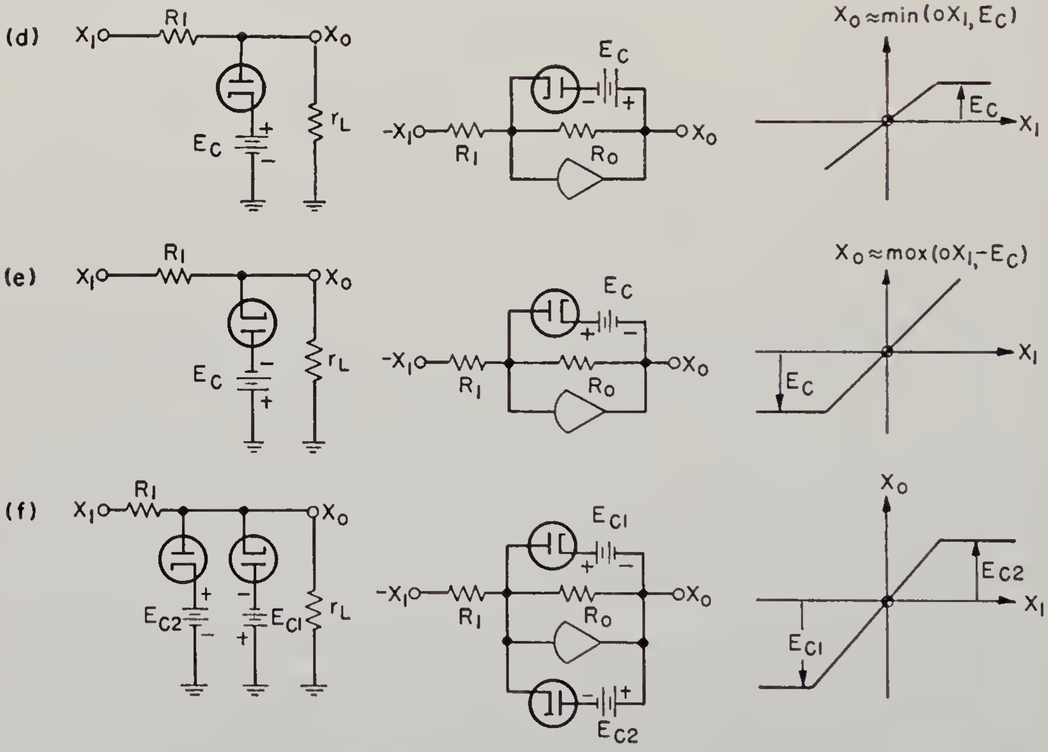


FIG. 6.23*d, e, f*. Shunt limiters and feedback limiters. The slope of the line segment is approximately equal to $r_L/(R_1 + r_L)$ for the shunt limiters, and R_o/R_1 for the feedback limiters. r_L is frequently an input resistor of an operational amplifier.

FIG. 6.23. Basic diode circuits for generating line-segment functions. Note that E_C, E_{C1}, and E_{C2} can be negative as well as positive, so that the breakpoints can be positioned at will.

functions directly; nonmonotonic functions are easily obtained as differences of two monotonic functions.

The diode universal function-generator circuits shown in Fig. 6.26 are designed for use with flexible multipurpose computers and permit a more

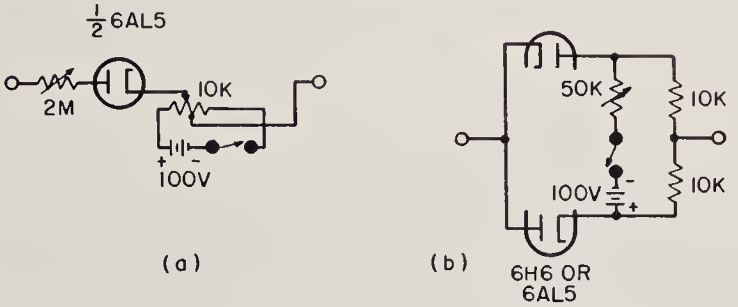


FIG. 6.24*a*, *b*. Diode "packages" such as these can be used in series, shunt, and feedback circuits. Small gas-tube-regulated power supplies may replace the batteries.

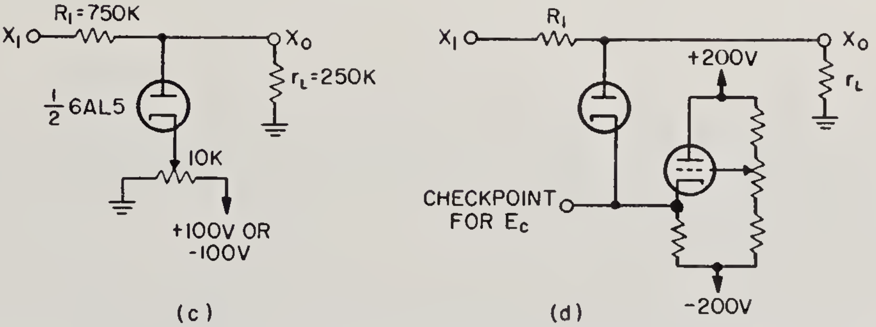


FIG. 6.24*c*, *d*. Bias arrangements for shunt limiters. The circuit of Fig. 6.24*d* uses a cathode follower as a bias source of low incremental impedance.

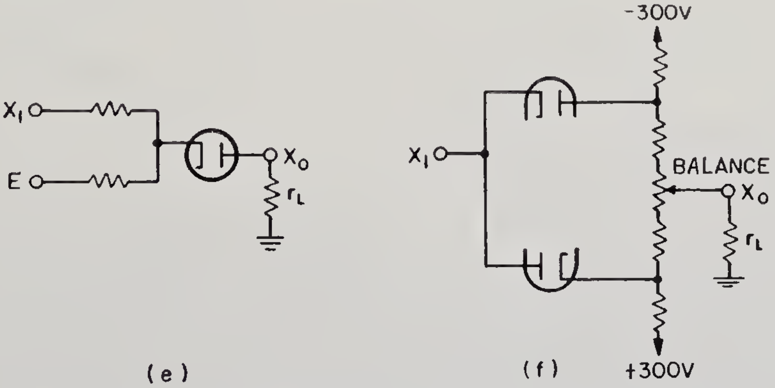
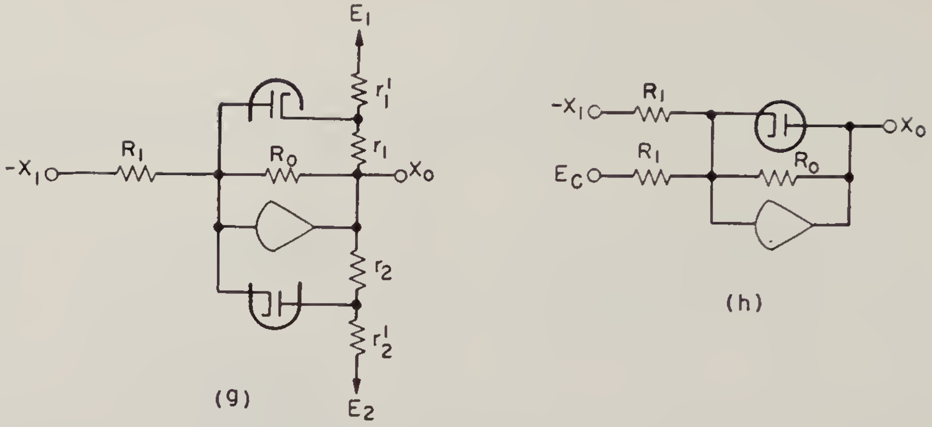


FIG. 6.24*e*, *f*. Bias arrangements for series-diode circuits.

systematic synthesis of the function to be generated. In each case, *one obtains the desired function by summing the output voltages of simple limiter channels* (see, in particular, Fig. 6.26*a*). One obtains nonmonotonic functions by reversing the signs of suitable terms of the sum with the aid of a phase inverter inserted either ahead of the corresponding limiter channels

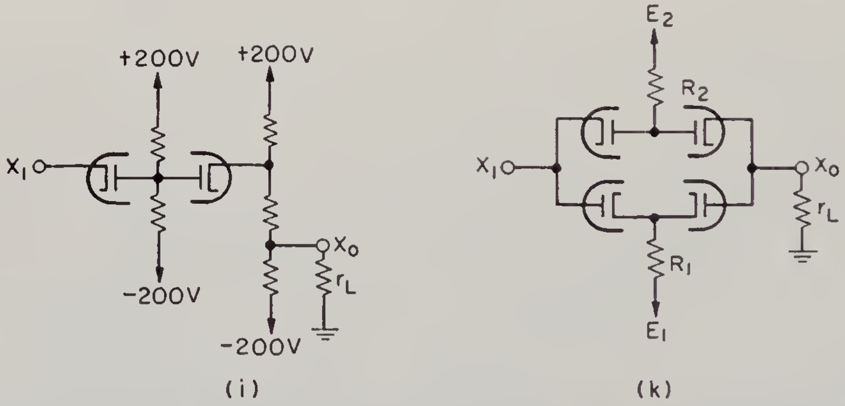


(g) (h)

FIG. 6.24*g*, *h*. Bias arrangements for feedback-diode circuits. The limiting values of X_o in Fig. 6.24*g* are

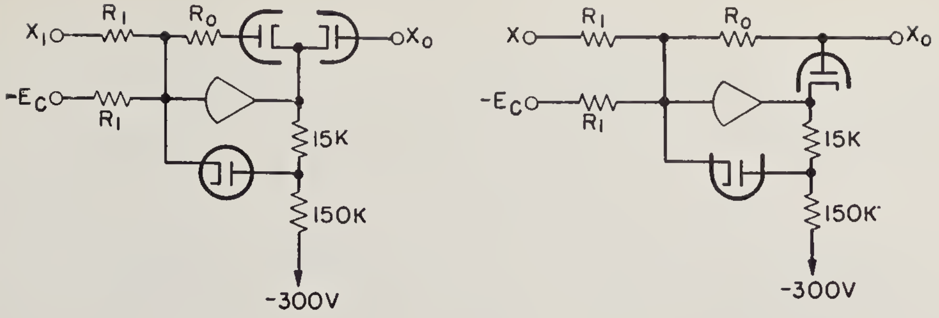
$$E_{C1} = \frac{r_1}{r'_1} E_1 \qquad \text{and} \qquad E_{C2} = \frac{r_2}{r'_2} E_2$$

assuming infinite amplifier gain and zero diode plate resistance (see also Fig. 6.23*f*). For good limiting, r_1 and r_2 must be small compared to R_1. In a typical circuit, $E_1 = -E_2 = 200$ volts or 300 volts. The circuit of Fig. 6.24*h* eliminates the impedance of the bias source for *one* diode.



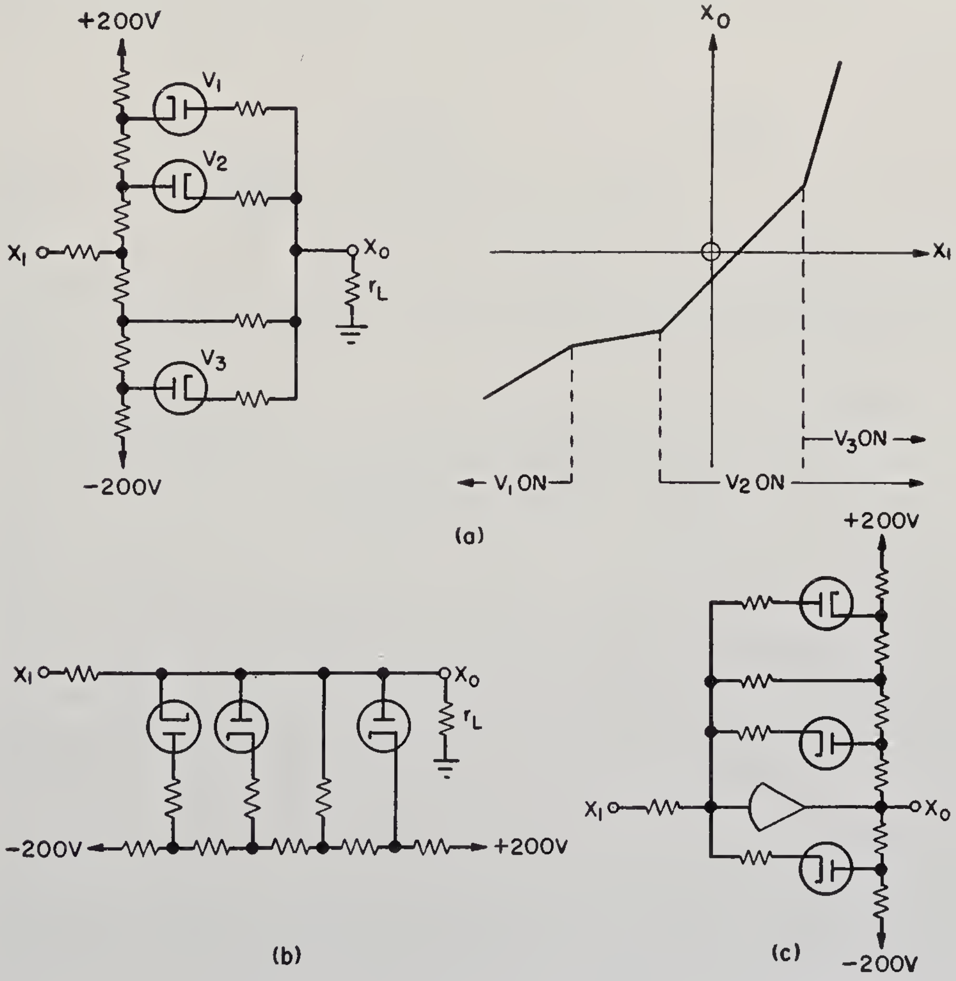
(i) (k)

FIG. 6.24*i*, *k*. Dual series limiters. In the "bridge" limiter of Fig. 6.24*k*, one matches $E_2 = -E_1$ and $R_2 = R_1$; the limiting values $\pm \dfrac{r_L}{R_1 + r_L} E_1$ of the output voltage X_o are then easily adjusted with relatively little effect on the slope of the characteristic (see also circuit 6.1*c*, Table 6, and R. J. Medkeff and R. J. Parent, A Diode Bridge Limiter for Use with Electronic Analog Computers, *Proc. AIEE*, **70**: Sec. T1-170, 1951).



(L) (m)

FIG. 6.24*l*, *m*. In each of these circuits, the high amplifier gain limits the output voltage sharply once the upper feedback diode begins to cut off. The lower feedback diode prevents amplifier overloads due to input voltages less than E_C. Note that the simpler circuit of Fig. 6.24*m* is preferable, since voltage feedback renders its gain essentially independent of the diode plate resistance (Morrill and Baum, *op. cit.*).



(a)

(b) (c)

FIG. 6.25. Diode function generators for monotonic functions. Many combinations of such circuits are possible; nonmonotonic functions may be obtained as differences of monotonic functions.

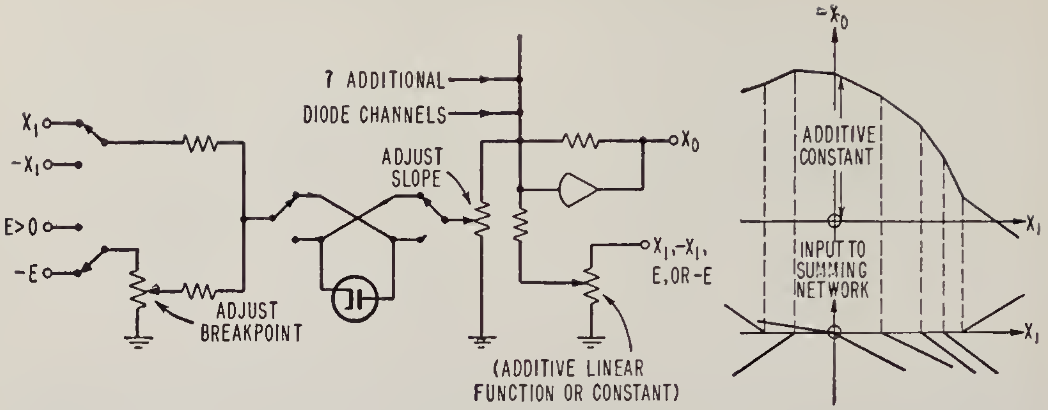


FIG. 6.26*a*. Simplified schematic diagram showing a series-limiter channel of the Reeves diode function generator. Different polarities of diode, input voltage, and bias voltage (see also Fig. 6.23*a*, *b*) are chosen by a four-deck, four-position "quadrant switch" yielding one of the four types of line segments shown at the right. Breakpoint and slope are set with a minimum of interaction on 10-turn helical potentiometers. The amplifier adds the output voltages of 8 such channels and an additional voltage aX_1 or B. One may combine function generators to obtain more than 9 segments.

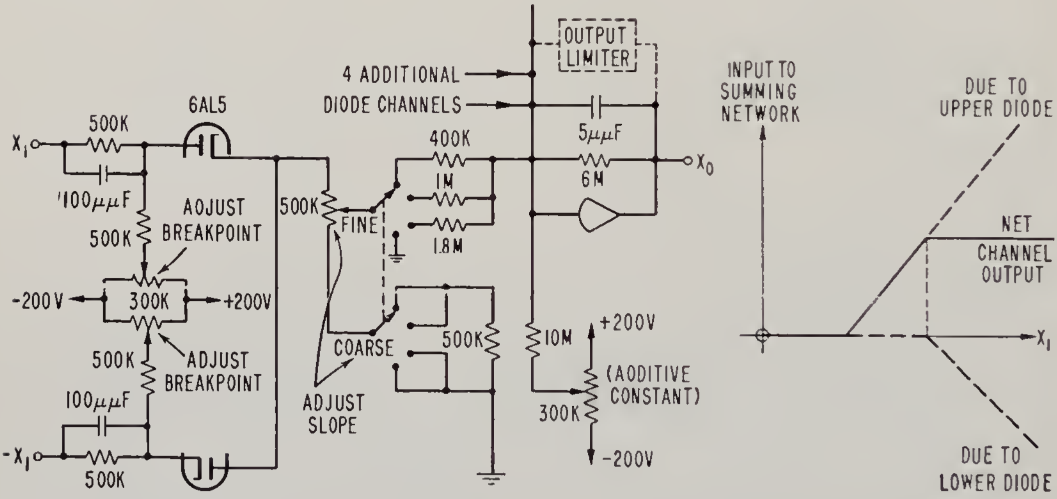


FIG. 6.26*b*. Each dual series-limiter channel of the Goodyear diode function generator generates line segments having equal and opposite slopes. The two breakpoints are staggered so that the channel output voltage will not contribute to the slope of any subsequent line segment. The additional circuitry facilitates slope adjustments and minimizes accumulation of diode drifts. The feedback output limiter indicated in broken lines is of the type shown in Fig. 6.24*g* and prevents calibration overloads.

(Figs. 6.26*a* and 6.26*b*), or following the limiter channels in question (Fig. 6.26*c*).

Built-up Diode Circuits.[1] Many simple function generators based on various combinations of the limiter circuits of Figs. 6.23 and 6.24 may be built up ("fabricated") on most multipurpose d-c analog computers. Such circuits can simulate the operating characteristics of many nonlinear devices. Diode limiter circuits can also be applied to integrators and other operational amplifiers (see also Secs. 3.3 and 3.4, and Table 6). Figure 6.27 shows a number of diode switching circuits useful in special applications (see also Secs. 3.9 and 7.7).

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[1] Morrill, C. D., and R. V. Baum, Diode Limiters Simulate Mechanical Phenomena, *Electronics*, **25**: 122, November, 1952.

Adjustment of Diode Function Generators. As a first step in setting up diode circuits to approximate a given function $X_o = F(X_1)$, a graph of the function must be fitted with the desired number of straight-line segments; this procedure yields the appropriate breakpoints and slopes. In the case of the function-generator circuits of Fig. 6.25, the required resistance values can then be computed from Kirchhoff's laws on the basis of ideal diode characteristics. Final adjustments should be made empirically, with the function-generator output connected to a servo plotting board, to a comparison potentiometer, or to a servo or digital voltmeter.

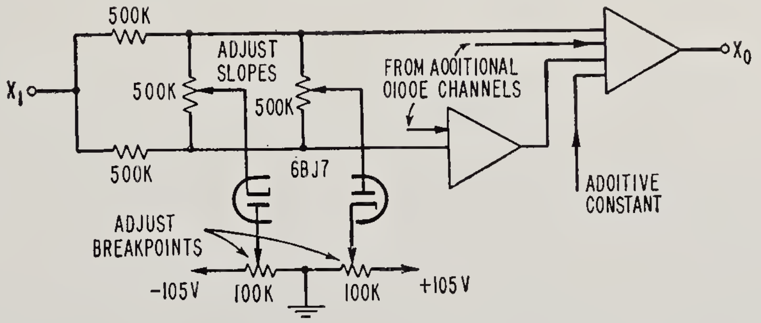

FIG. 6.26c. The Donner Model 35 diode function generator is an accessory package comprising 12 pairs of shunt limiters with regulated bias and filament supplies; two external d-c amplifiers are used. A similar circuit is used in the Beckman EASE computer (Sec. 8.4).

The adjustment of one of the universal function-generator circuits shown in Fig. 6.26 is especially simple. In such function generators, each limiter-channel output affects only function values corresponding to arguments X_1 beyond its breakpoint. Breakpoint and slope settings can be made by reference to *calibration curves* based on actual diode characteristics. One may, instead, set all slopes initially to zero and adjust breakpoints and slopes empirically for successively higher values of $|X_1|$. The entire procedure should be repeated once or twice in order to allow for possible interference between breakpoint and slope adjustments.

Miscellaneous Design Considerations. The nonideal switching characteristics of physical high-vacuum diodes (broken line in Fig. 6.22) tend to smooth the "corners" of empirically fitted polygonal approximation functions. Sharp limiting is, nevertheless, usually desirable in order to minimize the effects of variations in diode characteristics near the breakpoint. For good limiting, the rate of change of each diode plate voltage with respect to the input voltage X_1 just before conduction should be as large as practicable. The effects of changes in diode plate resistances on the slope of the output function are minimized through the use of relatively high circuit impedances in series with each diode.

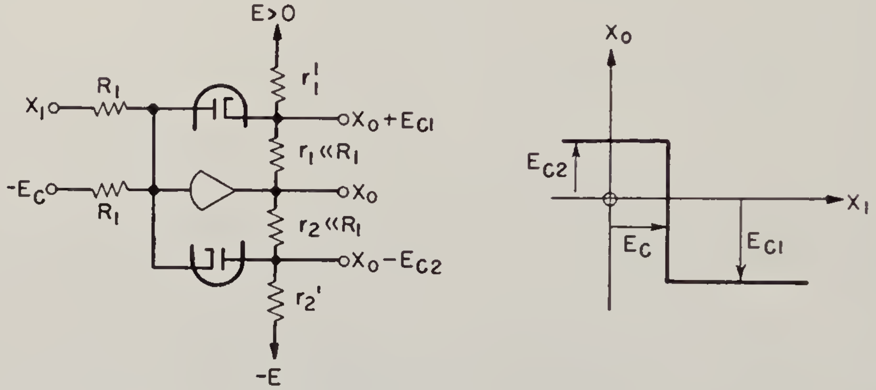

FIG. 6.27*a*. A feedback-limited high-gain amplifier used as a *comparator* for switching applications and for driving relays. $E_{C1} = \frac{r_1}{r_1'} E$, $E_{C2} = \frac{r_2}{r_2'} E$ (see also Sec. 6.6, Table 6; Morrill and Baum, *op. cit.*).

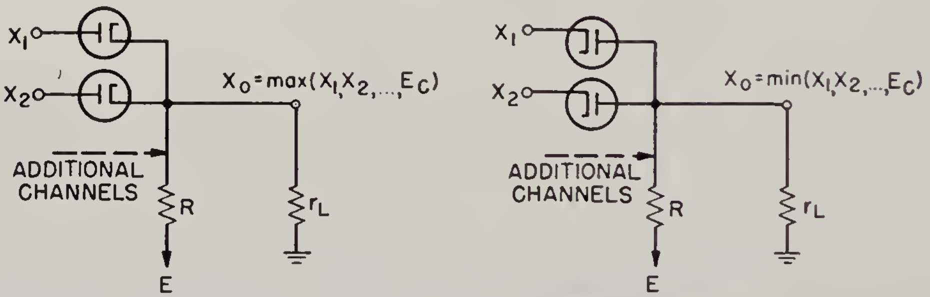

FIG. 6.27*b*. These diode circuits will select the largest or smallest of a set of input voltages. Diode and source resistances are assumed to be negligible compared to R and r_L. $E_C = \frac{r_L}{R + r_L} E$.

The physical design of diode circuits should ensure high leakage resistance between diode plates and cathodes. In series-diode and feedback circuits, leakage to ground is preferable to plate-cathode leakage (see also Secs. 5.9 and 8.3).

Thermionic diodes used in high-quality d-c analog computers require a regulated filament supply (usually of the saturable-reactor type) to stabilize the diode characteristics near the breakpoint. If necessary, variations in cathode emission and contact potential can be partially compensated by means of circuits similar in principle to that in Fig. 5.19*a*.[1]

Crystal diodes, which are small in size and require no filament supply, are often substituted for high-vacuum diodes. Unlike the latter (see Fig.

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[1] Chance *et al.*, *op. cit.*, Sec. 9.6.

6.22), crystal diodes tend to pass appreciable reverse currents at the relatively high inverse voltages (up to 200 volts) possible in limiter circuits.[1] Again, crystal characteristics tend to vary from unit to unit and as a function of temperature; the latter effect may be partially overcome through the use of a simple crystal oven. High-vacuum diodes or high-quality silicon-junction diodes (such as type 1N218), rather than point-contact crystal diodes, are usually employed in computer applications requiring component accuracies better than 1 per cent.

Frequency Response and Accuracy. Many function generators based on diode switching circuits can be used at signal frequencies in excess of 10 kc and are suitable for repetitive computers. Most frequently used functions can be generated with an accuracy between 0.1 and 5 per cent of full scale, depending on the number of segments used as well as on circuit design. The best diode function generators used in "slow" d-c analog computers permit accuracies better than 0.5 per cent of full scale at frequencies below 500 cps; the use of stable wire-wound resistance elements tends to limit the useful frequency range.

Computer Relays. High-speed computer relays (operational relays)[2] may be substituted for diode switches in many computer applications where clean switching is required (*e.g.*, in integrator input circuits) and where relay switching delays (1 to 20 msec) are not objectionable. The most useful computer relays are of the single- or double-pole double-throw polarized type, with or without a center-off position. Such relays may be driven by special differential-cathode-follower circuits similar to those shown in Figs. 7.16 and 7.17, by operational amplifiers, and, perhaps preferably, by comparator circuits of the type shown in Fig. 6.27*b*.

Computer setups involving logical decisions frequently employ relays. Some of the most useful relay applications are related to automatic programming functions (Sec. 7.7).

Other Switching Circuits. The use of *limiter amplifiers* has been generally abandoned in favor of diode limiter circuits. A single *neon bulb* (NE 51) can replace a pair of diodes and their associated bias supplies in certain applications where accuracy and precision of the breakpoint voltages are not important (Table 6).

6.8. COMPUTER SERVOMECHANISMS FOR D-C ANALOG COMPUTERS

Computer servomechanisms used in d-c analog computers are intended to position potentiometer shafts, resolvers (Sec. 6.11), and/or indicating dials in accordance with d-c input voltages. Two or more computing potentiometers may be ganged, so that each computer servomechanism can perform several multiplications by functions of its input voltage.

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[1] *Ibid.*, Sec. 3.15.

[2] Bennett, R. R., and A. S. Fulton, High-speed Relays in Electric Analog Computers, *Elec. Eng.*, **70**: 1083, December, 1951.

The feedback principle underlying the operation of computer servomechanisms has been described in Sec. 6.3; the methods used for the detailed analysis resemble those used to analyze feedback amplifiers in Sec. 5.4. Generally speaking, the accuracy with which a servomechanism can position its output shaft as a function of the input voltage will improve with increasing loop gain. Mechanical and electrical inertia of the servomechanism components prevent the servo output from following rapidly changing input signals without a phase lag. With high loop gains such phase lags may result in uncontrolled oscillations (hunting, instability). The phase lag may be counteracted to some extent by speed feedback from an induction generator or tachometer on the output shaft, or by special "lead networks" whose operation is similar to that of the lead networks described in Sec. 5.4 in connection with feedback amplifiers. Other equalizing networks may be added to provide higher loop gains at low frequencies. Such "integral" control will decrease the steady-state errors of the servomechanism. One can sometimes improve the frequency response of a computer servomechanism by inserting a suitable operational amplifier ahead of the servo input;[1] circuit 1.5, Table 1, may be used as such an "open-cycle controller."

The detailed theory as well as the design of small servomechanisms with specified requirements regarding stability, accuracy, and frequency response has been described very adequately in Vol. 21 of the MIT Radiation Laboratory Series.[2] This section is not intended to duplicate this material but will merely present a number of practical circuits suitable for d-c analog-computer applications.

Basic Circuit Arrangement. Figure 6.28 shows the block diagram of a typical computer servomechanism. The input d-c voltage and the signal feedback from the follow-up potentiometer are mixed to produce an "error signal" at the output of the mixing networks shown. Some integral control[3] can be introduced at this point in order to decrease the steady-state error of the servomechanism. The d-c error signal so obtained is used to modulate a 60- or 400-cps a-c voltage by means of a synchronous vibrator or chopper of the type described in Sec. 5.6.

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[1] Moore, J. R., lectures given at the University of California in Los Angeles during the academic year 1952–1953.

[2] Greenwood, Holdam, and MacRae, *op. cit.* A good discussion of the effects of nonlinearities and noise is given in GEDA Simulation Study of a Carrier-type Instrument Servomechanism, *Report GER*-5779, Goodyear Aircraft Corporation, Akron, Ohio, Apr. 21, 1954. See also H. Chestnut and R. W. Mayer, *Servomechanisms and Regulating Systems*, Vol. 2, Wiley, New York, 1955.

[3] See Greenwood *et al.*, *op. cit.*, for a detailed discussion of integral control and other equalizing techniques.

The a-c error signal produced in this manner is mixed with the a-c speed-feedback signal from an induction generator[1] at the input of a simple a-c amplifier. The amplifier power-output stage drives the control field of a two-phase induction motor.

Many variations of this basic circuit are possible. For example, a d-c tachometer, such as a small d-c motor with permanent-magnet field, can replace the induction generator, or electrical differentiation of the feedback error signal may be used (Figs. 6.30*c* and 6.33*b*).

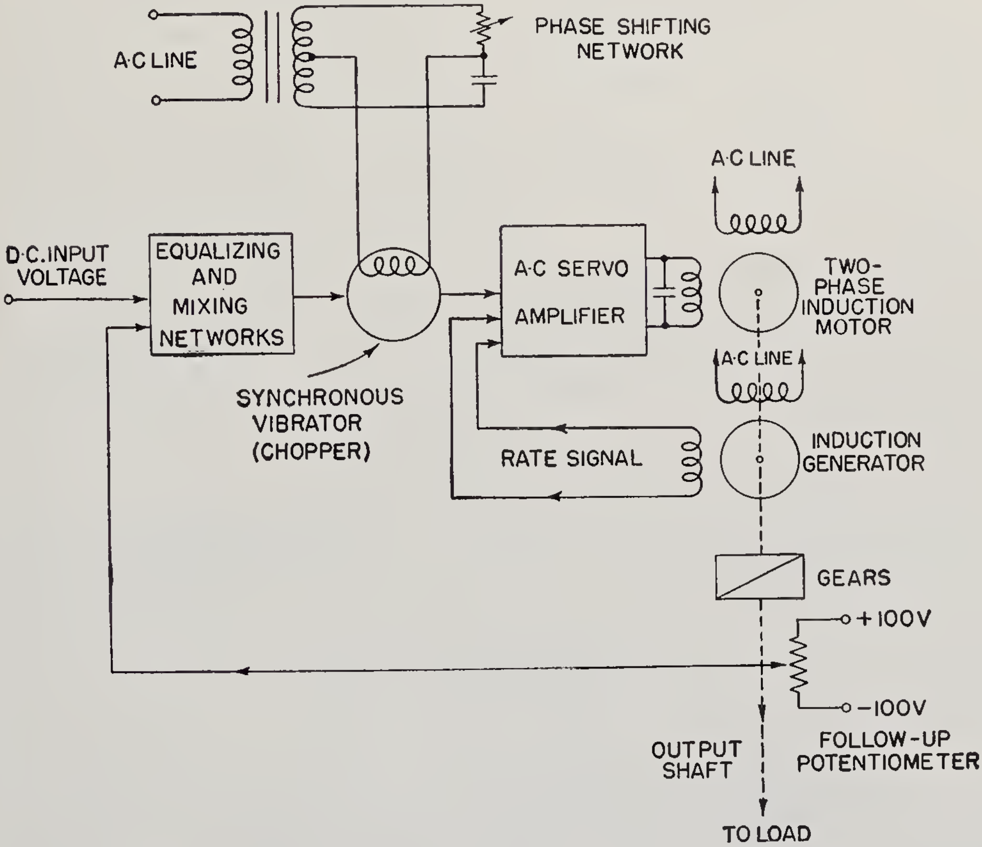

FIG. 6.28. Computer servomechanism.

Rate Servos. Some computer servomechanisms comprising a d-c tachometer can be connected so as to use only rate feedback, so that each servomechanism will integrate[2] as well as multiply. Such an arrangement permits unlimited continuous rotation of sine-cosine potentiometers or resolvers (Sec. 6.11) and the generation of other periodic functions, saves d-c integrators, and reduces the requirements on the servo high-frequency response. The accuracy of integration will, however, be lower than that obtainable with d-c integrators.

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[1] Refer to *ibid.* An induction generator may be considered as a transformer whose output voltage is proportional to the product of input voltage and rotor speed.

[2] Greenwood, Holdam, and MacRae, *op. cit.*

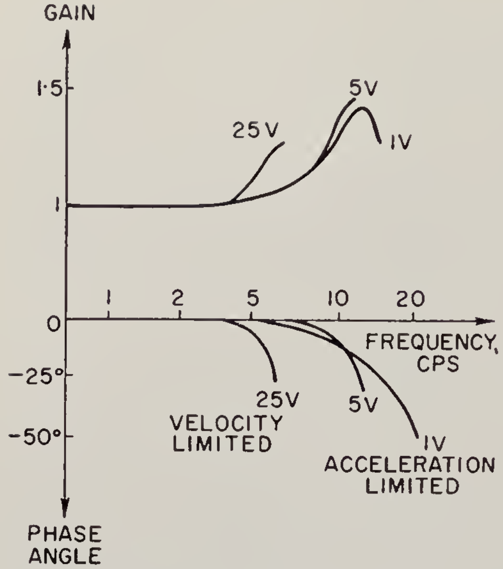

Fig. 6.29*a*. Frequency response of an experimental computer servomechanism at various amplitudes (velocity limit, 500 volts/sec; acceleration limit, 20,000 volts/sec²; based on Goodyear Aircraft Corporation data).

Servo Performance. The manufacturer's published frequency-response curves for computer servomechanisms usually refer to very small sinusoidal input voltages and measure the capability of the servomechanism to follow small high-frequency components of a composite signal. It is important to remember that computer servomechanisms are both velocity-limited (a typical velocity limit is 600 volts/sec) and acceleration-limited (a typical acceleration limit is 25,000 volts/sec²), so that "dynamic" errors tend to increase with increased signal amplitude. In particular, the phase error separating sinusoidal input and output voltages usually increases with sig-

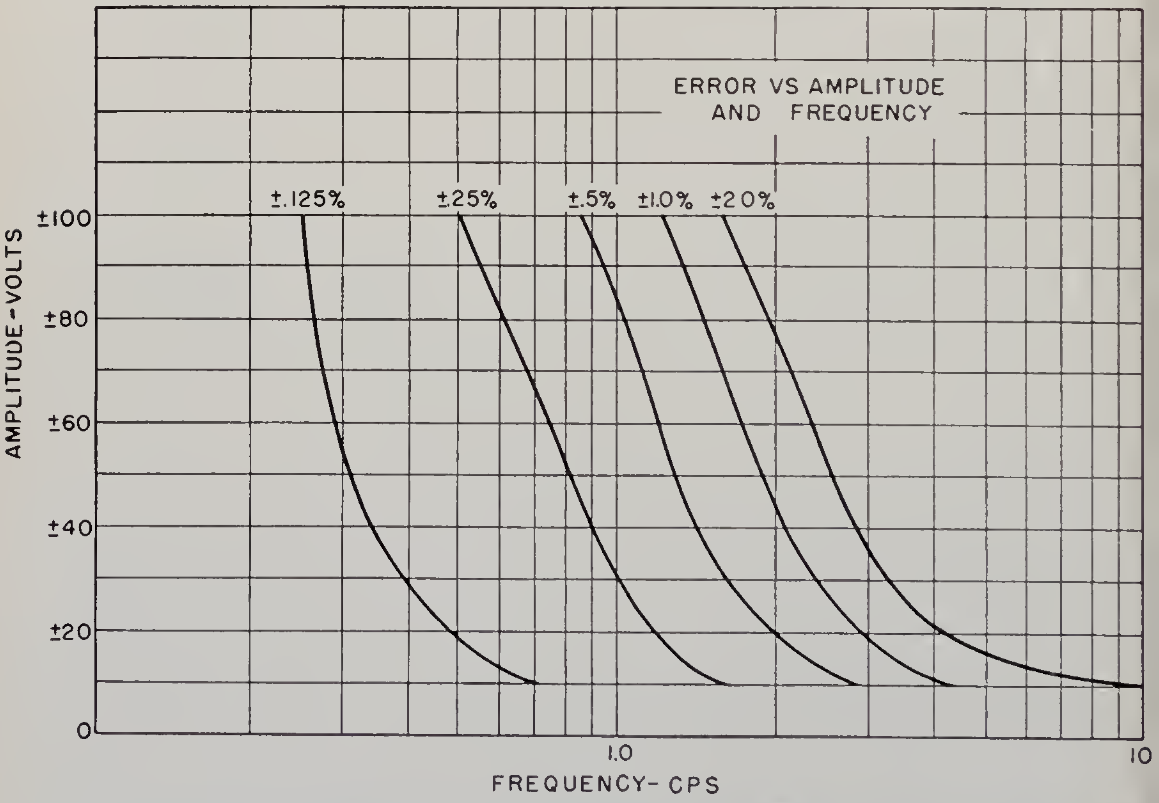

Fig. 6.29*b*. Maximum position error vs. amplitude and frequency for a commercial servomultiplier. The effects of potentiometer nonlinearity are not included (Electronic Associates, Inc.).

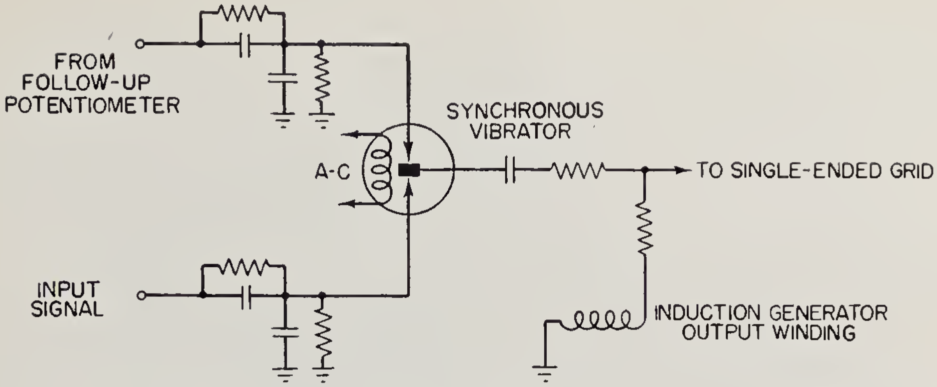

FIG. 6.30*a*. A single-ended input circuit with integral control. The component values will depend on the motor and load characteristics in each case.

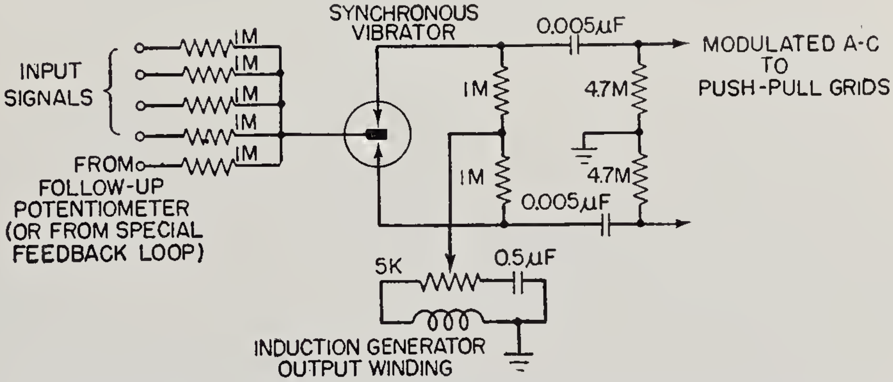

FIG. 6.30*b*. Push-pull input circuit with summing network used by the RAND Corporation (W. F. Gunning and A. S. Mengel, *Internal Report*, RAND Corporation, Santa Monica, Calif., Sept. 8, 1949). Note that the follow-up potentiometer, if used, is loaded by a resistance of 1 megohm.

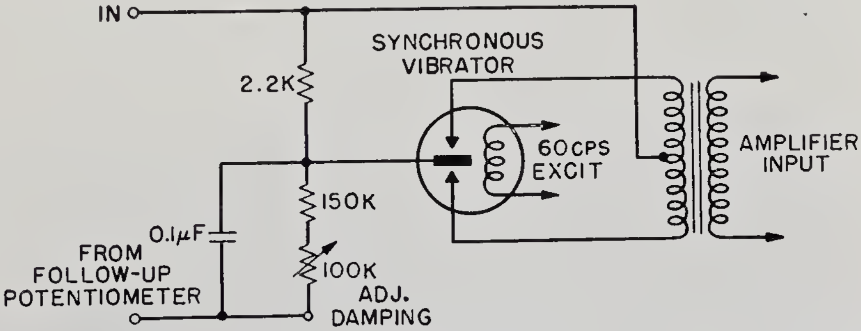

FIG. 6.30*c*. A transformer input circuit with error-rate network. No rate feedback is used; the follow-up potentiometer is not loaded (Electronic Associates, Inc.).

nal amplitude. Figure 6.29 illustrates the performance characteristics of a typical 60-cps computer servomechanism. The frequency response of the best commercially available high-performance 400-cps servomechanisms (Fig. 6.35) is similar to that shown in Fig. 6.29*a*, with the frequency scale contracted in approximately a 4:1 ratio.

Input Circuits. Figure 6.30 shows examples of input circuits which accomplish the purposes of mixing the input signal with the various feedback voltages, of introducing integral control, and of modulation. Both single-ended and push-pull circuits are used. The output voltage of an induction generator may be added to the error signal by either a series or a parallel arrangement. Many combinations of such circuits are possible.

In the input circuits of Fig. 6.30 feedback from a follow-up potentiometer is indicated; the feedback loop may, instead, include computing elements outside the servomechanism proper.

Servo Amplifiers. Practically any type of a-c amplifier having a maximum gain of about 100 db over the required frequency range and sufficient power output to drive the motor in question can be used as a servo amplifier.

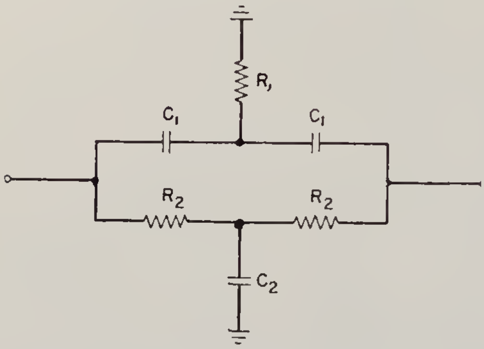

FIG. 6.31. Simple parallel-T third-harmonic suppressor. Useful circuit values for 180-cps rejection are $R_1 = 0.5M$, $R_2 = 1M$, $C_1 = 870\ \mu\mu f$, $C_2 = 1750\ \mu\mu f$. The rejection frequency f_r is given by

$$f_r = \frac{1}{\pi R_2} \frac{1}{\sqrt{2C_1C_2}}$$

with $4C_1R_1 = C_2R_2$.

As a relevant example, most public-address amplifiers having a 600-ohm output impedance will serve as excellent low-cost servo amplifiers. It is only necessary that they have the power rating required in each case and that their frequency response be reasonably free of phase shift over a frequency range of about 50 cycles on either side of the carrier frequency chosen (usually either 60 or 400 cps). Such amplifiers have the advantage of self-contained power supplies.

The following simple modifications will be found helpful in converting public-address amplifiers for servo applications:

1. *Limiting resistors* of 20,000 to 150,000 ohms connected in series with at least the output-tube grids will help to limit excessively large error voltages. The same purpose may be accomplished by means of selenium-diode shunt limiters (Sec. 6.7) in the input stage (see also Fig. 6.33*b*).
2. A *simple third-harmonic filter* of the parallel-T type inserted between two voltage-amplifier stages (two are needed for push-pull circuits) will reduce noise caused by the third harmonic of the carrier. Such third harmonics are usually generated in the induction generator. Figure 6.31 shows a parallel-T filter for use with a 60-cycle carrier; the equation for the rejection frequency is also indicated and will be found useful for the design of similar filters.

Figure 6.32*a* shows a modified public-address amplifier used as a servo amplifier with excellent results.

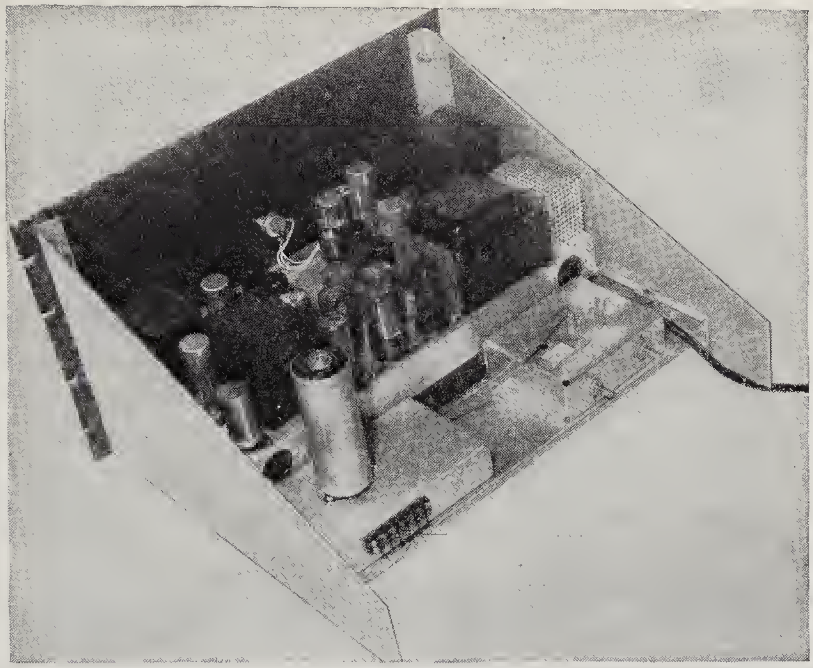

FIG. 6.32*a*. This servo amplifier (Curtiss-Wright Corporation, Airplane Division, Columbus, Ohio) is a modified 45-watt public-address amplifier. The input networks and the synchronous vibrator are arranged on a subchassis in the rear of the amplifier.

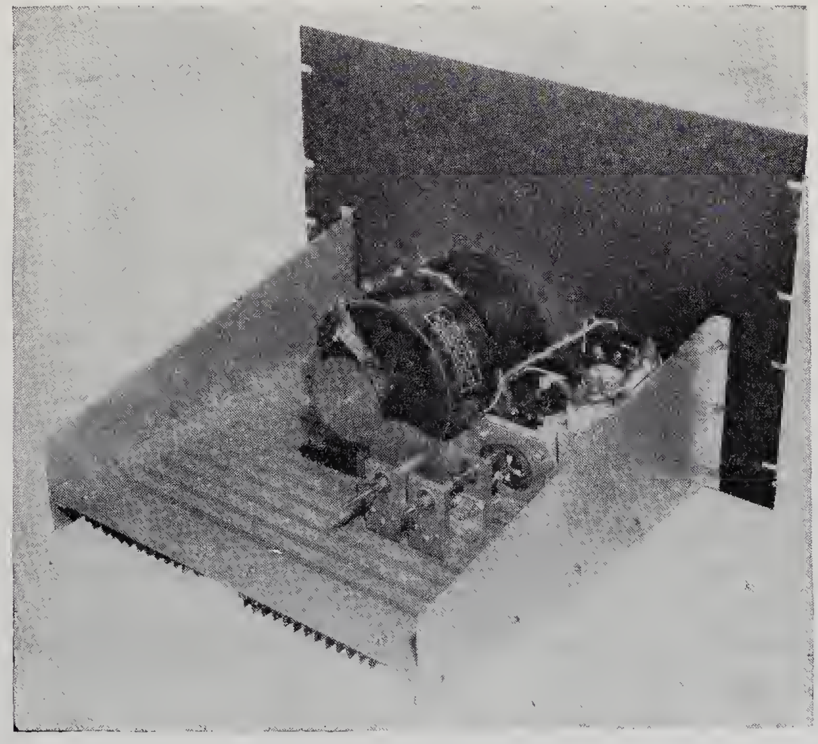

FIG. 6.32*b*. Mechanical components of the same computer servomechanism (Curtiss-Wright Corporation, Airplane Division, Columbus, Ohio) assembled on a slotted base plate for relay-rack mounting. Note the gear trains, bearing blocks, and resonating capacitors. A combination motor and induction generator is used.

Figure 6.33*a* shows the circuit of a push-pull amplifier specifically designed for servo application[1] in the RAND computer laboratory (see Sec. 8.4). Figure 6.33*b* shows a single-ended amplifier circuit.

The use of an input transformer ahead of the servo amplifier permits one to operate the chopper at a lower impedance level; this will tend to reduce the noise introduced in the input circuit. The benefits to be gained in this manner may, however, not justify the cost of a well-shielded input transformer.

The cost of the servo amplifier can be further reduced by eliminating the output transformer as well; this is made possible through the use of modern split-field induction motors whose high-impedance control-field windings can be connected directly into the plate circuits of push-pull power amplifiers (Fig. 6.33*b*). In such circuits, any d-c unbalance in the push-pull output stage tends to cause a spurious torque at the carrier frequency.

Servomotors, Induction Generators, and Gear Trains. An increasing number of manufacturers produce small induction motors suitable for computer applications; some of these motors are combined with an induction generator in an integral assembly. Induction generators or tachometers should always be coupled directly to the driving motor, since the differentiation tends to increase the effects of gear backlash.

The *stalled torque* of a servomotor must be greater than the maximum static friction torque necessary to start the load through its gear train. The *maximum (zero-torque) speed* for each power rating must be greater than the maximum speed required by the computer application. For each given gear ratio, the high-frequency response of the servomechanism is improved by increasing the *torque-to-inertia ratio* of the servomotor; this is the ratio between the stall torque and the moment of inertia of the rotor assembly.

The speed-torque characteristics and the *power rating* of the servomotor as well as the *gear ratio* to be used between motor and load will be determined by the speed-torque requirements of the load in each case. A motor having as high a torque-to-inertia ratio as possible will then be chosen. Suitable torque-to-inertia ratios for 60-cycle motors will be above 20,000 radians/sec^2. Although 400-cycle motors may have still higher torque-to-inertia ratios, their stall torques for a given power rating may be lower than those of 60-cycle motors.

Very substantial increases in servomotor torque can usually be obtained by operating the motors at higher than rated power for the short time of each computer run. A relay actuated by operating the recorder switch or the "compute" switch on the computer control panel (see Sec. 7.2) can be

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[1] Gunning, W. F., and A. S. Mengel, *Report RM* 236, RAND Corporation, Santa Monica, Calif.

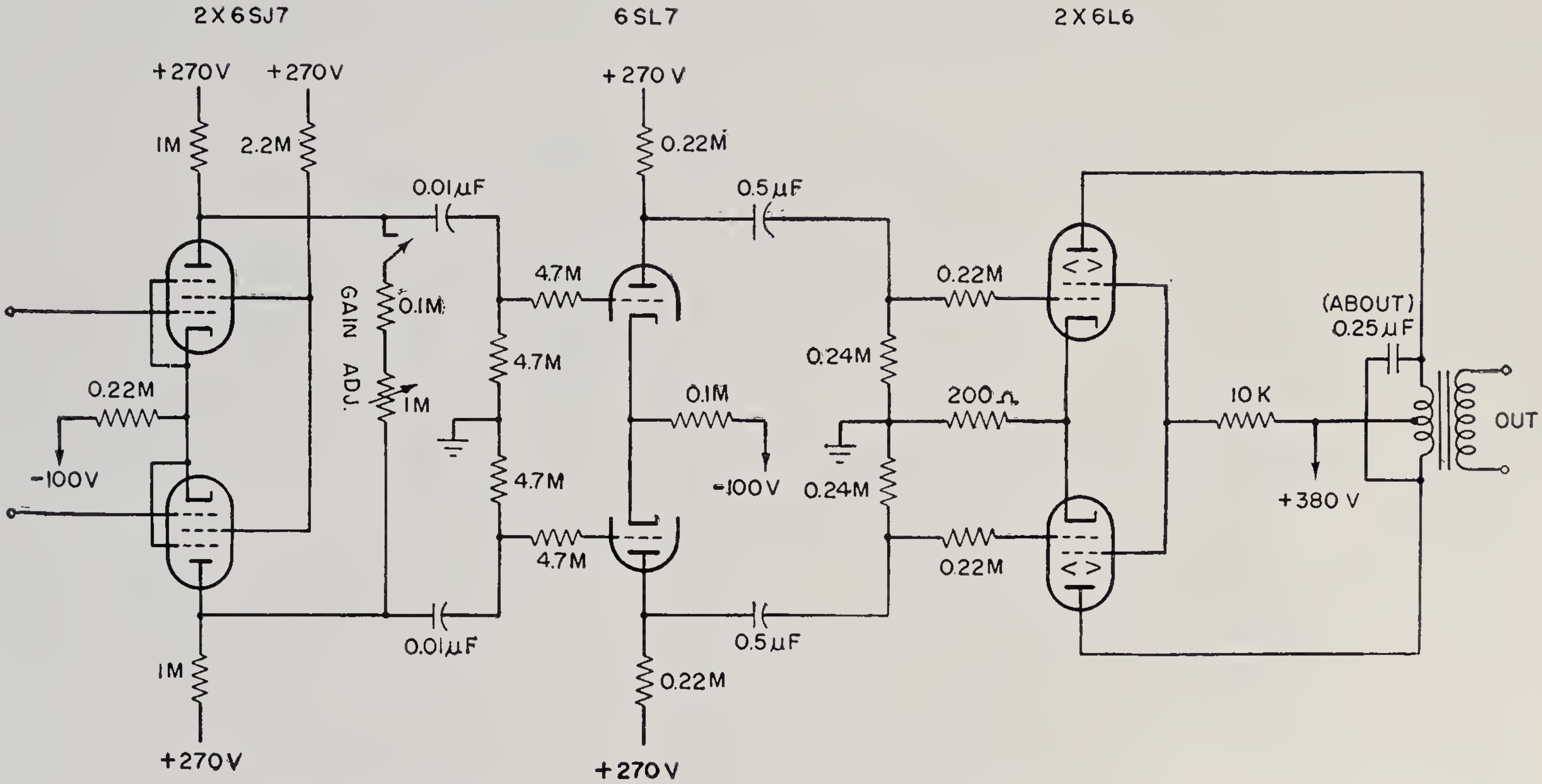

Fig. 6.33*a*. Servo-amplifier circuit from the RAND analog computer (Gunning and Mengel, *op. cit.*). This circuit may be used together with the input circuit of Fig. 6.30*b*.

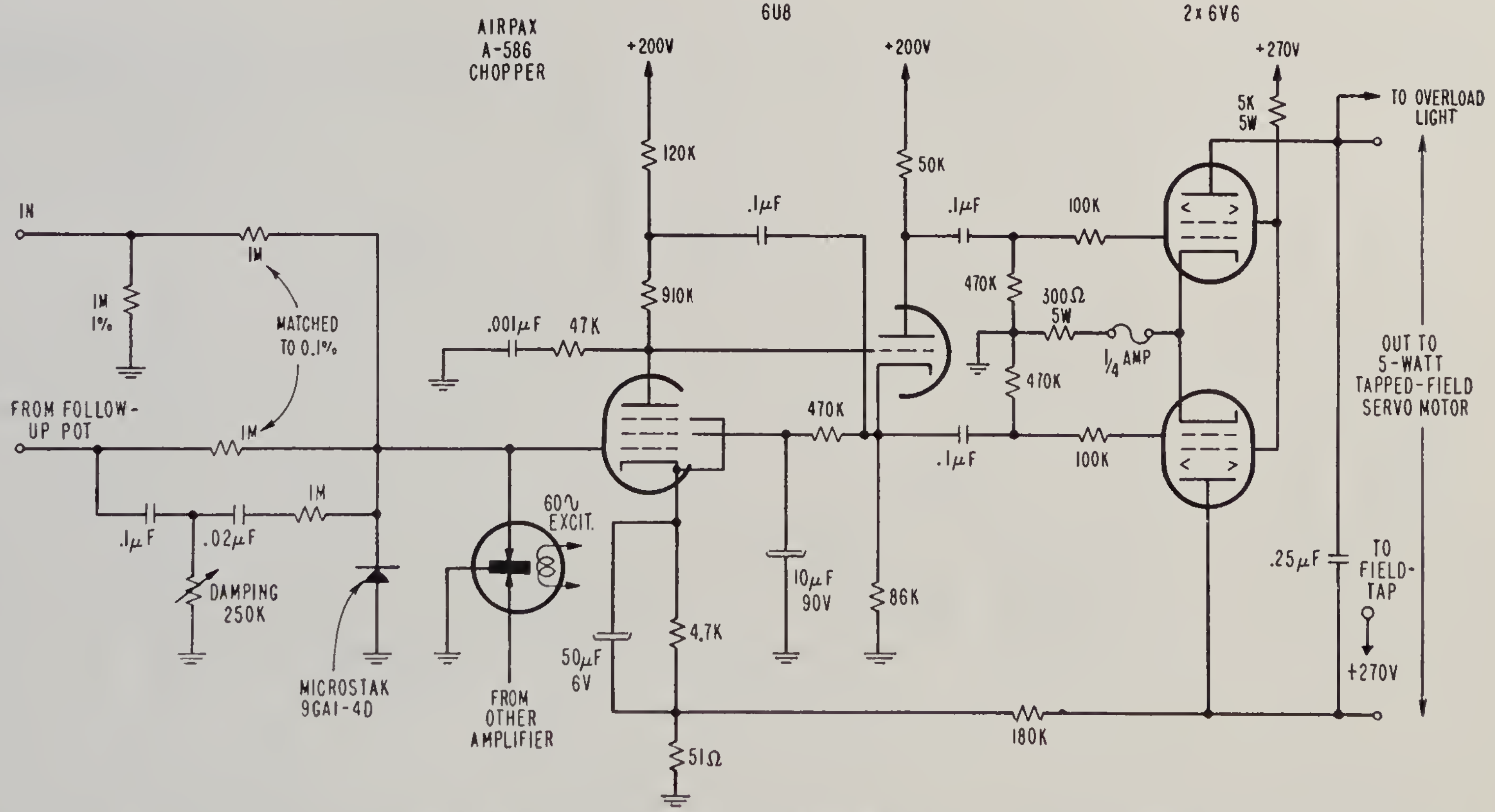

FIG. 6.33*b*. This 60-cps servo amplifier uses only three tubes and shares its vibrator with another amplifier. The 5-watt split-field servomotor drives four helical potentiometers (including the follow-up potentiometer) and requires no output transformer. Rate feedback is obtained by electrical differentiation. Static error is within the 0.1 per cent linearity of the follow-up potentiometer (Reeves Instrument Corporation, New York City).

made to increase the reference-field voltage from 150 to 190 per cent of the rated value for the duration of the computer run only. The power output of the servo amplifier must be similarly increased. At least for computer runs of less than 3-min duration, the motor stall torque may be increased to between two and four times its rated value without damage to the motor.

Precision-type gears, if possible in matched sets, should be used in all computer applications; the gears should be carefully meshed in order to minimize binding and backlash. Since the inertia of large gears may impair the servo frequency response, gear trains consisting of more than two gears should be used for the higher gear ratios.[1]

Phasing and Equalization. The servomotor control field is usually resonated at the carrier frequency by means of a capacitor in parallel with the primary winding of the output transformer to reduce noise. The carrier phase in the control field must differ by 90 deg from that in the reference field. The reference fields of all servomotors in a computer are usually connected directly to the a-c line. The carrier phase in the control fields may be adjusted by means of a phase-shifting network, such as the one shown in Fig. 6.28 in the a-c supply used for the synchronous vibrators.

The values of the forward gain, rate, and integral feedback of a servomechanism for stable operation with good frequency response are usually computed roughly; the exact values are then obtained through final adjustments of the equalizing networks. Note that the same servomechanism will require different adjustments in the circuits of Figs. 6.8*a* and *b*. These adjustments may be made with the aid of a servomechanism tester[2] by checking the servo response to sinusoidal input voltages. If such a device is not available, a d-c step-function voltage may be applied to the servo input; the equalizing networks are then adjusted until the servo output follows the input voltage as closely as possible without excessive overshoot. Some other methods for testing servomechanisms are discussed in Vol. 21 of the MIT Radiation Laboratory Series.[3]

Mechanical Construction. Figure 6.32*b* shows the electromechanical components of a laboratory-constructed servomechanism set up on a *slotted base plate* which allows for a wide variety of component arrangements. As an alternative, Fig. 6.34 shows a method of attaching servo components to *drilled base plates* by means of dogs and screws. The dogs shown in Fig. 6.34 are easily made by drilling and cutting stock pieces of aluminum having the profile required. The holes in the base plate itself may or may not be tapped, although the former makes for more convenient assembly. Bases and shafts may be lined up by means of a simple T square or similar device. Both slotted and drilled base plates as well as suitable bearing blocks, motor mounts, etc., for small servomechanisms are available commercially from a number of manufacturers.

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[1] James, H. M., N. B. Nichols, and R. S. Phillips, *Theory of Servomechanisms*, MIT Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947, pp. 131–133.

[2] Korn, G. A., and T. M. Korn, Modern Servomechanism Testers, *Elec. Eng.*, September, 1950.

[3] Greenwood, Holdam, and MacRae, *op. cit.*

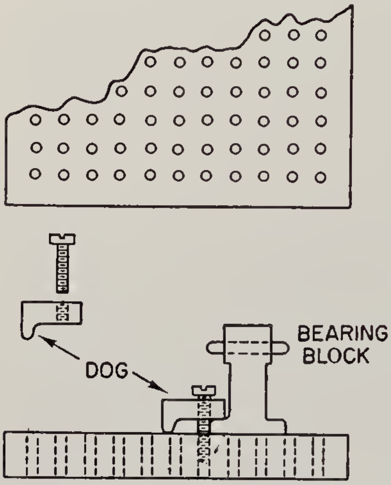

Fig. 6.34. Means of attaching servo components to a drilled and tapped base plate. Note that the components may be turned in any direction on the base plate.

Figure 6.35 shows the compact construction of a commercial computer servomechanism.

Limit Stops and Overload Lights. If helical-type potentiometers are to be driven by servomechanisms, special precautions must be taken to prevent damage to the internal stops in the potentiometers caused by accidentally driving the wiper assembly into these stops. The stall torque of a servomotor, when applied to the potentiometer through a gear train, can be very appreciable and will almost certainly destroy the potentiometer. Special systems of mechanical stops and/or limit switches have been devised for this purpose. In an alternative scheme, the follow-up potentiometer winding is extended beyond its end terminals; large erroi voltages (plus 300 volts and minus 300 volts) applied to the end resistances will then "repel" the potentiometer wiper away from the limit stops. Perhaps the easiest solution is to use single-turn potentiometers. Such potentiometers may be obtained with accuracies and resolutions better than 0.1 per cent of full scale for resistance values between 10,000 and 100,000 ohms; they can be fitted with "bridges" which permit continuous rotation.

Each computer servomechanism should be provided with an overload bulb (see also Secs. 5.6 and 7.2) which indicates excessive error voltages in the servo amplifier (see, for instance, Fig. 6.33*b*).

Other Types of Servomechanisms. The more powerful types of computer d-c amplifiers are capable of driving very small d-c servomotors directly. Such amplifiers are easily fitted with equalization networks by the usual operational-amplifier techniques (Sec. 1.5) and permit the design of attractive wide-band servomechanisms, particularly in special-purpose computers.[1] Special limiters may be needed to avoid overloads in the automatic-balancing channels of the d-c amplifiers used.

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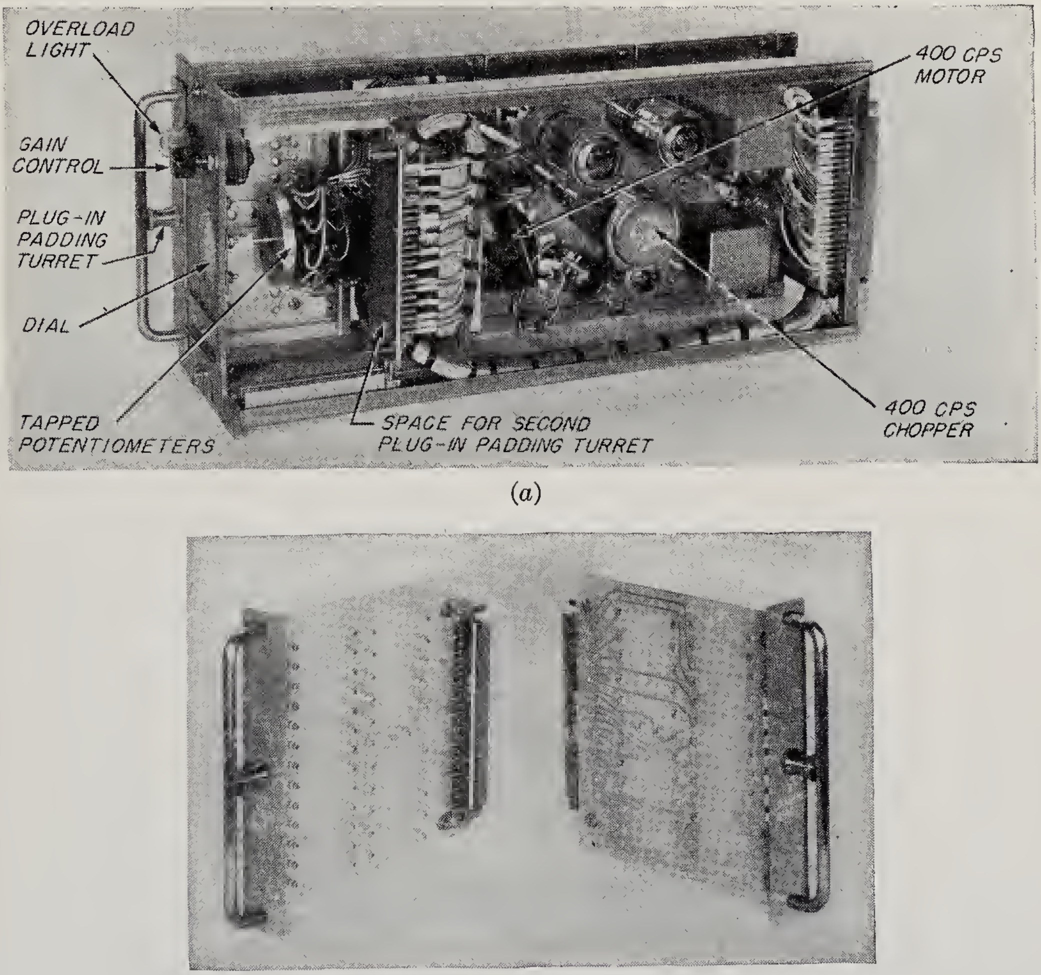
[1] Hosemann, J. H., and A. D. Pendleton, Computing Amplifier Subs for Servo Amplifier, *Control Eng.*, March, 1955.

Relay servomechanisms using a sensitive polarized relay in a circuit like that of Fig. 6.36*a* to control d-c motors may be useful in certain special applications because of their relative simplicity. The frequency response of such devices may be made reasonably flat up to 2 or 3 cps. In order to minimize the effects of static friction in the relay, an a-c voltage (the so-called dither voltage) is introduced into the relay coil so as to vibrate the relay armature continuously. The dither frequency is made higher than the highest frequency to which there is appreciable servo response. If the contacts of the sensitive relay cannot handle the motor current, the sensitive relay can control a power relay which in turn controls the motor. Although relay circuits may be less complex than vacuum-tube circuits, these devices are generally not so reliable as vacuum-tube amplifiers, and



(*a*)

(*b*)

FIG. 6.35. A high-performance computer servomechanism. A 400-cps servomotor drives four linear two-tap potentiometers and two nine-tap potentiometers. The upper plug-in tap-voltage-setting turret has been removed; two such turrets are shown in Fig. 6.35*b*. Note the taper-pin connections. Frequency response has a 40 per cent amplitude rise at about 40 cps; dynamic errors are less than 0.5 per cent at 2 cps. Maximum velocity, 1,500 volts/sec; maximum acceleration, 60,000 volts/sec^2 (Reeves Instrument Corporation, New York City; see also Sec. 8.4).

their sensitivity and frequency response may not be satisfactory except in special cases.

If an a-c power source is available, a-c motors are usually preferred to d-c motors in order to eliminate the maintenance problems associated with brushes and commutators. The a-c amplifiers used for controlling two-phase motors require a high-voltage d-c supply which may or may not be self-contained in each servo amplifier. If servomotors are operated at high power levels, the d-c supplies needed for amplifiers using vacuum tubes may become bulky and expensive. In such cases, recourse

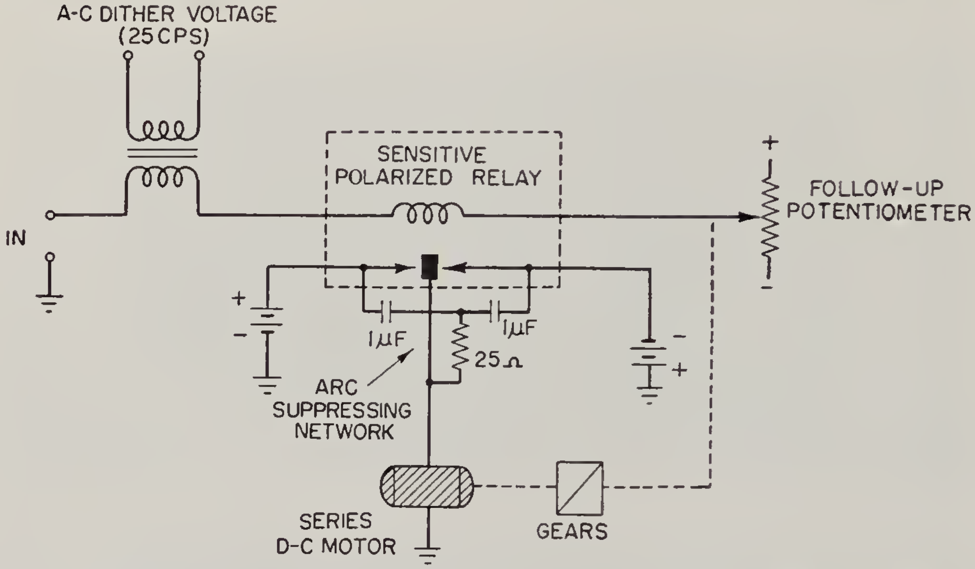


Fig. 6.36*a*. Simple relay servomechanism with dither. The optimum dither frequency and amplitude will depend on the relay spacing, sensitivity, and time delay.

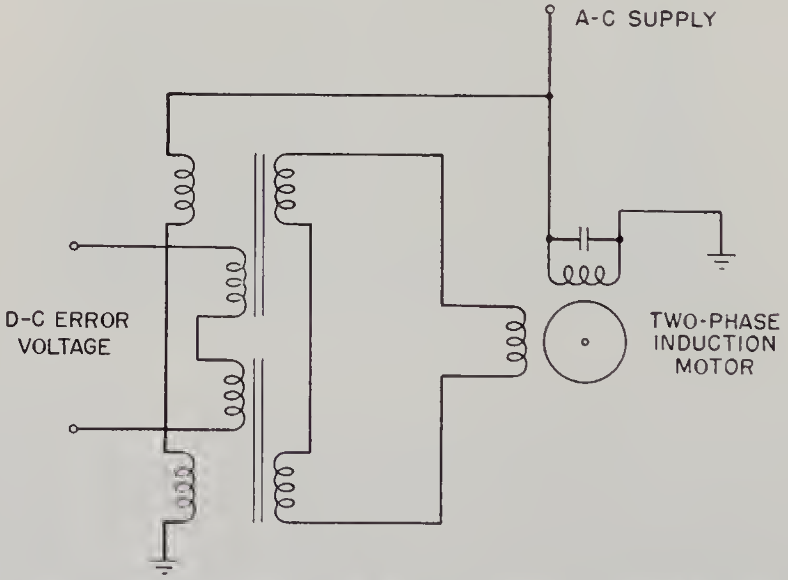


Fig. 6.36*b*. Use of a saturable transformer to control the a-c voltage applied to a motor field in accordance with a d-c input voltage.

may be had to thyratron control of a-c motors,[1] but careful shielding and isolation of the thyratrons will be necessary in order to prevent their interaction with other computing elements. The use of *saturable reactors* for direct control of the alternating currents in the motor fields may be preferable. Figure 6.36*b* shows how the alternating current in a motor field may be effectively controlled by a d-c input voltage. Although some internal noise is inherent in saturable-reactor modulators, its results can be effectively reduced by including a d-c preamplifier in the servo loop. *Ordinary operational amplifiers may conceivably be adapted for this purpose.* The use of saturable reactors will eliminate the need for synchronous vibrators and the attendant maintenance problems as well as the requirement for high-power d-c supplies. A d-c tachometer or electrical differentiation may be used for speed feedback.

Finally, serious consideration should be given to the use of commutated-potentiometer servomechanisms of the type described in Sec. 6.3.

6.9. SERVO-DRIVEN POTENTIOMETERS AND RELATED CIRCUITS

The potentiometers used with computer servomechanisms must satisfy the requirements of mechanical ruggedness, freedom from microphonism, accuracy, resolution, and stability already discussed in Sec. 4.1 in connection with constant-coefficient-setting potentiometers. In addition, servo-driven potentiometers must satisfy special requirements directly related to their utilization in the servomechanism, *viz.*:

1. The static and dynamic torques needed to position the potentiometer wiper must be as low as consistent with reliable operation. A static torque of 2 oz-in. per potentiometer may be considered as a reasonable value.
2. The potentiometer windings must sustain the wiper friction for a reasonable life period without changing their properties. At the rotational speeds involved in the normal operation of most d-c analog computers, a useful life of at least 300,000 revolutions may be expected.
3. A minimum of noise should be generated by the motion of the wiper over the potentiometer winding.

The construction of potentiometers satisfying these requirements is a highly developed art described extensively in the literature (see also Fig. 4.2).[2]

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[1] Greenwood, Holdom, and MacRae, *op. cit.*, Chap. 12.

[2] Blackburn, J. F., *Components Handbook*, Chap. 8, by F. E. Dole, MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949; Duncan, D. C., Characteristics of Precision Servo Computer Potentiometers, *Trans. AIEE Conference on Feedback Systems*, Atlantic City, N.J., 1951; Schmidt, H. A., The Precision Potentiometer as a Voltage Divider, *Product Engineering—Annual Handbook of Product Design for 1954.*

Tapered Function Potentiometers. Potentiometers used to multiply an input voltage by a function of the shaft displacement are constructed so that the fraction of the total resistance tapped off by the wiper is a nonlinear function of the shaft displacement. Wire-wound elements for function potentiometers are given the proper "taper" (or rate of resistance change with respect to shaft displacement) by suitable changes in the shape of the card, the pitch of the winding, and/or the wire diameter in accordance with the desired function (Fig. 6.37). The construction of tapered function potentiometers is discussed in detail in Vol. 17 of the MIT Radiation Laboratory Series;[1] tapered wire-wound function potentiometers for a number of frequently used functions are commercially available.

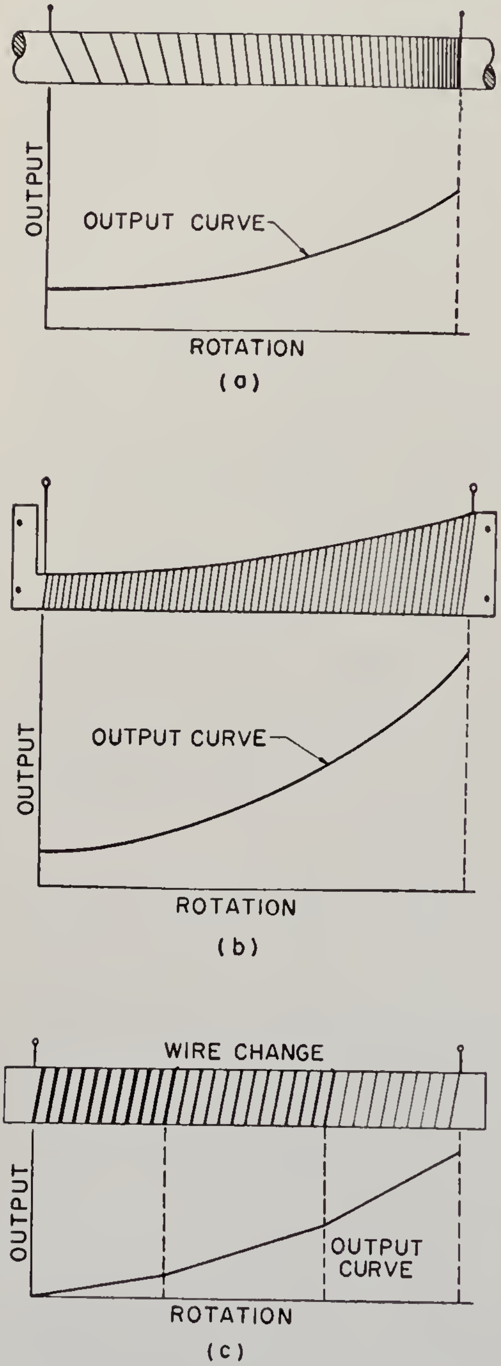

FIG. 6.37. Wire-wound function potentiometers produced by variable wire spacing (*a*), a tapered potentiometer card (*b*), and changes in wire size and/or composition (*c*). Combinations of these techniques may be used (copyright 1954, HELIPOT® Corporation—reprinted by permission).

Simple tapered function potentiometers can generate only functions which vary monotonically between two values determined by the voltages at the two input terminals of the potentiometer winding. Nonmonotonic functions can, however, be generated by employing one or more *taps* in the potentiometer winding; the output voltage into a load of infinite impedance will, then, vary monotonically between each pair of voltages at the taps (see, *e.g.*, Fig. 6.51*f*).

Function Potentiometers Based on Linear Resistance Elements with Controlled Wiper Displacement. A linear potentiometer winding may be used for function generation if the displacement of the potentiometer wiper is made to vary as the desired function of an input shaft rotation. A cam or a linkage can be used to transmit the input shaft motion in the desired manner. A stylus linked to a potentiometer through a linkage, a rack-and-pinion drive, or a worm-gear arrangement may even be used for manual following of any reasonable curve plotted on a moving chart. The potentiometer can then generate a voltage as a function of the chart displacement. The usefulness of this *input table* or *curve follower* is not restricted to the generation of a single function; it may be regarded as a type of "universal function generator."

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[1] See footnote on p. 313.

Servo-driven Automatic Curve Followers. The practical *automatic curve follower* shown in Fig. 6.38 utilizes the servo-positioned drum of a servo recorder (Sec. 7.3). The drum is rotated about its axis by a servomechanism whose input voltage is the argument of the function to be generated. A linear resistance element constituting the potentiometer winding is mounted parallel to the drum axis. A wire is cemented over a graph of the curve to be tracked, and the graph paper is placed on the drum with the independent variable axis in the direction of drum rotation. As the drum rotates in correspondence to the independent variable, the linear resistance element is always parallel to the drum axis directly over the correct value of the dependent variable on the graph paper.

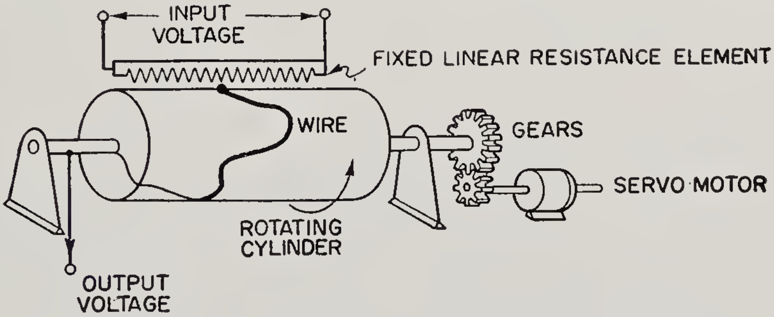

Fig. 6.38. Reeves-type function generating potentiometer.

The wire on the curve then acts as the wiper of a linear potentiometer tapping off the proper fraction of an input voltage applied across the linear resistance element as the latter is pressed against the drum and wire by light spring pressure. As the drum rotates, this wiper moves across the potentiometer in a manner determined by the given curve. With a resistance element of 20,000 ohms resistance and linear to within 0.1 per cent, it is possible to multiply a voltage by any reasonable function of another voltage with an accuracy of about 0.15 per cent of full scale.[1]

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[1] Gunning and Mengel, *op. cit.* The preparation of the wire for any given function is simple. A length of 0.015-in.-diameter nichrome wire is first straightened by

All types of servo recorders (plotting boards, Sec. 7.3) can be converted into automatic curve followers through the use of simple accessory components. The desired curve is drawn with conducting ink; a capacitive, inductive, or potentiometer-type pickoff replaces the recording pen and forces the pen servomechanism to follow the dependent variable. The d-c output may be taken from the follow-up potentiometer associated with the pen servomechanism.[1] Such utilization of a servo plotting board is more convenient than the scheme of Fig. 6.38; one may, for instance, record a function with conducting ink for playback in a later phase of the computation. On the other hand, the frequency range of the dependent variable is now limited by the response of both pen *and* carriage or drum servomechanisms (Sec. 7.3).

Automatic curve followers can generate a wide variety of functions and are thus particularly useful for flexible multipurpose computers. A typical application is the Fourier analysis of experimental data (Fig. 3.28*a*).

Reduction of Errors Due to Potentiometer Loading[2] and Manufacturing Inaccuracies. The output voltage X_o of a potentiometer connected to a finite load resistance r_L is no longer proportional to the resistance tapped off between the zero-voltage point on the potentiometer and the potentiometer wiper. The resulting *loading error* was derived for a special case in Sec. 4.1; Fig. 6.39 specifies the loading errors for the three most important potentiometer circuits. The circuit of Fig. 6.39*c* is, in general, preferable to that of Fig. 6.39*b*.

If the linear follow-up potentiometer of a servomechanism (Fig. 6.8) is loaded by a finite resistance r_L, the fractional servo shaft displacement Θ is no longer accurately proportional to the servo input voltage X_2, but

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stretching it beyond the yield point. The wire is then fitted to the curve drawn on a piece of graph paper lying flat on a table. The wire is held to the paper by means of small split pins and is covered with a layer of cement (one part General Cement No. 30-8 Radio Service Cement to one part No. 31-8 Solvent) which secures it to the paper even after the pins are removed. When the cement has dried, the top surface of the wire is laid bare through the use of abrasive cloth. The graph paper is then ready to be placed on the drum. Curve followers of this type are manufactured by the Reeves Instrument Corporation, New York City, which has also experimented with small permanent function potentiometers based on the same principle. See S. A. Frost, Compact Analog Computer, *Electronics*, July, 1948.

[1] Vance, P. R., and D. L. Haas, An Input-output Unit for Analog Computers, *Proc. IRE*, **41**: 1483, 1953.

[2] Nettleton, L. A., and F. E. Dole, Reducing Potentiometer Loading Error, *Rev. Sci. Instr.*, **18**: 332, 1947; Bradley, F. R., and R. D. McCoy, Computing with Servo-driven Potentiometers, *Tele-Tech and Electronic Industries*, September, 1952; Gilbert, J., Here's a Shortcut in Compensating Potentiometer Loading Errors, *Control Eng.*, February, 1955; —, Compensating Function Potentiometers for Loading Errors, *Control Eng.*, March, 1955.

$\Theta = \frac{X_2}{Y}[1 + E(\Theta)]$, where $E(\Theta)$ is given in Fig. 6.39. This effect must be remembered if the servomechanism in question is to drive indicating dials or function potentiometers (see also Sec. 6.10). Figure 6.30*c* shows a servo input circuit which does not load the follow-up potentiometer.

The loading errors can be neglected in some low-accuracy computations. In a typical application $r = 20K$, $r_L = 1M$, so that the loading errors will be less than 0.6 per cent in Figs. 6.39*a* and *b*, and less than 0.3 per cent in Fig. 6.39*c*. If such errors are to be tolerated, the accuracy (linearity, resolution, etc.) of the potentiometer used usually need not exceed 0.1 per cent.

Computing errors due to potentiometer loading may be reduced or made negligible by the following procedures:

1. In servomultipliers employing linear potentiometers, one may eliminate the loading errors by *loading multiplying and follow-up potentiometers with equal resistances.* This method does not apply to servo-driven function potentiometers.
2. The *bootstrap circuit* of Fig. 6.40 will eliminate loading effects for linear or nonlinear potentiometers, usually at the cost of one additional phase-inverting amplifier.
3. Special *isolation amplifiers* may be used.[1]

The following procedures may be used singly or in combination:

4. The use of *end resistances* (Figs. 6.41*a* and *b*) restricts the effective range of the quantity F in Fig. 6.39 and reduces the fractional errors due to loading. The resulting unfavorable scale change may increase the effects of other errors.
5. Multiplying or function-generating potentiometers may be given a taper which corrects the loading errors associated with a fixed load (*preloading*).
6. *Tapped potentiometers* (Sec. 6.10) permit convenient corrections for loading errors and winding imperfections. Figures 6.41*c* and *d* illustrate a simple preloading method. In the circuit of Fig. 6.41*c*, $R = 0.31r_L$ and $F_o = 0.74$ are useful design values.[2]
7. The function actually generated by the potentiometer may fit $kF(X_2)$ better than $F(X_2)$, where k is a suitable constant. Once k is chosen for best fit (least maximum error), one obtains

$$X_o = X_1F(X_2) = \frac{X_1}{k}[kF(X_2)]$$

by a simple coefficient change.

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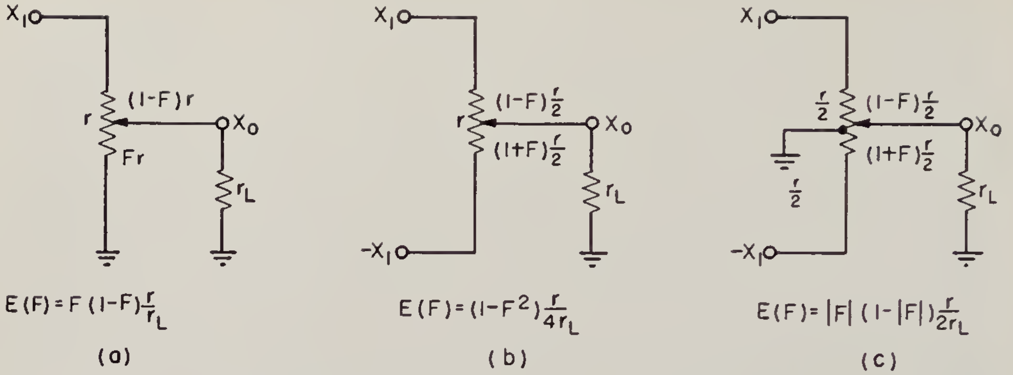
[1] Bradley, F., and R. McCoy, Isolation Circuits for Analog Computers, *Electronics*, May, 1952.

[2] Gilbert, *op. cit.*



$E(F) = F(1-F)\frac{r}{r_L}$ $\quad E(F) = (1-F^2)\frac{r}{4r_L}$ $\quad E(F) = |F|(1-|F|)\frac{r}{2r_L}$

(a) (b) (c)

FIG. 6.39. Effects of potentiometer loading. For each of the circuits shown, $F \equiv F(\Theta)$ is a function of the fractional shaft displacement Θ; $F \equiv \Theta$ for simple multiplication (linear potentiometer). In each case

$$\frac{X_o}{X_1} = \frac{F}{1 + E(F)} = F\left[1 - \frac{E(F)}{1 + E(F)}\right] \approx F[1 - E(F)]$$

The approximate expression is useful if $r/r_L \leq 0.1$.

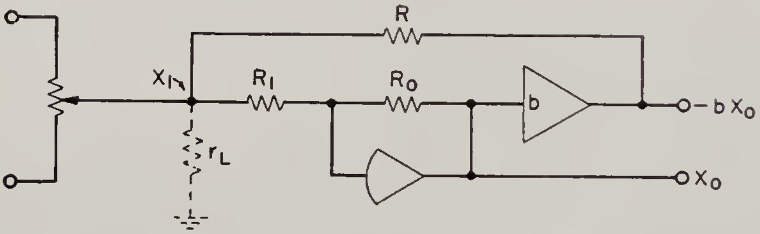


FIG. 6.40. This bootstrap circuit supplies all current drawn by the load, and thus eliminates all loading effects, if

$$R = \frac{r_L}{r_L + R_1}(bR_o - R_1)$$

Most frequently, the input resistor R_1 of the operational amplifier is the only load, so that $r_L = \infty$, and $R = bR_o - R_1$. The bootstrap principle has many other applications (Secs. 4.4 and 7.4).

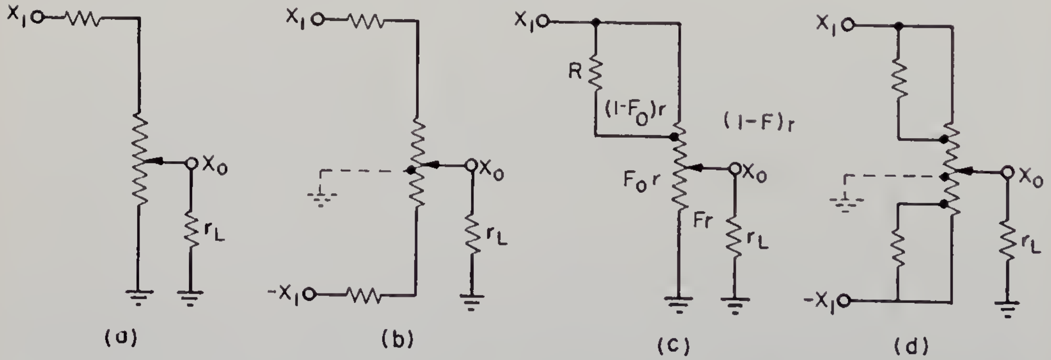


FIG. 6.41. Reduction of potentiometer-loading errors by means of end resistances (a, b) and by preloading (c, d). The two methods may be combined.

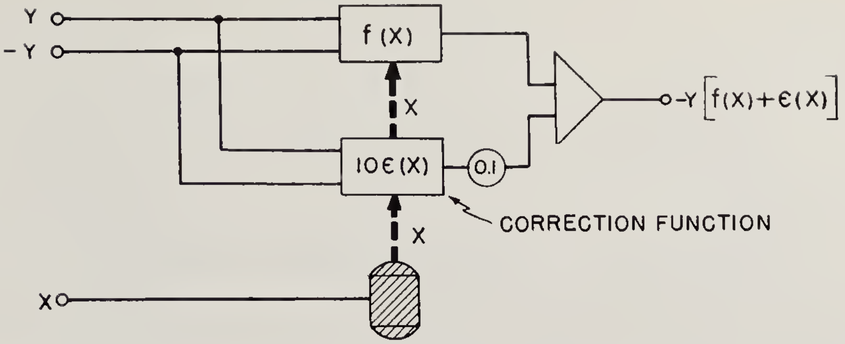


FIG. 6.42. Use of a correction function generator to improve the accuracy of a servo multiplier or function generator. The correction function generator may be a tapped potentiometer (Sec. 6.10), so that the additive correction function can be set empirically. Note that the effects of small imperfections in the correction function generator will be very small. Analogous correction methods may apply to electronic function generators; such devices cannot, of course, correct errors due to drift or parameter changes (Gunning and Mengel, *op. cit.*).

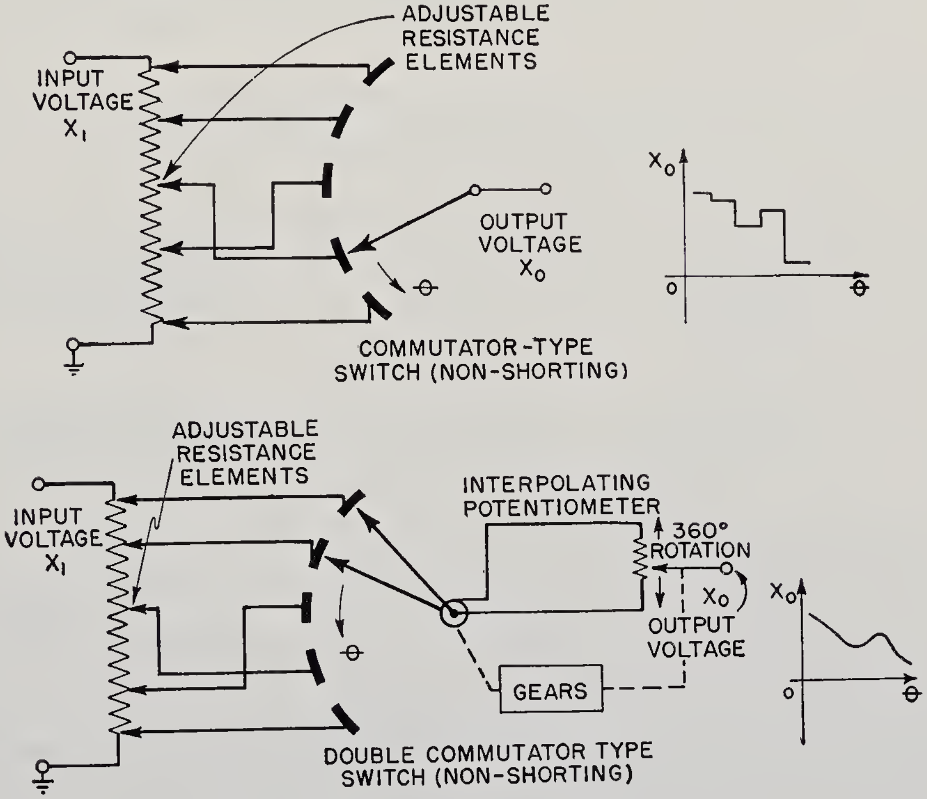

ADJUSTABLE RESISTANCE ELEMENTS
INPUT VOLTAGE X₁
OUTPUT VOLTAGE X₀
X₀
0
θ
θ
COMMUTATOR-TYPE SWITCH (NON-SHORTING)
ADJUSTABLE RESISTANCE ELEMENTS
INPUT VOLTAGE X₁
INTERPOLATING POTENTIOMETER
360° ROTATION
X₀
OUTPUT VOLTAGE
X₀
θ
GEARS
0
θ
DOUBLE COMMUTATOR TYPE SWITCH (NON-SHORTING)

FIG. 6.43. Switch-type function generators.

8. Figure 6.42 shows a *correction function generator*[1] which will correct errors due to loading and winding imperfections. Many similar arrangements are possible.

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[1] Gunning and Mengel, *op. cit.*; see also L. Cahn, Automatic Compensation of Potentiometer Loading Errors, *Helinews*, Helipot Corporation, South Pasadena, Calif., June, 1953.

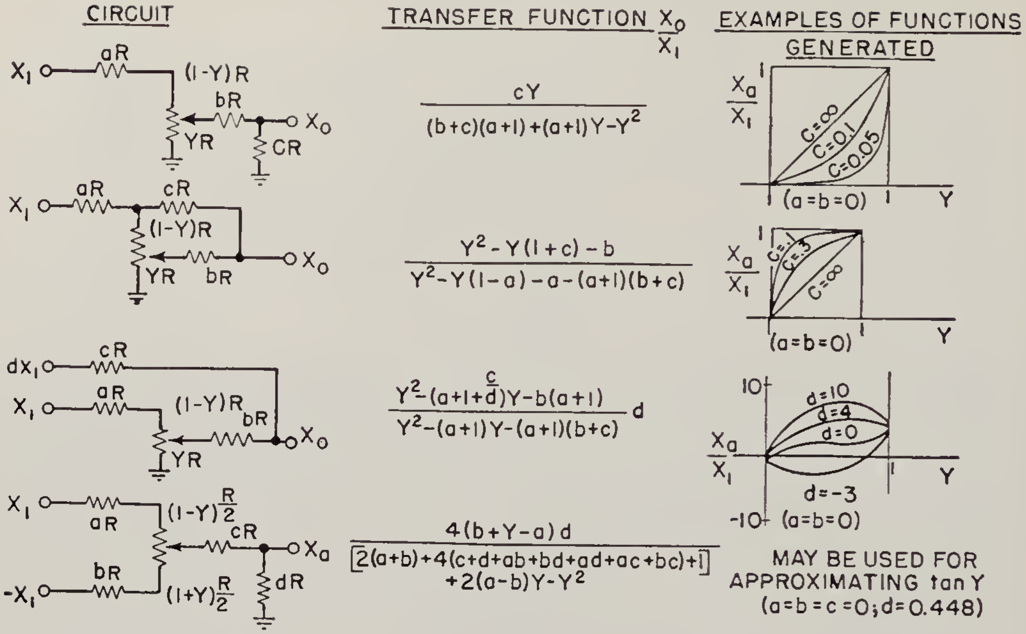
CIRCUIT
TRANSFER FUNCTION $\frac{X_0}{X_1}$
EXAMPLES OF FUNCTIONS GENERATED
$\frac{cY}{(b+c)(a+1)+(a+1)Y-Y^2}$
C=∞
C=0.1
C=0.05
(a=b=0)
$\frac{Y^2-Y(1+c)-b}{Y^2-Y(1-a)-a-(a+1)(b+c)}$
C=1
C=3
C=∞
(a=b=0)
$\frac{Y^2-(a+1+\frac{c}{d})Y-b(a+1)}{Y^2-(a+1)Y-(a+1)(b+c)}d$
d=10
d=4
d=0
d=-3
(a=b=0)
$\frac{4(b+Y-a)d}{[2(a+b)+4(c+d+ab+bd+ad+ac+bc)+1]+2(a-b)Y-Y^2}$
MAY BE USED FOR APPROXIMATING tan Y
(a=b=c=0; d=0.448)

FIG. 6.44*a*. Use of loaded linear function potentiometers for approximating various nonlinear functions. The potentiometer setting is determined by the machine variable Y (Greenwood, Holdam, and MacRae, *op. cit.;* see also Table 5).

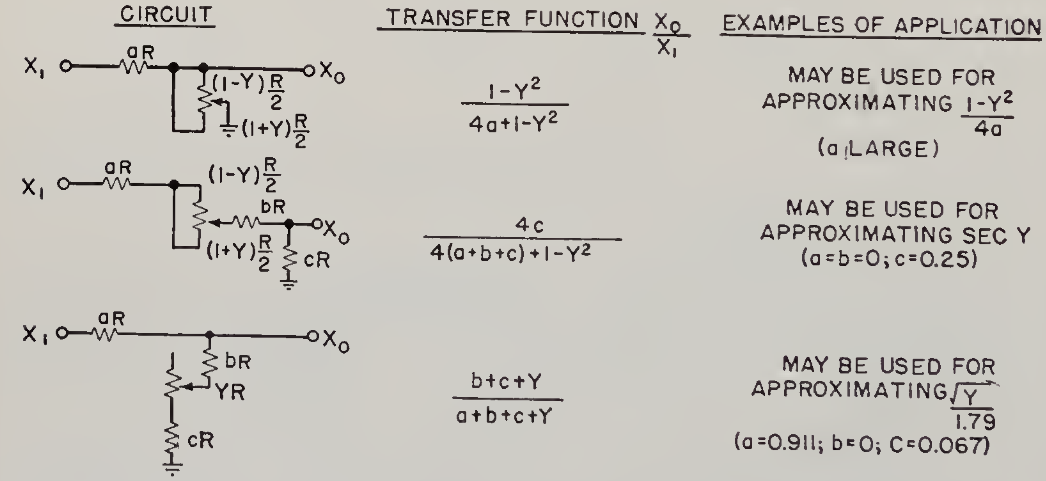
CIRCUIT
TRANSFER FUNCTION $\frac{X_0}{X_1}$
EXAMPLES OF APPLICATION
$\frac{1-Y^2}{4a+1-Y^2}$
MAY BE USED FOR APPROXIMATING $\frac{1-Y^2}{4a}$
(a LARGE)
$\frac{4c}{4(a+b+c)+1-Y^2}$
MAY BE USED FOR APPROXIMATING SEC Y
(a=b=0; c=0.25)
$\frac{b+c+Y}{a+b+c+Y}$
MAY BE USED FOR APPROXIMATING $\frac{\sqrt{Y}}{1.79}$
(a=0.911; b=0; C=0.067)

FIG. 6.44*b*. Miscellaneous circuit arrangements permitting the use of simple linear potentiometers for generating nonlinear functions (Chance *et al., op. cit.;* Greenwood, Holdam, and MacRae, *op. cit.*). The networks are adjusted in each case to match the desired functions. The potentiometer setting is determined by the machine variable Y (see also Table 5).

Methods 4, 5, 7, and 8 apply particularly to special-purpose computers. *Methods 6, 7, and 8 serve to reduce the effects of manufacturing inaccuracies in the potentiometer winding, such as nonlinearity, as well as loading errors.*

Rotating-switch Function Generators. If it is sufficient to approximate a given function by a step function (the step function may, for instance, be smoothed out by subsequent integration), a function potentiometer may be replaced by a simple tap switch rotated in accordance with the argument of the given function (see Fig. 6.43*a*). The tap voltages are adjusted in accordance with the values of the given function for the corresponding shaft displacements. For better accuracy, a linear "interpolating potentiometer" may be connected between successive taps by means of two switch arms in the manner shown in Fig. 6.43*b*. The interpolating potentiometer is geared up from the switch shaft so that it makes a complete revolution while each switch arm moves between successive taps.

Use of Servomechanisms to Vary the Transfer Functions of Operational Amplifiers. Useful and accurate servo multipliers and function generators can be constructed by using a servo-driven potentiometer or tap switch to vary the feedback or input resistor of an operational amplifier, usually a simple phase inverter (see, *e.g.*, Sec. 2.7, Fig. 6.57, and Table 5). Such multipliers and function generators do not require loading corrections.

Function-generator Circuits Based on Linear Potentiometers.[1] Figure 6.44 shows how one may generate or approximate a wide variety of functions by loading linear potentiometers, by returning the potentiometer load to various points in the circuit, and by other special connections. A number of especially useful circuits, some of which involve operational amplifiers, are tabulated in Table 5. Such methods of function generation are particularly applicable to special-purpose computers.

6.10. TAPPED POTENTIOMETERS AS FUNCTION GENERATORS[2,3]

A useful and flexible type of function potentiometer is easily constructed as follows. A linear potentiometer is tapped 7 to 99 times. A set of *adjustment networks* (padding networks) permit the voltages at each tap to be adjusted; the tap voltages are set to approximate values of a desired function corresponding to the respective angular displacements. The potentiometer will then approximate linear interpolation between successive tap voltages (Fig. 6.45). Figure 6.46 shows a number of practical circuits. The adjustment networks contain small variable potentiometers or rheostats, so that the function to be generated may be changed at will. If no such changes are desired, fixed resistors may be substituted for the variable ones. In either case, networks corresponding to different functions may be mounted on interchangeable plug-in turrets or panels (Figs. 6.35 and 6.47*d*). The design of the adjustment networks may be limited by the current rating of the tapped potentiometer and/or of the source (operational amplifier) which drives the potentiometer.

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[1] Levenstein, H., Generating Non-linear Functions with Linear Potentiometers, *Tele-Tech*, October, 1953; Bolie, V. W., Synthesis of Function Generators by Continued-Fraction Theory, *Tele-Tech*, March 1956. See also references to Table 5.

[2] Korn, G. A., The General Difference Analyzer, Curtiss-Wright *Report P* 537-V-46, 1948.

[3] Korn, G. A., Design and Construction of Universal Function Generating Potentiometers, *Rev. Sci. Instr.*, **21**: 77–81, 1950. The authors are grateful to P. G. Redgemont of the British Admiralty Signal and Radar Establishment for pointing out to them that tapped function potentiometers were used in England as early as 1933.

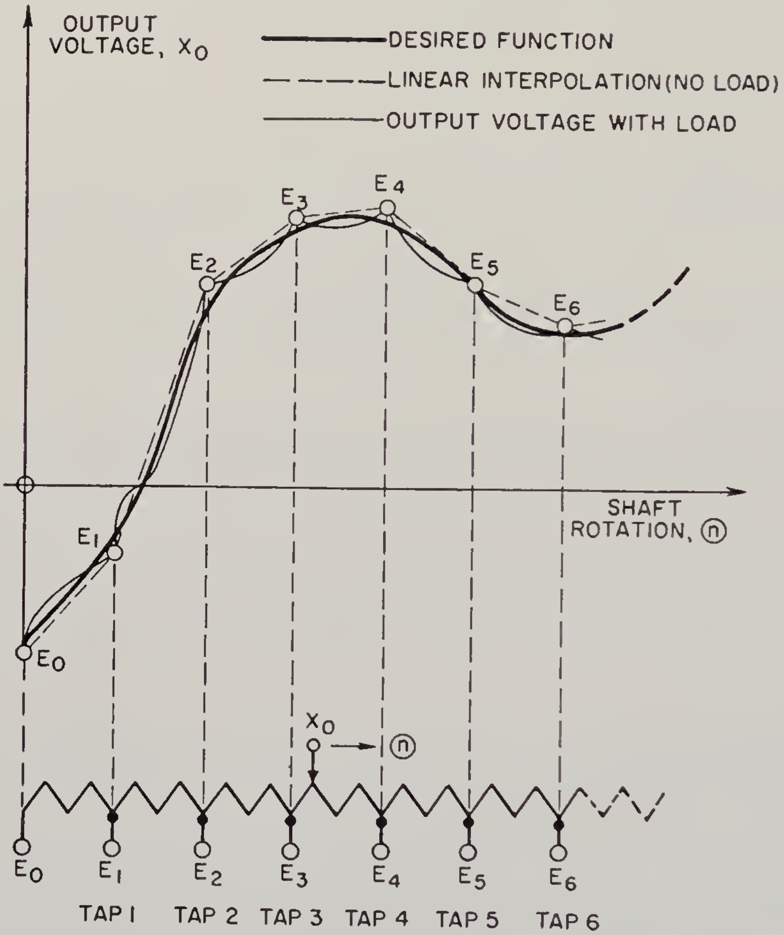

FIG. 6.45. Approximation of a desired function by means of a tapped potentiometer. For best results, the tap voltages may have to be set slightly above or below the function values corresponding to the tap locations. The effects of loading have been exaggerated in the diagram to show the method of curve fitting used for best accuracy. One can obtain even better interpolation by averaging the voltages from two or three close-spaced potentiometer wipers.

The following advantages of tapped function potentiometers should be noted:

1. Tapped function potentiometers possess a high degree of flexibility. Adjustment networks are easily set up for a wide variety of functions (including empirical functions) and may be stored and reused.

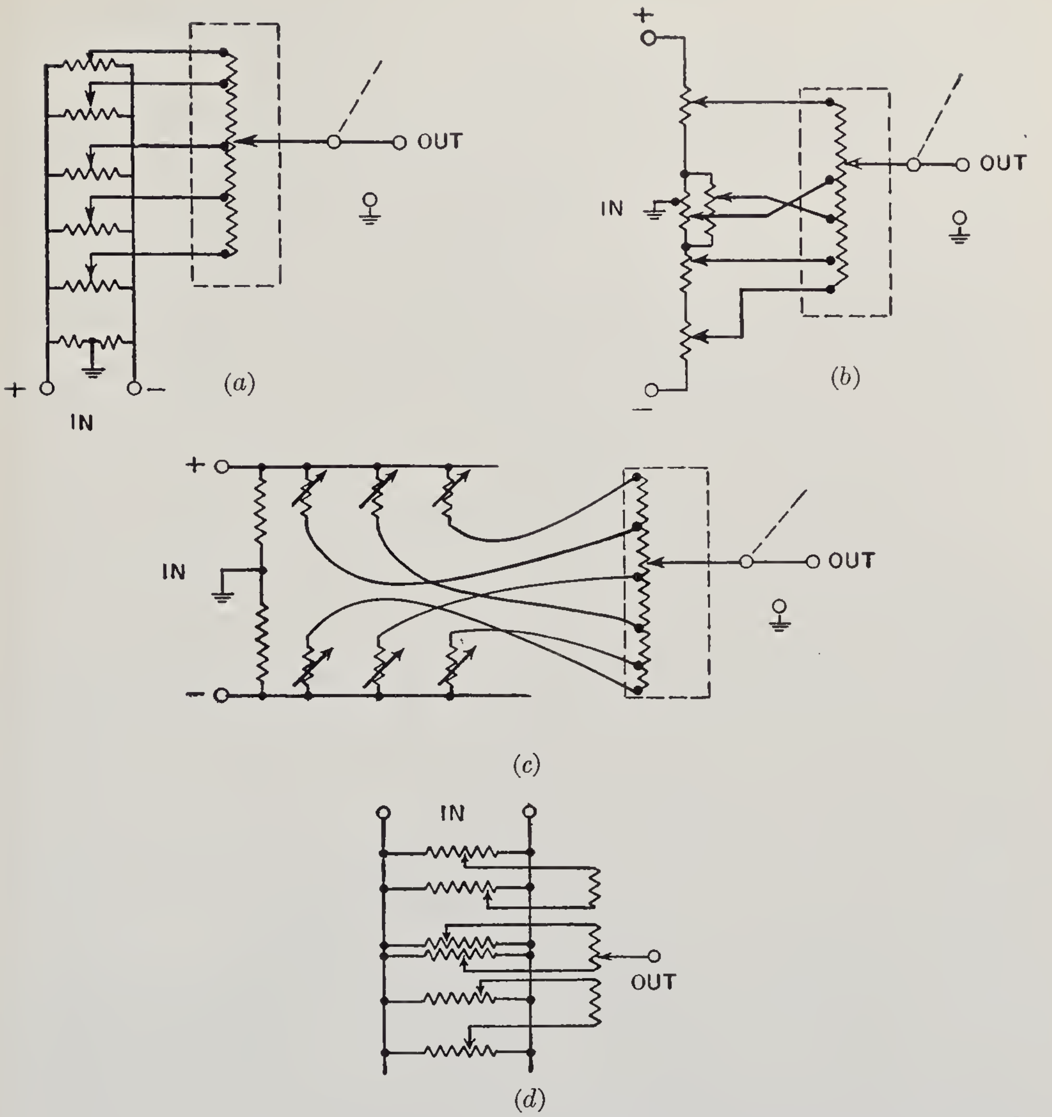

Fig. 6.46. Tapped-potentiometer function generators and adjustment networks. Typical resistance values for the adjustment potentiometers or rheostats in (*a*) and (*c*) are between $5r$ and $500r$, where r is the total resistance of the tapped potentiometer (10,000 to 100,000 ohms). The voltage rise between adjacent taps must not exceed the power rating of the potentiometer winding; alternate taps may be protected by fuses. (*d*) shows a split tapped potentiometer which permits the generation of functions containing step-function terms.

2. Potentiometers of relatively low cost can be used, since accuracy does not depend critically on potentiometer linearity or tap locations.
3. Potentiometer loading (Sec. 6.9) will not introduce errors in the tap voltages, since they are adjusted with the load connected.

The accuracy with which a given function can be approximated by means of a tapped potentiometer will depend on the slope and curvature of the function itself as well as on the number of taps and on the potentiometer load. A wide variety of useful functions can be approximated to within 0.5 per cent if ten or more taps are available.

Construction of Tapped Potentiometers. Most types of commercial wire-wound potentiometers can be ordered with taps suitable for function generation; the tap leads are usually welded to a single turn of the potentiometer wire. Figure 6.47 indicates several methods for tapping existing wire-wound potentiometers in the laboratory. Figure 6.47*c* shows an experimental 40-tap potentiometer constructed in the manner of Fig. 6.47*b*. Figures 6.35*b* and 6.47*d* illustrate the construction of adjustment networks.

Calibration of Adjustment Networks. To "set up" a given function on a tapped potentiometer, one begins by plotting the desired graph of output voltage vs. servo input voltage. The input voltages corresponding to the tap locations must be marked on the graph; servo input voltages corresponding to equally spaced taps will *not* be equally spaced if the servo follow-up potentiometer is loaded (Sec. 6.9). The desired tap voltages may then be determined to fit the given function in the manner of Fig. 6.45.

Since the tap voltages are to be "set" with the load connected, the only effect of potentiometer loading will be to reduce the absolute value of the output voltage *between taps* below that corresponding to ideal linear interpolation. In most practical applications this effect is small; the analysis of page 329 permits one to limit the loading effect by a suitable choice of resistance values. Figure 6.45 shows how possible errors due to the "droop" of the output function between taps can be minimized by proper curve fitting.

Once the desired tap voltages are known, suitable resistance values for the adjustment networks can be computed from Kirchhoff's laws; each tap must have the correct voltage with the load connected. Since the computation tends to become quite cumbersome, it may be preferable to determine the desired resistance values empirically. The potentiometer output voltage is measured, with the load connected, by means of a comparison potentiometer (Fig. 4.6), and the padding resistances are adjusted successively until the correct output function is obtained. In general, the various adjustments will interact, so that several complete cycles of adjustment are necessary.

The more elaborate calibration scheme of Fig. 6.48 avoids most of the interaction between tap adjustments. The adjustment networks (not shown) associated with the 1st, 2d, . . . tap are adjusted *in numerical order*. In the diagram, tap 1 has already been adjusted. Switch section S_1 permits one to apply equally spaced positioning voltages[1] to the

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[1] If the servo follow-up potentiometer is loaded (Sec. 6.9), one may position the potentiometer wiper exactly on the equally spaced taps by changing the positioning network or by loading the arm of S_1 with a suitable resistance.

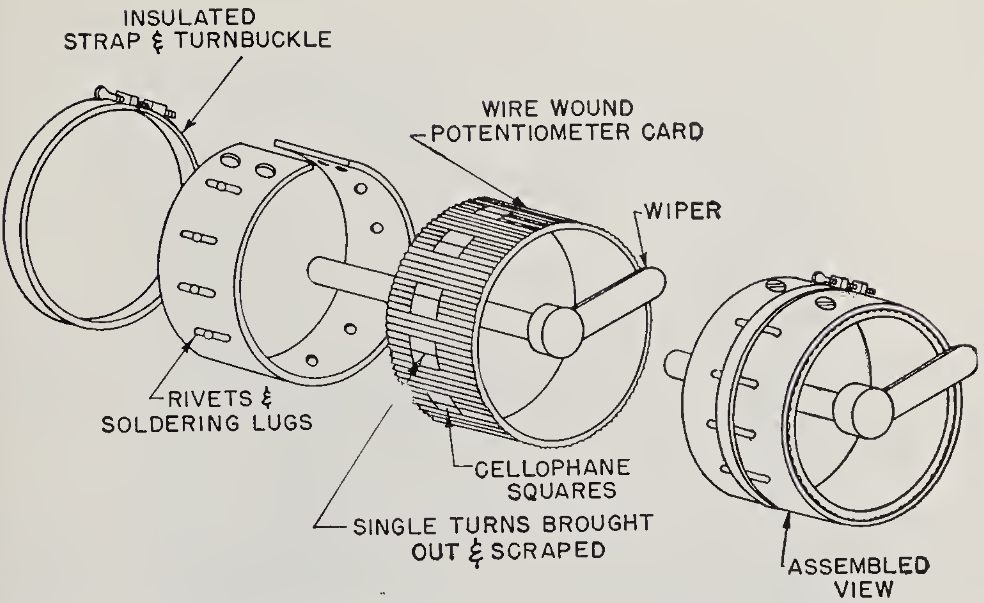
INSULATED STRAP & TURNBUCKLE
WIRE WOUND POTENTIOMETER CARD
WIPER
RIVETS & SOLDERING LUGS
CELLOPHANE SQUARES
SINGLE TURNS BROUGHT OUT & SCRAPED
ASSEMBLED VIEW

Fig. 6.47*a*. Exploded view of a wire-wound potentiometer modified by tapping Single turns of wire are isolated by cellophane strips and are contacted by plated rivets attached to the potentiometer shell. In another similar arrangement[1] soft solder forms a bond between each scraped potentiometer wire and a piece of copper foil soldered to the connecting lead.

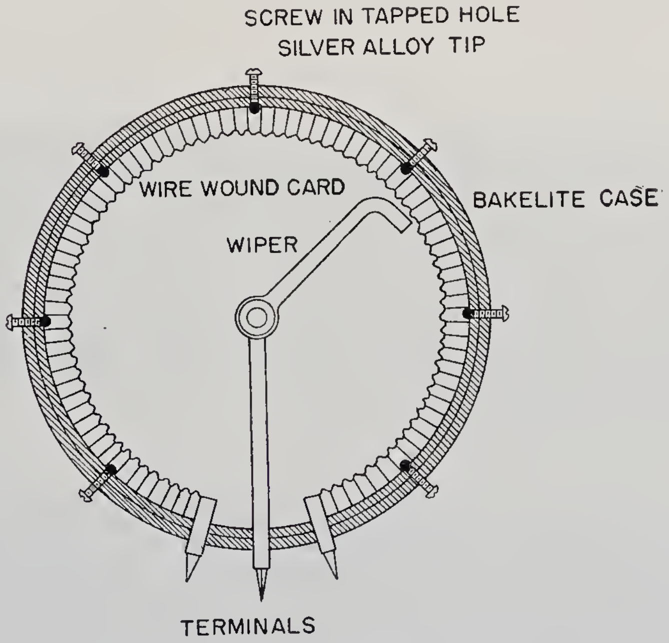

Fig. 6.47*b*. A method for tapping wire-wound potentiometers with thick plastic shells. The potentiometer card must be removed before the holes are drilled. The location of a helical card may be found by placing the potentiometer shell in a lathe and cutting a light thread of the proper pitch.

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[1] Private communication from R. J. Sullivan, Chief, Potentiometer Research Unit, Air Force Cambridge Laboratories, who was kind enough to point this out to the authors.

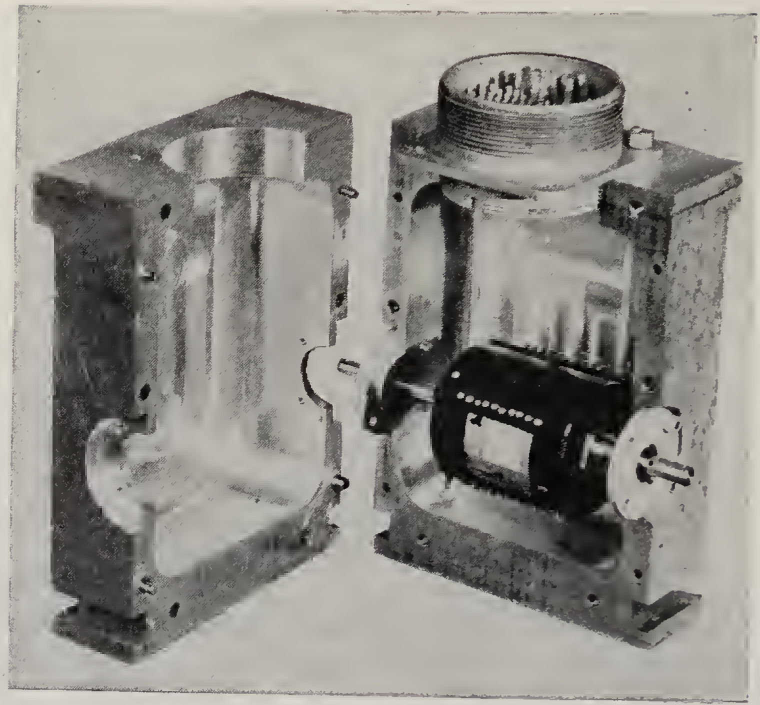

Fig. 6.47*c*. Photograph of an experimental 40-tap helical-type potentiometer (Boeing Airplane Company).

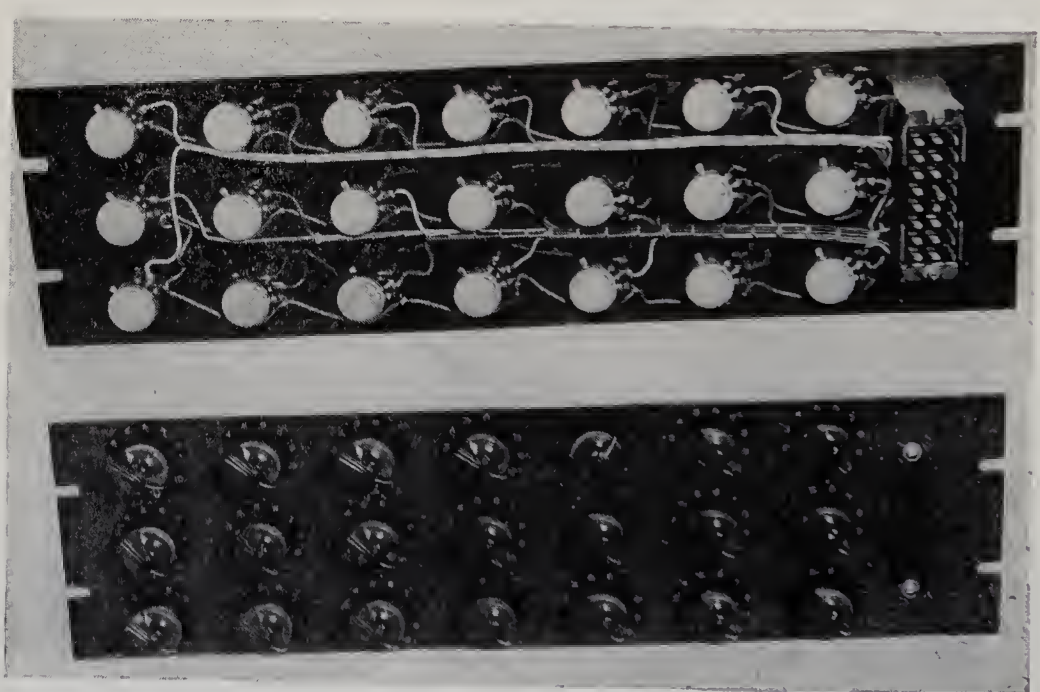

Fig. 6.47*d*. Front and back views of a plug-in panel bearing adjustment potentiometers for a 20-tap function potentiometer. Note the shaft locks (Curtiss-Wright Corporation).

servomechanism driving the tapped potentiometer. To adjust tap 2, the potentiometer wiper is positioned near tap 2 to approximate correct loading. While tap 2 is being adjusted, the d-c amplifiers shown temporarily approximate the correct voltage at tap 3. The selector switch is next advanced to adjust taps 3, 4, . . . ; it is seen that each tap is adjusted *with the load connected and with the two adjacent taps near their final correct voltages.* A single set of comparison potentiometers, two amplifiers, and a balance meter serve all tapped potentiometers of a computer installation by means of plug-in connections. Individual positioning networks and selector switches are packaged together with the adjustment networks for each tapped potentiometer.

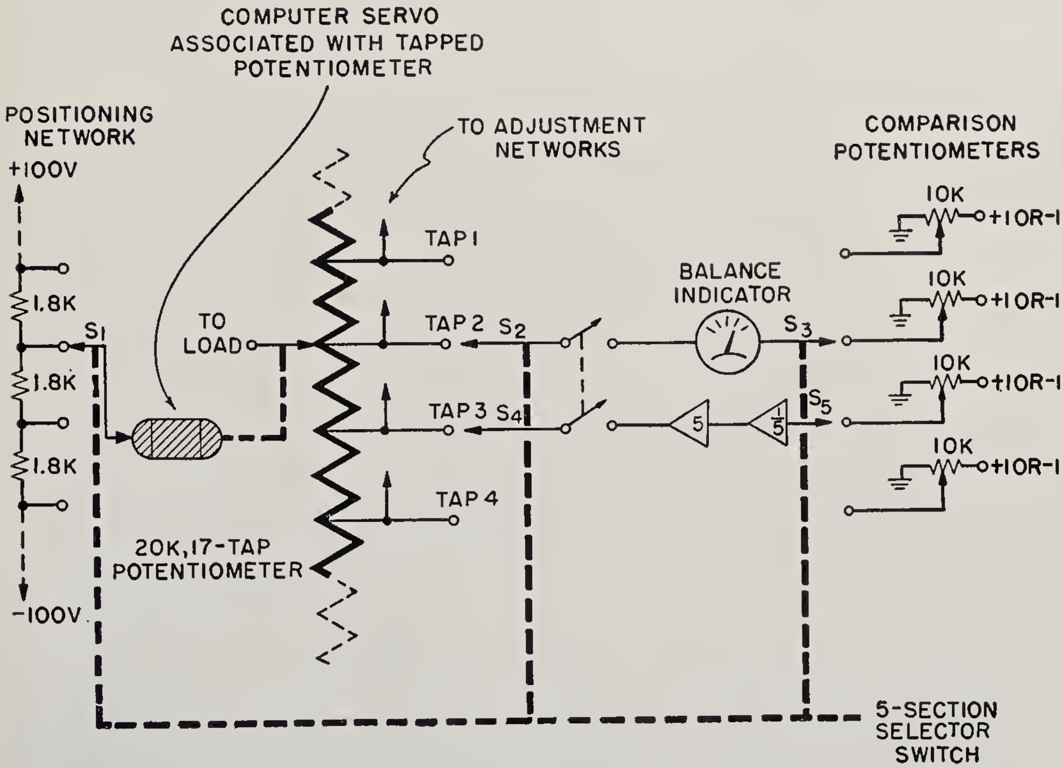

Fig. 6.48. Calibration of tapped-potentiometer function generators in the PACE computer (Electronic Associates, Inc.). The adjustment networks (not shown) are similar to those of Fig. 6.46*c*. The balance meter is protected by limiting rectifiers. The 4-megohm input impedance of the d-c amplifier shown does not load the comparison potentiometer appreciably.

The Effect of a Resistive Load Connected to a Tapped-potentiometer Function Generator.[1] The proper design of tapped-potentiometer function generators and their adjustment networks involves the determination of the minimum load resistance which will still yield tolerably linear interpolation. For the analysis of this basic design problem, use is made of the fact that Thévenin's theorem[1] permits the representation of the tapped potentiometer and its associated networks by the *equivalent circuit* indicated in Fig. 6.49.

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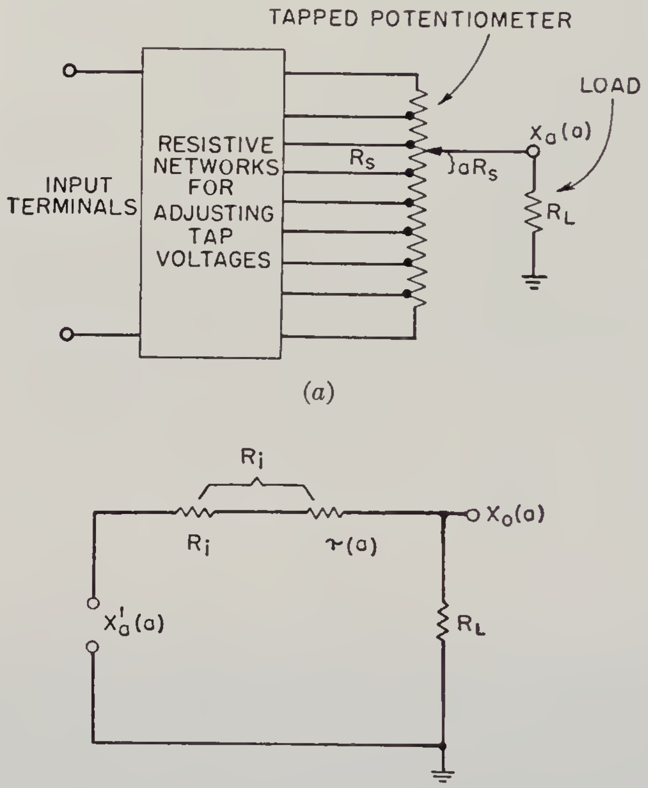
[1] Korn, G. A., Design and Construction of Universal Function Generating Potentiometers, *Rev. Sci. Instr.*, **21**: 77, 1950. Equation (6.35) corrects an error in the reference.



(a)

(b)

FIG. 6.49. A tapped-potentiometer function generator with adjustment networks (*a*), and an equivalent circuit (*b*).

Let X_o' be the open-circuit voltage measured between the potentiometer output terminal and ground if the load resistance R_L is removed. According to Thévenin's theorem, the potentiometer circuit is then equivalent to a voltage source of voltage X_o' connected in series with the resistance R_i' seen by the load if all real voltage sources are replaced by short circuits.

Consider now the variation of the internal resistance R_i' and the open-circuit voltage X_o' as the sliding contact is moved through any one segment of the tapped winding. Let a denote the fraction of the segment in question traversed by the wiper, with $a = 0$ at that tap for which the value of

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[1] Smythe, W. R., *Static and Dynamic Electricity*, p. 129, McGraw-Hill, New York, 1939.

R_i' is lower. It is possible, for each winding segment, to consider R_i' as the sum of a fixed term R_i and a term $r(a)$ variable with the shaft rotation measured by a. Thus,

$$R_i' = R_i + r(a) \qquad \text{with } r(a) \geq 0 \qquad r(0) = 0 \qquad (0 \leq a \leq 1) \tag{6.31}$$

If the tapped potentiometer is a linear one, it is, moreover, clear that the open-circuit voltage X_o' must be a linear function of a:

$$X_o'(a) = (1 - a)X_o'(0) + aX_o'(1) \tag{6.32}$$

With reference to Fig. 6.49*b*, Ohm's law yields the output voltage X_o of the function potentiometer with the load connected, *viz.*,

$$\begin{aligned} X_o &= X_o' \frac{R_L}{R_i' + R_L} = X_o'(a) \frac{R_L}{R_i + R_L + r(a)} \\ &= X_o'(a) \frac{R_L}{R_i + R_L} \left[1 - \frac{r(a)}{R_i + R_L + r(a)} \right] \end{aligned} \tag{6.33}$$

Since X_o' varies linearly with a, it is seen that, for reasonably linear interpolation, r will have to be made small compared with $R_i + R_L$. This will now be discussed on a quantitative basis.

Let $X_{o(\text{lin})}$ be the result of a correct linear interpolation between the two tap voltages $X_o(0)$ and $X_o(1)$:

$$X_{o(\text{lin})} = (1 - a)X_o(0) + aX_o(1) \tag{6.34}$$

The deviation $|X_o - X_{o(\text{lin})}|$ from true linear interpolation between tap voltages may be estimated by the relation

$$\begin{aligned} |X_o - X_{o(\text{lin})}| &= \frac{R_L}{R_L + R_i} \left| aX_o'(1) \frac{r(1)}{R_L + R_i + r(1)} - X_o'(a) \frac{r(a)}{R_L + R_i + r(a)} \right| \\ &\leq \left| \frac{2R_L R_s}{(R_L + R_i)^2} X'_{\text{max}} \right| \leq \left| \frac{2R_s}{R_L} X'_{\text{max}} \right| \end{aligned} \tag{6.35}$$

where X'_{max} is the larger of the two quantities $|X_o'(0)|$ and $|X_o'(1)|$, and where R_s is the resistance of a winding segment between taps.

6.11. REPRESENTATION OF VECTORS IN DIFFERENT COORDINATE SYSTEMS

Many applications require the description of the same vector quantities in terms of two or more different coordinate systems. Practically important examples are furnished by descriptions of the forces acting on an aircraft. These forces (drag, thrust, lift, gravity, etc.) may be described in terms of coordinate axes fixed with respect to the earth, the aircraft, or the relative wind (see also Chap. 3). A study of the transformations of the components of *two-dimensional* vector quantities will illustrate all the necessary analog-computer techniques used for such transformations.

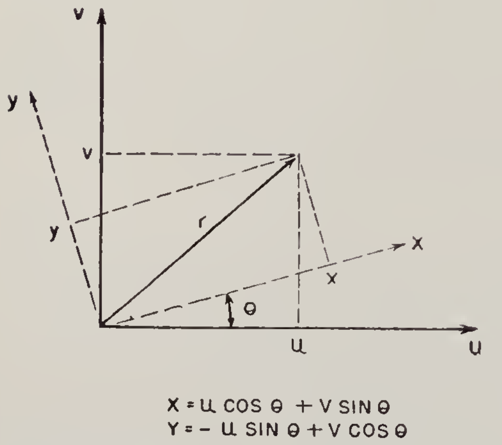

FIG. 6.50*a*. Rotation of rectangular coordinate system.

Representation of a Vector in Different Coordinate Systems: Rotation of Coordinate Axes. Figure 6.50*a* shows a vector which may be described by its components x,y along the x,y coordinate axes as well as by the components u,v along the u,v coordinate axes. Both the x,y and u,v reference frames are rectangular cartesian coordinate systems having a common origin; the angle θ between the x axis and the u axis is defined as shown. Under these conditions, the vector components x,y are related to the components u,v by the *transformation equations*

$$\left.\begin{aligned} x &= u\cos\theta + v\sin\theta \\ y &= -u\sin\theta + v\cos\theta \end{aligned}\right\} \qquad (6.36)$$

If machine variables X, Y, U, V corresponding to the vector components x, y, u, v, respectively, are chosen with equal scale factors a so that

$$X = ax \qquad Y = ay \qquad U = au \qquad V = av \qquad (6.37)$$

the *machine equations* corresponding to (6.36) are simply

$$\left.\begin{aligned} X &= U\cos\theta + V\sin\theta \\ Y &= -U\sin\theta + V\cos\theta \end{aligned}\right\} \qquad (6.38)$$

If the angle θ can be represented by a servo shaft displacement equal to θ, the relations (6.38) can be established in a d-c analog computer in the manner shown in Fig. 6.50*b*. The multiplications by trigonometric functions are performed by means of function potentiometers on the θ shaft, and summing amplifiers are used for addition.

The resistance windings, or "cards," of many commercially available sine and cosine potentiometers usually cover only 180 deg of shaft rotation. Figures 6.51*a* and *b* show how such sine or cosine potentiometers may be connected up to multiply a voltage by trigonometric functions of angles in the first and second quadrants (from 0 to 180 deg). By connecting such potentiometers into the computer circuit in the manner shown in Figs. 6.51*c* and *d*, input voltages can be multiplied by sine and cosine functions of angles between minus and plus 90 deg. If. as is frequently the case, the range of the angle θ exceeds 180 deg, two sine potentiometers in the circuit of Fig. 6.51*e* may be used. Both potentiometer wipers can rotate continuously. They are electrically connected and mechanically ganged in such a manner that one of the potentiometers is used for values of the angle θ in the fourth and first quadrants and the other one for values of θ in the second and third quadrants.

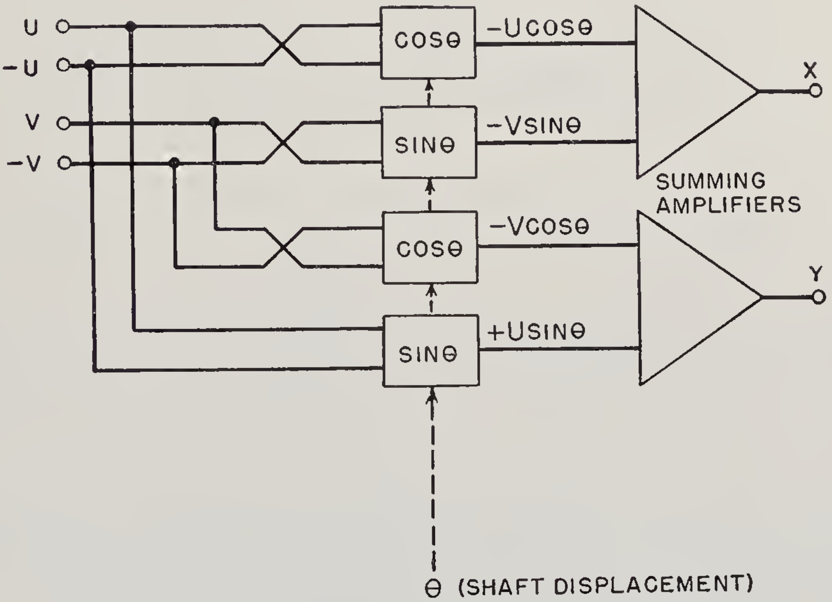

FIG. 6.50*b*. D-c analog-computer setup for Eq. (6.38).

Figure 6.51*f* illustrates the construction of a commercial sine-cosine potentiometer designed especially for coordinate transformations. The potentiometer card shown is wound into cylindrical shape to permit continuous wiper rotation. Several such cylindrical cards can be mounted concentrically. The entire potentiometer assembly is immersed in oil to decrease wear and noise.

The rectangular-card sine-cosine potentiometer[1] shown schematically in Fig. 6.51*g* not only permits continuous rotation but also yields two output voltages; one output voltage is proportional to the sine and the other to the cosine of the potentiometer-shaft displacement. The rectangular-card potentiometer consists of a closely wound rectangular resistance winding or card rotated by the input shaft and four contacts or wipers attached to the fixed potentiometer case. As the card is rotated, each contact effectively travels over the card in a circle about the axis of rotation.

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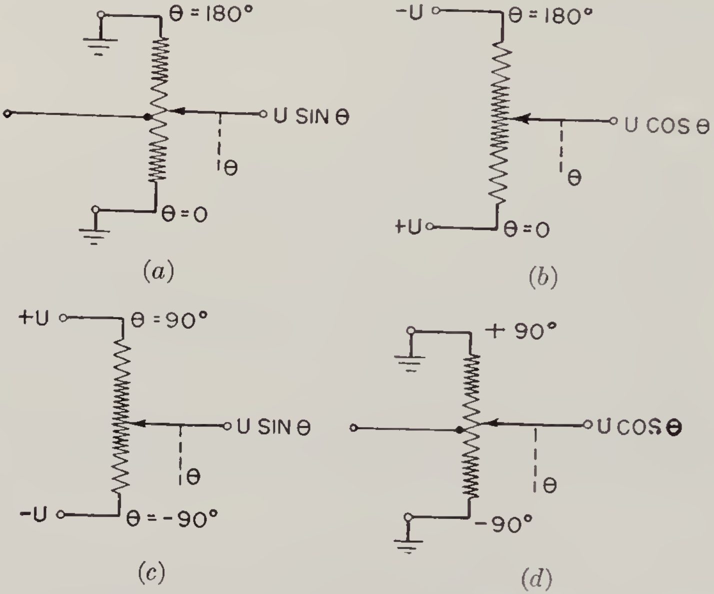
[1] The rectangular-card potentiometer was designed at the MIT Radiation Laboratory, Cambridge, Mass., and is now commercially available from the Gamewell Company, Newton Upper Falls, Mass. A detailed description of this device will be found in Greenwood, Holdam, and MacRae, *op. cit.*, pp. 111–116.

The voltage at each contact will be proportional to the sum of the voltage at the center of the card and a sinusoidal function of the angular displacement of the card. The two voltage differences between the pairs of diametrically opposite contacts shown in Fig. 6.51g will be proportional to the sine and cosine of the angle of rotation, since the center voltage will be canceled out on taking the difference. Each voltage difference may be conveniently obtained by means of a differential amplifier (see Sec. 5.3). As long as a differential output circuit of this kind is used, the axis of rotation need not even pass through the exact geometrical center of the contacts,[1] since any off-center errors will be canceled out. The use of



(a) (b) (c) (d)

FIG. 6.51a, b, c, d. Connections of sine and cosine potentiometers for operation between 0 and 180 deg (a, b) and between minus 90 and plus 90 deg (c, d). Note the two types of tapered windings.

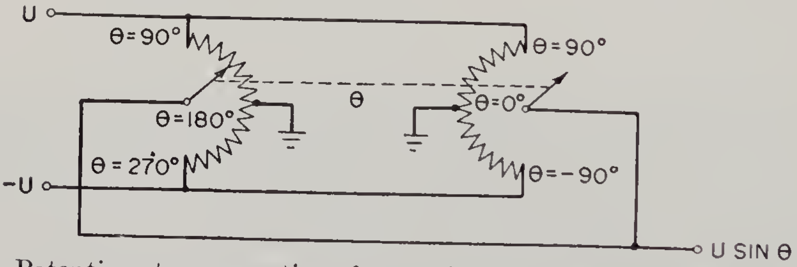


FIG. 6.51e. Potentiometer connections for continuous rotation. Each potentiometer card is used for alternate 180 deg of angular displacement. Integral multiples of 360 deg may be added to each angular displacement indicated in the diagram.

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[1] For a detailed discussion of the problems involved in manufacturing rectangular-card potentiometers, see P. Rosenberg, Sinusoidal Potentiometers Types RL10CB RL10CD, RL10E, RL14, *RL Report* 423, Aug. 16, 1943.

differential output circuits of some type[1] with rectangular-card potentiometers is recommended in all computer applications.

Each rectangular-card potentiometer can replace between two and four ordinary tapered potentiometers; the functions are generated with an accuracy of the order of 0.6 per cent of full scale. Such potentiometers are, however, not too easy to construct because of the close mechanical tolerances required, and their construction makes it difficult to gang several rectangular-card potentiometers on a common shaft.

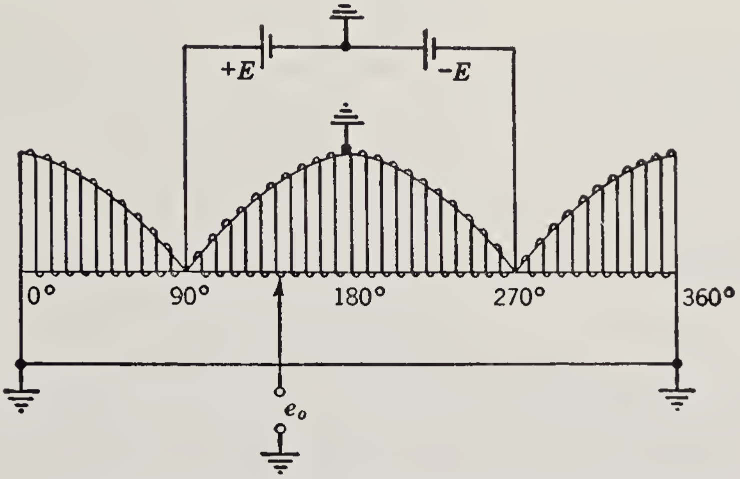

Fig. 6.51*f*. Sine-cosine potentiometer designed for continuous rotation (from W. Soroka, *op. cit.*). Commercial potentiometers of this type may be preloaded (Sec. 6.9) and permit accuracies of 0.15 to 0.3 per cent of full scale.

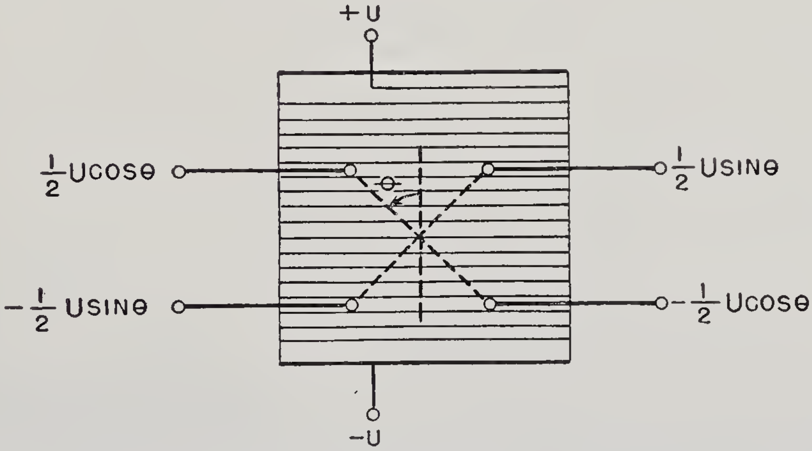

Fig. 6.51*g*. Rectangular card sine-cosine potentiometer. The axis of rotation is perpendicular to the plane of the diagram.

Another method for obtaining the transformations (6.38) with an analog computer involves the use of a-c computing elements. Two a-c voltages of equal frequency and phase may be modulated[1] with d-c voltages U and V, respectively. If these a-c voltages are impressed on the two stator windings of a special rotatable transformer called a *resolver* (see Fig. 6.52), whose rotor displacement is equal to θ, the rotor output a-c voltages will be modulated by the voltages X and Y given by the transformation equations (6.38). D-c voltages proportional to X and Y may then be obtained through the use of phase-sensitive demodulators.[2] Accuracies better than 0.5 per cent can be obtained with such arrangements, but the modulators and demodulators as well as the resolvers have to be designed with great care to obtain such accuracy. Resolvers will be less susceptible to mechanical wear than potentiometers, but the modulators, demodulators, and resolver driver amplifiers required with computer circuits using resolvers will add to the cost and complexity of the equipment. Such circuits are, therefore, most useful if one modulator can be used for several multiplications, or if the a-c output can be used directly for driving an a-c servomechanism.

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[1] A circuit somewhat different from that of Fig. 6.51*g* is shown in Greenwood, Holdam, and MacRae, *op. cit.*

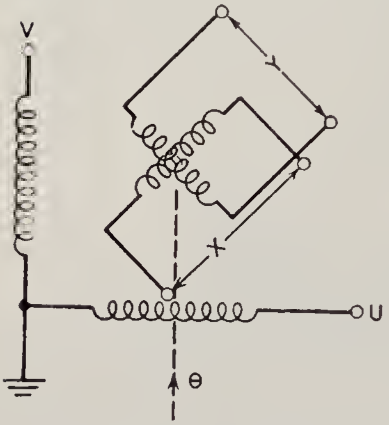

FIG. 6.52. Use of a resolver to set up Eqs. (6.38). The values indicated are amplitudes of a-c voltages.

If the computer in question works on a time scale so fast that computing servomechanisms cannot be used, the angle θ will have to be represented by a voltage in the machine, and a more complicated combination of electronic function generators and multipliers will be required. The trigonometric functions might also be generated by implicit computation.

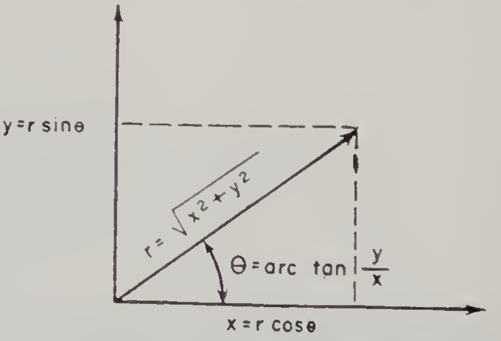

FIG. 6.53. Transformation from polar to rectangular cartesian coordinates.

Transformation from Polar to Rectangular Cartesian Coordinates. If the polar coordinates r and θ of a two-dimensional vector are given, it is readily seen from Fig. 6.53 that the transformation equations to rectangular coordinates are

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad (6.39)$$

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[1] Chance *et al.*, *op. cit.*, Chaps. 12 and 14.

[2] *Ibid.*

Assuming that equal scale factors are used to introduce the machine variables X, Y, and R corresponding to x, y, and r, respectively, it is possible to obtain a set of corresponding machine equations

$$\left.\begin{aligned} X &= R\cos\theta \\ Y &= R\sin\theta \end{aligned}\right\} \tag{6.40}$$

If a servo shaft displacement equal to the angle θ can be made available in the computer, the multiplications indicated in the transformations (6.40) may be performed by means of function potentiometers as shown in Fig. 6.54*a* or by means of a resolver as shown in Fig. 6.54*b*. In "fast" computers, electronic multipliers and function generators will again be required to establish the transformation.

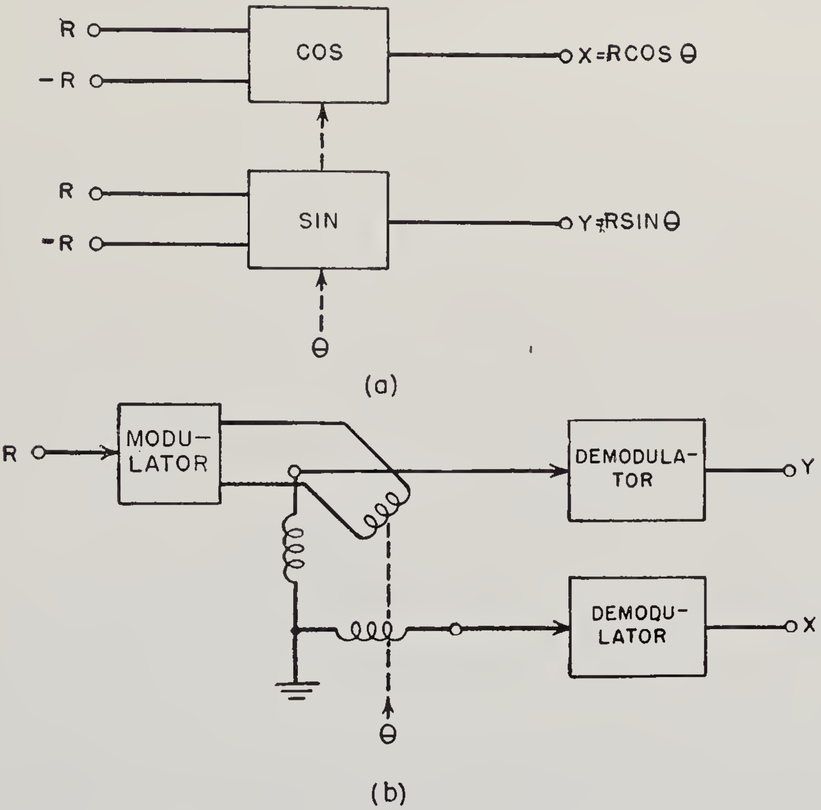


FIG. 6.54. Transformation from polar to rectangular coordinates by means of function potentiometers and resolvers.

Rectangular-to-polar Transformations. The equations of transformation from rectangular to polar coordinates are obtained by solving Eqs. (6.39) for r and θ. One finds

$$\left.\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\frac{y}{x} \end{aligned}\right\} \tag{6.41}$$

If equal scale factors are used for r, x, and y, the corresponding machine equations are

$$\left.\begin{aligned} R &= \sqrt{X^2 + Y^2} \\ \theta &= \arc \tan \frac{Y}{X} \end{aligned}\right\} \tag{6.42}$$

Whereas these relations could be established directly on an analog computer, it may be more convenient to obtain R and θ by *implicit solution* of the equations

$$\left.\begin{aligned} -X \sin \theta + Y \cos \theta &= 0 \\ X \cos \theta + Y \sin \theta &= R \end{aligned}\right\} \tag{6.43}$$

which are equivalent to the transformations (6.42).

The first equation (6.43) does not contain R and may be solved implicitly for θ. Assuming that θ may again be represented by a servo shaft displacement equal to θ, the implicit solution of

$$-X \sin \theta + Y \cos \theta = 0 \tag{6.44}$$

may be accomplished by the *vector servomechanism* or *inverse resolver* shown at the top of Fig. 6.55*a*. Suppose that initially the servo shaft displacement is not equal to the correct value θ, so that Eq. (6.44) does not hold but

$$-X \sin \theta + Y \cos \theta = \epsilon$$

In the circuit of Fig. 6.55*a*, *a voltage equal to the "error" ϵ is computed and used to drive a servomechanism which will position the θ shaft so as to minimize the error ϵ.* The shaft displacement will, thus, be forced to assume the correct value θ. Function potentiometers on the θ shaft are then used to compute the voltage R according to the second equation (6.43). Potentiometers of the rectangular-card type will be found useful in this circuit.

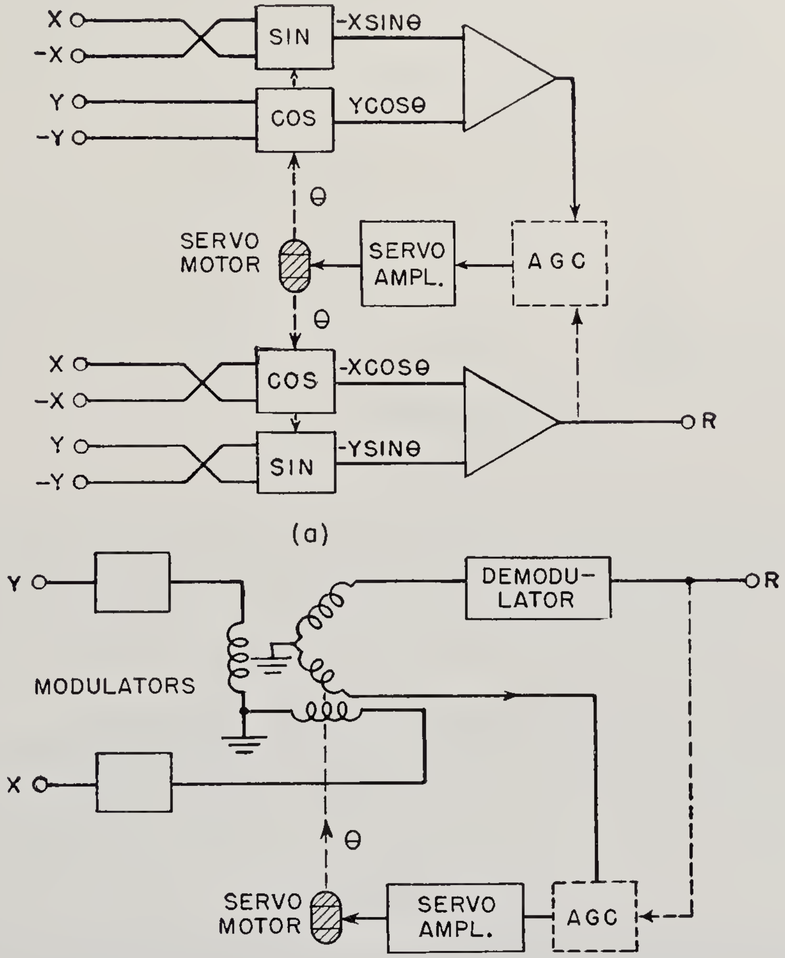
Figure 6.55*b* shows a similar arrangement in which the function potentiometers have been replaced by a single resolver driven by an a-c servo.

It may be seen that the loop gain of each vector servo shown in Fig. 6.55 is not constant but depends on the values of X and Y; in fact, the loop gain is proportional to the value of

$$R = \sqrt{X^2 + Y^2}$$

Since only servomechanisms with constant loop gain can be designed for optimal performance, it is expedient to include *automatic-gain-control (AGC) circuits* (shown in broken lines in Fig. 6.55) in the servo loop. Figure 6.56 shows a practical AGC circuit which may be used to decrease the servo loop gain as R increases. Such a device will tend to keep the loop gain constant over some range of values of R; it is clear that no vector servo can operate if $X = Y = R = 0$.

The circuit of a vector servo without AGC is shown in Vol. 21 of the MIT Radiation Laboratory Series.[1] The circuit of a vector servo with single-ended inputs is also shown in that volume.[2] Accuracies attainable with vector servos are of the order of ¼ to 1 per cent in amplitude of the radius vector R and 0.5 deg in angle.



(a)

(b)

FIG. 6.55. Block diagrams of vector servomechanisms (inverse resolvers).

The reader is reminded that the innocent-looking coordinate transformations (6.38) and (6.43) may involve the computation of a small difference of two large terms. The resulting possibility of large percentage errors can spell the doom of computations requiring good accuracy, such as trajectory computations (Sec. 3.5). Computations which must involve such transformations should be checked with particular care.

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[1] Greenwood, Holdam, and MacRae, *op. cit.*, p. 479.

[2] *Ibid.*, p. 163.

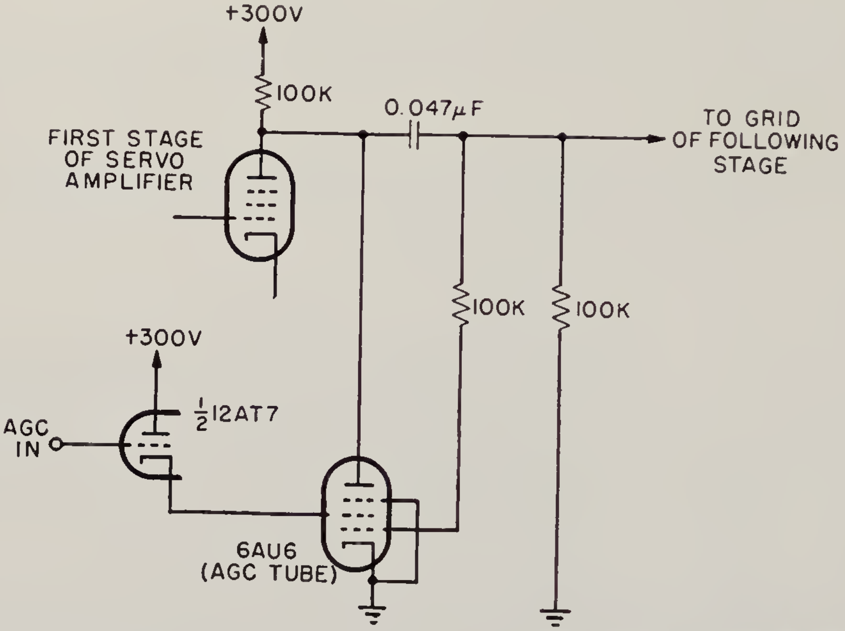

Fig. 6.56. AGC circuit inserted between two single-ended servo-amplifier stages. A (positive) AGC voltage applied to the screen of the AGC tube will increase its transconductance. The resulting increased degeneration reduces the amplifier gain. The circuit is useful for AGC voltages between 1 volt and 100 volts (Electronic Associates, Inc.).

6.12. DIVISION AND GENERATION OF INVERSE FUNCTIONS

Division. Computing elements capable of division must accept two input voltages X_1 and Y, say, and produce an output voltage

$$X_o = (\text{constant}) \frac{X_1}{Y}$$

which will increase in absolute value as the absolute value of the divisor Y is decreased. Hence, in view of the given limit on the magnitudes of the computer voltages, *no division circuit will be able to compute X_1/Y for absolute values of the divisor Y smaller than some minimum value $|Y|_{\min}$.* This proposition must hold no matter what type of computing element is used for division. It is seen that, in general, the quotient X_1/Y cannot be computed correctly while the divisor Y crosses zero or changes sign during the computation.

Division by Reciprocal Multiplication. The division X_1/Y is equivalent to a multiplication $X_1(1/Y)$ by the function $1/Y$ of Y. All types of multipliers and function generators may be used for division in this manner. The product $X_1(1/Y)$ may, in particular, be obtained from a servo-driven function potentiometer (Secs. 6.9 and 6.10); the servomechanism used may perform multiplications by other functions of Y at the same time.

Division by Means of Servomechanisms and Other Feedback Devices. In principle, all multiplying circuits based on the feedback principle of Fig. 6.7 can be used for division: their output voltage X_o is proportional to the reciprocal $1/Y$ of the input voltage Y. In particular, the servomultipliers shown in Fig. 6.8 can be used to divide an input voltage X_1 or X_2 by an input voltage Y applied to the follow-up potentiometer.

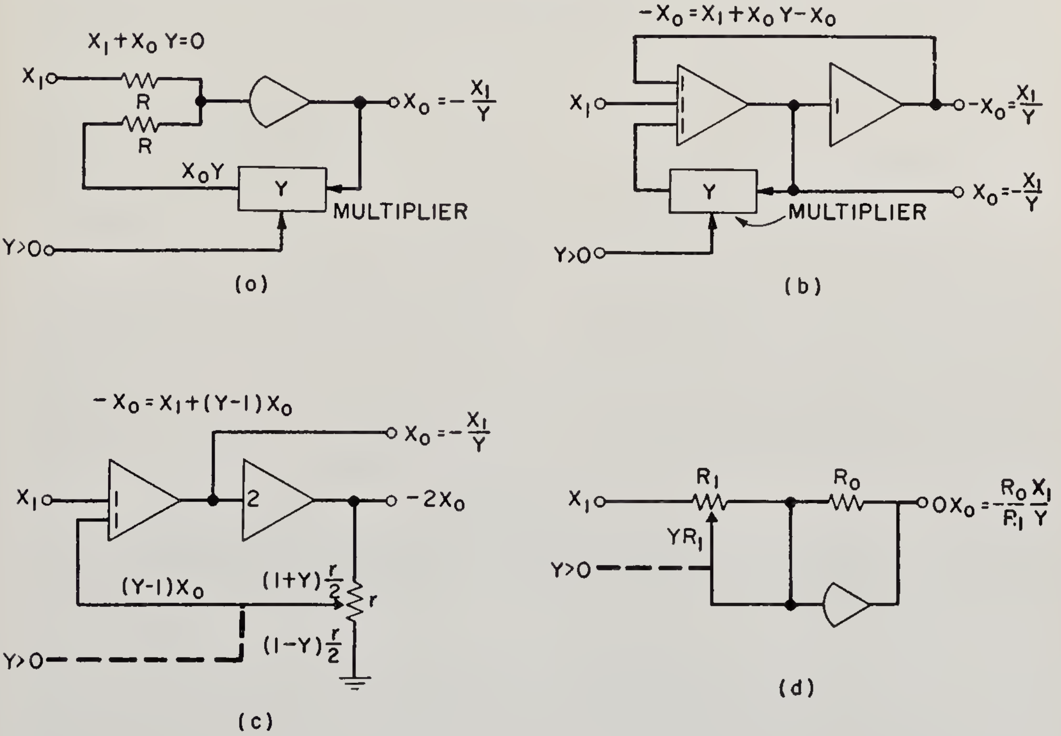

Fig. 6.57. Division circuits (see text).

Division methods of this type are, in general, not recommended because any change in the variable Y in Fig. 6.7 will directly affect the loop gain and thus compromise the stability, accuracy, and frequency response of the feedback circuit. This may be particularly serious in the case of computer servomechanisms, whose frequency response is often marginal even under the best conditions. An automatic-gain-control (AGC) circuit (Fig. 6.56) inserted into the differential amplifier or servo amplifier can eliminate such difficulties; if the divisor Y is applied to the AGC input, the AGC circuit will keep the loop gain reasonably constant over a range of values of Y.

Division by Implicit Computation. Use of Operational Amplifiers. Figures 6.57*a* and *b* illustrate the implicit computation of the quotient $X_o = -X_1/Y$ by continuous (approximate) solution of the machine equation $X_1 + X_oY = 0$. Note that this machine equation must be satisfied by the *initial values* of X_1, X_o, and Y (see also Sec. 2.11). In Fig. 6.57*b*, the high-gain amplifier has been replaced by a regenerative loop comprising an ordinary summing amplifier and a phase inverter.

The computer setups shown in Figs. 6.57*a* and *b* are most frequently used with servomultipliers. Linear servo-driven potentiometers used in these circuits will require loading corrections obtained, *e.g.*, by loading the servo follow-up potentiometer (Sec. 6.9). In the case of the circuit of Fig. 6.57*b*, the method of Fig. 6.40 may apply.

Figures 6.57*c* and *d* show two other computer setups permitting division by means of a linear servo-driven potentiometer. The arrangement of Fig. 6.57*c* is not recommended, since no simple loading correction is possible. The operational-amplifier division circuit of Fig. 6.57*d* permits accurate division and requires no loading correction, but the input impedance is low if Y is small.

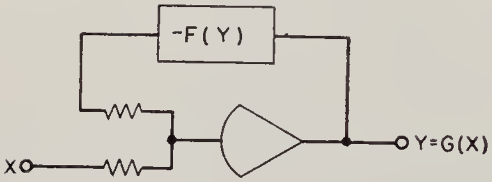

FIG. 6.58. Circuit for generating the inverse function $Y = G(X)$ of a given function $F(Y)$ by continuous approximate solution of the equation $X - F(Y) = 0$. The initial value of Y must equal the correct initial value of $G(X)$. A necessary condition for stability is $\frac{\partial F}{\partial Y} < 0$.

Generation of Inverse Functions (see also Sec. 2.11). The division circuits shown in Figs. 6.57*a* and *b* are special cases of the more general circuit of Fig. 6.58. If a function $F(Y)$ of a machine variable Y can be generated and possesses a suitable inverse function $G(X)$, then one can obtain $G(X)$ through continuous implicit solution of the equation

$$F[G(X)] - X = 0$$

in the manner of Fig. 6.58 if the correct initial value of $G(X)$ is known.

6.13. GENERATION OF FUNCTIONS OF TWO OR MORE VARIABLES

A voltage F, say, which is equal to a given function of not one but of several machine variables X, Y, . . . can usually be obtained or at least approximated by means of judicious combinations of summing and/or multiplying circuits with devices generating functions of a single variable. Thus one can obtain the function

$$F = F(X,Y) = \sin (X + Y)$$

by first adding the voltages X and Y and then generating the sine of their sum. Functions of several variables given by empirical data (tables or curves) can also frequently be approximated by combinations of sums and products of functions of a single variable. Numerical interpolation methods may be used to approximate a given function by a polynomial.[1,2] The switching scheme of the step multiplier (Sec. 6.3) can be generalized to permit generation of functions of two or more variables. In special-purpose computers, a three-dimensional cam[3] may be used to position a potentiometer shaft in accordance with two servo input voltages.

Electrical Interpolation Schemes. A function $F(X,Y)$ of two machine variables X and Y may be generated as a function of the shaft displacement X for several fixed values of Y by a number of function potentiometers. A tapped potentiometer whose shaft displacement is proportional to Y can then be used for *interpolating* approximate values of $F(X,Y)$ as shown in Fig. 6.59*a*. Instead of using separate function potentiometers for different values of Y, one can sometimes use a single cylindrical-type function potentiometer (Fig. 6.38) with separate contact wires corresponding to each function $F(X,Y = \text{constant})$.[4] The tapped function potentiometer may also be replaced by one of the commutating-switch arrangements shown in Fig. 6.43.

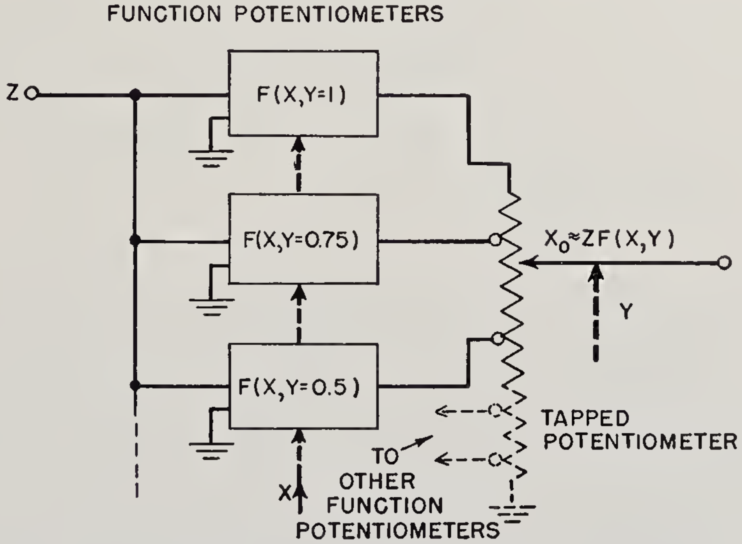

FIG. 6.59*a*. Circuit for multiplying a voltage Z by a function $F(X,Y)$ of two machine variables.

In another interpolation scheme, contour lines of the desired function $F(X,Y)$ are drawn on a sheet of semiconducting paper representing the XY plane. If the correct function values are established on these equipotential lines in the manner of Fig. 6.46, an ordinary servo plotting

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[1] Householder, A. S., *Principles of Numerical Analysis*, McGraw-Hill New York, 1953.

[2] Jerrard, R. P., and G. T. Jacobi, Generation of a Function of Two Variables, *ACM Annual Computer Conference*, Cambridge, Mass., September, 1953.

[3] Meissinger, H. F., Representation of Functions of Several Variables on REAC Equipment, *CYCLONE Symposium No. I*, Reeves Instrument Corporation, New York City, 1951.

[4] Gunning and Mengel, *op. cit.*

board (Sec. 7.3) can be used to position a probe which will interpolate function values between the equipotential lines.

The Variable-density-type Cathode-ray Function Generator.[1] In this function generator, a translucent mask of variable density is placed between the screen of a cathode-ray tube and a photocell (Fig. 6.59*b*). The photocell output voltage will then vary as a function $F(X,Y)$ of the horizontal and vertical deflection voltages X and Y applied to the cathode-ray tube. The accuracy of such function generators is limited to between 1 and 10 per cent by the difficulty of positioning the beam accurately. The frequency response is excellent (up to 100 kc), and the device is suitable for use in repetitive computers. In particular, functions like (constant)XY, constant(X^2Y), etc., may be generated by a variable-density-type cathode-ray function generator which can thus serve as a simple electronic multiplier. Masks having opacity variations over a range of 200 to 1 can be produced by photographic means.

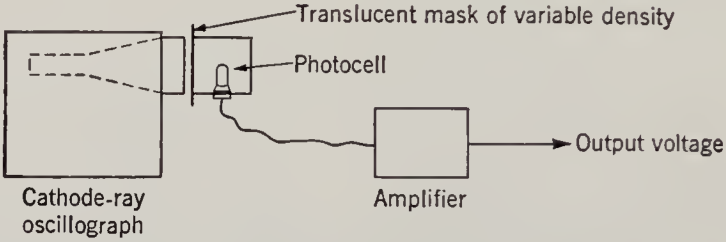

FIG. 6.59*b*. A simple variable-density-type cathode-ray function generator. The two input voltages are the horizontal and vertical deflection voltages of a direct-coupled cathode-ray oscillograph.

A related more sophisticated type of cathode-ray function generator[2] employs a dual-beam cathode-ray tube. While the first beam travels over the function mask, a feedback circuit continuously positions the second beam on a calibrated comparison mask so that the light outputs from the two beams match; the deflection voltage of the comparison beam is the output voltage. A second feedback loop regulates the beam intensities.

Use of Diode Function Generators. The diode function-generator circuits described in Sec. 6.7 can be generalized to generate functions of two or more variables. Thus, if a variable voltage Y replaces the constant reference voltage E in the function-generator circuit of Fig. 6.26*a*, the function-generator output voltage X_o will be a broken-line function of X for each value of Y (Fig. 6.60*a*), *i.e.*, X_o is a function $X_o = F(X,Y)$ of both X and Y.[3]

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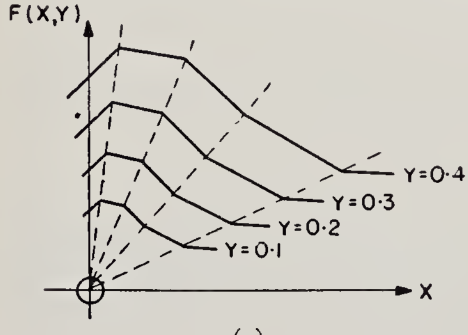
[1] Wallmann, H., An Electronic Integral-transform Computer and the Practical Solution of Integral Equations, *J. Franklin Inst.*, **250**: 45, 1950.

[2] "Variable Density Film Yields Three Functions," *Control Eng.*, May, 1956.

[3] Meissinger, H. F., An Electronic Circuit for the Generation of Functions of Several Variables, *lecture presented at the IRE National Convention*, New York, 1955.

Since the breakpoint voltage of the ith limiter channel will be b_iY, where b_i is a constant, all breakpoints generated by any one limiter channel must lie on a straight line through the origin. These *breakpoint loci* bound segments of equal slope and are indicated in broken lines in Fig. 6.60*a*.

One may generalize the above procedure by adding a function of X or Y to the output voltage X_o. One can obtain an even more general class



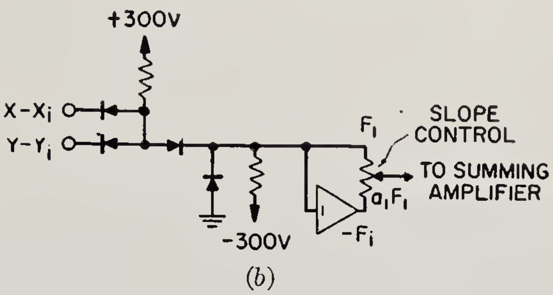
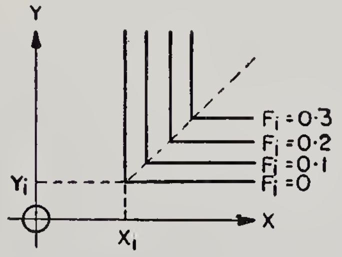


FIG. 6.60*a*. A function $F(X,Y)$ of two machine variables X and Y generated by the diode function generator of Fig. 6.26*a* if the constant reference voltage E is replaced by the variable voltage Y.

FIG. 6.60*b*. Generation of a function $F_i(X,Y) = \max\,[\min(X - X_i,\ Y - Y_i),0]$ by a double selection circuit. A very general class of functions may be approximated by sums $\sum\limits_i a_iF_i$.



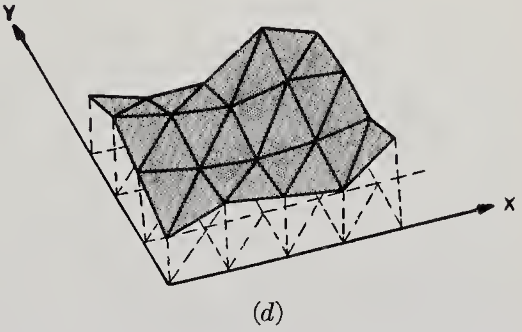


FIG. 6.60*c*. Contour lines of a function $F_i(X,Y) = \max\,[\min(X - X_i,\ Y - Y_i),0]$ generated by the circuit of Fig. 6.60*b*.

FIG. 6.60*d*. Approximation of a function $F(X,Y)$ by a sum of functions F_i of the type shown in Fig. 6.60*c*. Note the triangular facets.

of functions $F(X,Y)$ by introducing suitable functions $b_iG(Y)$ or $G_i(Y)$ of Y as bias voltages in the limiter channels of Fig. 6.26*a*, so that the breakpoint loci are no longer straight lines.

To "set up" a given function $F(X,Y)$, one first draws a set of parametric curves as in Fig. 6.60*a*. One may locate the breakpoint loci roughly after joining points of equal slope on the parametric curves; Meissinger[1] has also developed a graphical method yielding the desired breakpoint voltages for special classes of functions $F(X,Y)$. Once the breakpoints are established, the slopes can be set for a representative curve of the family. The subsequent fine adjustment of the function generator may be a somewhat involved procedure.

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[1] *Ibid.*

Function generators of the type shown in Fig. 6.26*a* attempt to represent suitable functions $F(X,Y)$ of two machine variables X, Y as sums of channel output functions $\max \{[X - G_i(Y)], 0\}$. A somewhat more generally applicable scheme for generating functions of two variables is due to Philbrick,[1] whose elegant circuit also simplifies the setup procedure. Philbrick's more complicated (and, in general, more numerous) selection channels (Fig. 6.60*b*) produce functions

$$F_i(X,Y) = \max [\min (X - X_i, Y - Y_i), 0].$$

$F(X,Y)$ is represented as a sum $\Sigma a_i F_i(X,Y)$ of such functions, which may be regarded as two-dimensional generalizations of the elementary line segments discussed in Sec. 6.7. The contour lines of Fig. 6.60*c* illustrate the nature of a typical channel output $F_i(X,Y)$.

To "set up" a desired function $F(X,Y)$, one first chooses the breakpoints (X_i,Y_i). One may then simply adjust the slope controls to obtain the desired function values $F(X_i,Y_i)$ for successively increasing values of X_i and Y_i. There will be little interaction between the adjustments of individual channels since each channel output voltage F_i is equal to zero for $X < X_i$, $Y < Y_i$ (this is exactly true only for ideal diode characteristics). Figure 6.60*d* shows a three-dimensional graph of a function $F(X,Y)$ generated by a Philbrick-type function generator.

The accuracy of the diode function-generator circuits described depends, again, on the nature of the function generated. Accuracies better than 1 per cent at frequencies in excess of 200 cps have been obtained in representative applications, and these results are capable of improvement. The basic circuits shown permit a vast number of variations.

Functions of Three or More Variables. Functions of three or more variables can be generated as combinations of functions of two variables; thus $F = F(X_1,X_2,X_3)$ may be expressed in the form $F = \Phi[X_1,Y(X_2,X_3)]$. The Philbrick method described above can, in principle, be extended directly to the case of 3, 4, . . . variables through the use of more complicated selection circuits; the number of diodes required will increase dramatically[2] with the number of independent variables.

[1] Memorandum reports by G. A. Philbrick, G. A. Philbrick Researches, Inc., Boston, Mass.: Continuous Electronic Representation of Nonlinear Functions of *n* Variables, dated Nov. 1, 1951; Analog Component for the Embodiment of Nonlinear Functions of Two Variables, dated Apr. 10, 1952; Analog Component for Functions of *n* Variables, dated Apr. 25, 1952.

[2] Philbrick, *ibid.*

CHAPTER 7

AUXILIARY CIRCUITS AND COMPUTER OPERATION

7.1. INTRODUCTION OF INITIAL CONDITIONS. HOLD CONDITION

The necessity of introducing initial conditions of the machine variables into any differential analyzer was pointed out in Secs. 1.2 and 2.7. Differential equations are relations between *changes* of variables, and the initial conditions define the initial values from which these changes of variables must start. The number of initial conditions which must be given with a system of ordinary differential equations is equal to the *order* of the given set of differential equations; each variable contributes the order of its highest-order derivative.

In computer circuits using capacitors to establish relations between changes of machine variables, the number of initial conditions to be introduced will, in general, be equal to the number of capacitors used. The initial conditions for the machine variables will be established if, for instance, each capacitor is given a correct *initial charge* (Fig. 7.1). Since enforcement of the given differential equations by the computing elements would make it impossible to keep the variables set at their initial values, it is necessary to make the computer *inoperative* in the RESET or "initial" condition.

If, as is most frequently the case, parallel-feedback integrators are used to relate changes of machine variables, an initial condition corresponding to each *integrator* must be established. This may be accomplished by two basic methods:

1. Each integrating capacitor may be kept charged initially, so that each integrator output voltage will start to vary from the correct initial value.
2. All integrating capacitors may be discharged before computation; a step-function voltage equal in value to the respective desired initial-condition voltage must then be added to each integrator output voltage at the start of the computation (Fig. 7.2).

The second method requires that the output voltages of all the integrators be forced to a reference level, usually zero, before computation, the initial-condition voltages being supplied as step functions equal in magnitude to the correct initial output voltages. Although this second method is somewhat easier to instrument, it can be used only with computers having recorders and computing elements capable of following a step function with sufficient accuracy. This requirement is satisfied by repetitive computers which are discussed in Sec. 8.5.

Since "slow" d-c analog computers may incorporate servomechanisms and recorders which cannot follow a step function very accurately, it is desirable to set the output voltages of all computing elements to the correct initial values *before computation*. The gain controls and reference levels of the recording devices may then be adjusted using the known initial-condition voltages to calibrate the recorder scales.

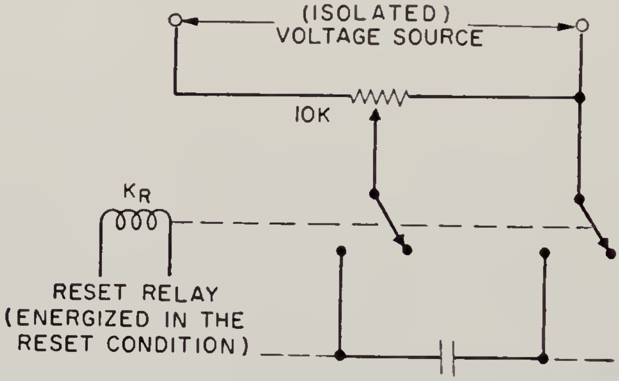

FIG. 7.1. A simple method for establishing an initial charge on a capacitor associated with a computing network or operational amplifier. The voltage source used may be a battery or a small VR-tube-regulated power supply. All reset relays are deenergized at the start of a computer run.

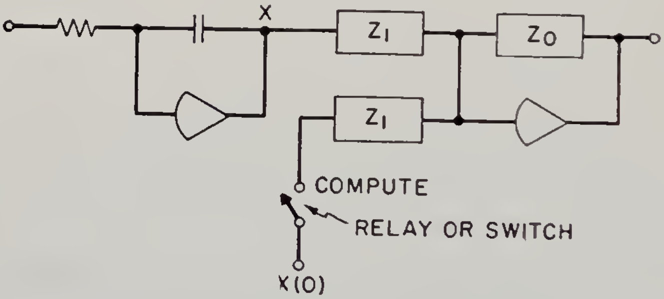

FIG. 7.2. Introduction of an initial-condition voltage into the operational amplifier following an integrator.

A Practical Circuit for Setting up Initial Conditions. The practical circuit of Fig. 7.3 forces the output voltage X_o of each parallel-feedback integrator to assume a desired initial value X_{o0} by charging the integrating capacitor from a reference power supply (Sec. 7.4) common to all integrators.

The relays K_R and K associated with each integrator are energized in the RESET condition, so that the computer is made inoperative, and the initial-condition voltage E_0 is applied to the integrator (Fig. 7.4*a*). The resistor r_L decreases the integrator time constant, so that the integrating capacitor can be charged rapidly. The output voltage X_o* of the resulting circuit (Fig. 7.4*b*) is given by Eq. (4.40); thus

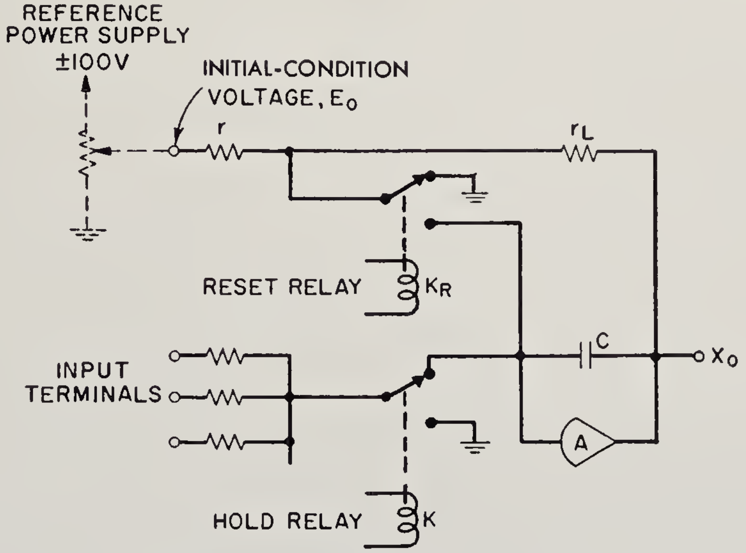

FIG. 7.3. Practical RESET and HOLD circuits for a parallel-feedback integrator. Both K_R and K are energized in the RESET condition; K alone is energized in the HOLD condition (see also Fig. 7.4). Typical circuit values are $r = r_L = 0.1M$, so that $X_{o0} = -E_0$.

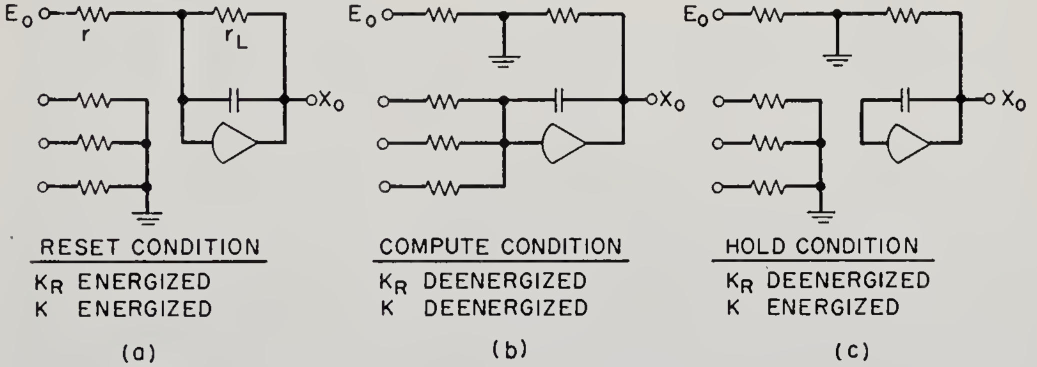

FIG. 7.4. Configuration of the parallel-feedback integrator of Fig. 7.3 in the RESET condition (*a*), in the COMPUTE condition (*b*), and in the HOLD condition (*c*).

$$X_o = \frac{A}{(1-A)\frac{r}{r_L}+1}\,\frac{1}{\frac{(1-A)rC}{(1-A)\frac{r}{r_L}+1}P+1}\,E_0 \qquad (7.1)$$

Since E_0 is a constant voltage, the reasoning of Sec. 4.5 shows that the integrator output voltage X_o will approach the value

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* The reader should remember that X_o is given by Eq. (7.1) only in the RESET condition.

$$X_{o0} = E_0 \frac{A}{(1 - A)\dfrac{r}{r_L} + 1} \approx -E_0 \frac{r_L}{r} \qquad \left(|A| \gg 1, \left|A \frac{r}{r_L}\right| \gg 1\right) \tag{7.2}$$

the more rapidly the smaller the value of r_LC is; X_{o0} is the desired initial value of the integrator output voltage. As a rule, one chooses $r = r_L$, so that $X_{o0} = -E_0$. *The reversal in sign must be remembered in actual computer setups.* Note also that the setting of the initial-condition-setting potentiometer in Fig. 7.3 requires a *loading correction* (Sec. 4.1).

With all initial conditions established in this manner, a computer run is initiated by means of a "compute" switch which deenergizes the relays K_R and K associated with each integrator.[1] In the COMPUTE condition, the initial-condition-setting circuit is effectively removed from the integrator, which is now in its normal operating condition and connected to the other computing elements (Fig. 7.4*b*). The output voltage X_o will then vary from the preset initial value X_{o0} as prescribed by the given differential equations.

Initial Settings of Computing Elements Other than Integrators. Summing amplifiers and phase inverters need not be made inoperative in the RESET condition. Their output voltages will automatically assume correct initial values determined by the initial values of the integrator output voltages. This is also true for computer servomechanisms whose shaft positions are determined by the values of their input voltages.

Time shafts, such as those used to drive function potentiometers, are sometimes driven by clock motors instead of servomechanisms. Such shafts must be *reset* to the shaft position corresponding to $\tau = 0$ before each computer run (see also Sec. 7.2).

HOLD Condition. At the end of each computer run, a differential-equation solver is returned to the RESET condition. It is, however, sometimes desirable to stop the computer during the course of a computation in such a manner that all the machine variables are held to the values they had at the instant the computation was stopped. This may be useful for checking the values of the machine variables; in special cases, also, different coefficients, functions, or interconnections may be introduced for subsequent parts of a computer run (Sec. 7.7).

In the HOLD condition, the relay K in Fig. 7.3 is energized while the relay K_R is deenergized. The integrating capacitor is then disconnected from the input resistors (Fig. 7.4*c*) and will tend to "hold" its charge. The junction of the input resistors is grounded[2] and thus still looks essentially like the grid of a high-gain feedback amplifier to any preceding computing elements.

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[1] The relays are deenergized rather than energized in the COMPUTE condition so as to minimize noise pickup, particularly if a-c relays are used.

[2] This ground connection is omitted in some machines.

More specifically, the integrator time constant in the HOLD condition is R_LC, where R_L and C are, respectively, the leakage resistance and the capacitance of the integrating capacitor. The integrating circuit will thus lose its charge at the initial rate of $100/R_LC$ per cent per second and will integrate any d-c amplifier offset voltage E_e at the same rate. The resulting absolute error after τ_H seconds is less than $(|X_o| + |E_e|)\tau_H/R_LC$. For high-quality integrating capacitors ($R_LC \geq 5 \times 10^5$ sec), the error is less than 0.021 volts per 100 sec even if $X_o = 100$ volts and $E_e = 2$ mv. It is seen, however, that, unlike mechanical and electromechanical differential analyzers, *a d-c analog computer cannot be kept in the* HOLD *condition for an indefinite time.*

In low-cost computers without permanently installed hold relays, selected integrators may still be placed in the HOLD condition by means of suitably patched computing relays (Sec. 6.7). It is also possible to use diode switches (Sec. 6.7) for the same purpose (as in the backlash circuit of Table 6) if the diode back resistances are sufficiently high.

Summing amplifiers and servomechanisms will be kept in the HOLD condition by the integrator output voltages, but provisions must be made to stop all time motors or clock motors associated with the computer. If such motors have a tendency to coast when stopped, they must be provided with brakes effective in the HOLD condition.

With the computer in the HOLD condition, the interrupted computer run may be continued on deenergizing the hold relays K; the machine will then return to the COMPUTE condition.

7.2. CONTROL CIRCUITS

The purpose of the control circuits is to initiate, stop, and supervise the computing operation; to indicate certain gross errors; and to protect the computer from overloads. The design of these circuits is a very important problem. The control-circuit design may be considered as a connecting link which ties all the various computing elements and the associated equipment together into one integrated operating unit, a computing machine. The computer must be easy to use for the operating personnel. The control operation must be foolproof as well as practical and convenient from the "human-engineering" viewpoint, for the control circuits must be used over and over, and often by nontechnical personnel.

Control Functions. The *control and indicating devices* of a typical d-c analog differential-equation solver will comprise the following:

1. *Power-supply controls* (switches and/or push buttons) for turning the computer power supplies on and off.
2. *Operating controls* (switches and/or push buttons) which permit the operator to start and stop the operation of computing elements and

recording equipment. *The most important operating controls are those governing the action of the reset and hold relays on the integrators* (Sec. 7.1).

NOTE: The operating controls may be physically separated from the power-supply controls in order to avoid errors.

3. *Control relays and automatic timers* may be used to operate the controls automatically (Sec. 7.7).
4. *Indicator lights* for indicating computer "conditions" such as

   SAFE
   STAND-BY
   RESET (READY)
   RECORDER ON
   COMPUTE
   HOLD
   MOTOR ON
   BALANCE CHECK

   Such indicator lights help to supervise the control operations. Large bull's-eye-type indicator lights can be more easily seen from a remote-control position and are also a joy for visiting firemen. There is no strict convention governing the colors of the various indicator lights, but these colors should be chosen so that no possibility of confusing one light with another exists.
5. *Overload indicators* (lights, buzzers, etc.) warn the operator when an overload occurs anywhere in the computing circuits.
6. *Remote-control stations* enable the operator to control the computer from any location in the computing room.
7. *Dials* serve to indicate clock-motor or servomechanism shaft displacements corresponding to the values of various machine variables.
8. Control devices for associated equipment, running-time meters, etc.

A Simple Control System for a D-c Analog Computer. The physical operation of the controls just outlined can be demonstrated best by analyzing a basic control system for a simple differential equation solver. Such a control system is shown in Fig. 7.5. The control sequence for placing the computer in operation is indicated from left to right, and all switches and indicator lights are listed on the diagram.

1. Reading from left to right, 110-volt, 60-cycle line power is applied through line fuses and the *stand-by switch* to the filament transformers and to the control-relay circuits. The computer is now in the STAND-BY condition.
2. The *high-voltage switch* energizes the high-voltage power supplies. The computer is now READY for operation.
3. Just before a computer run, the operator may or may not start the recorder paper feed by turning the *recorder switch* on.

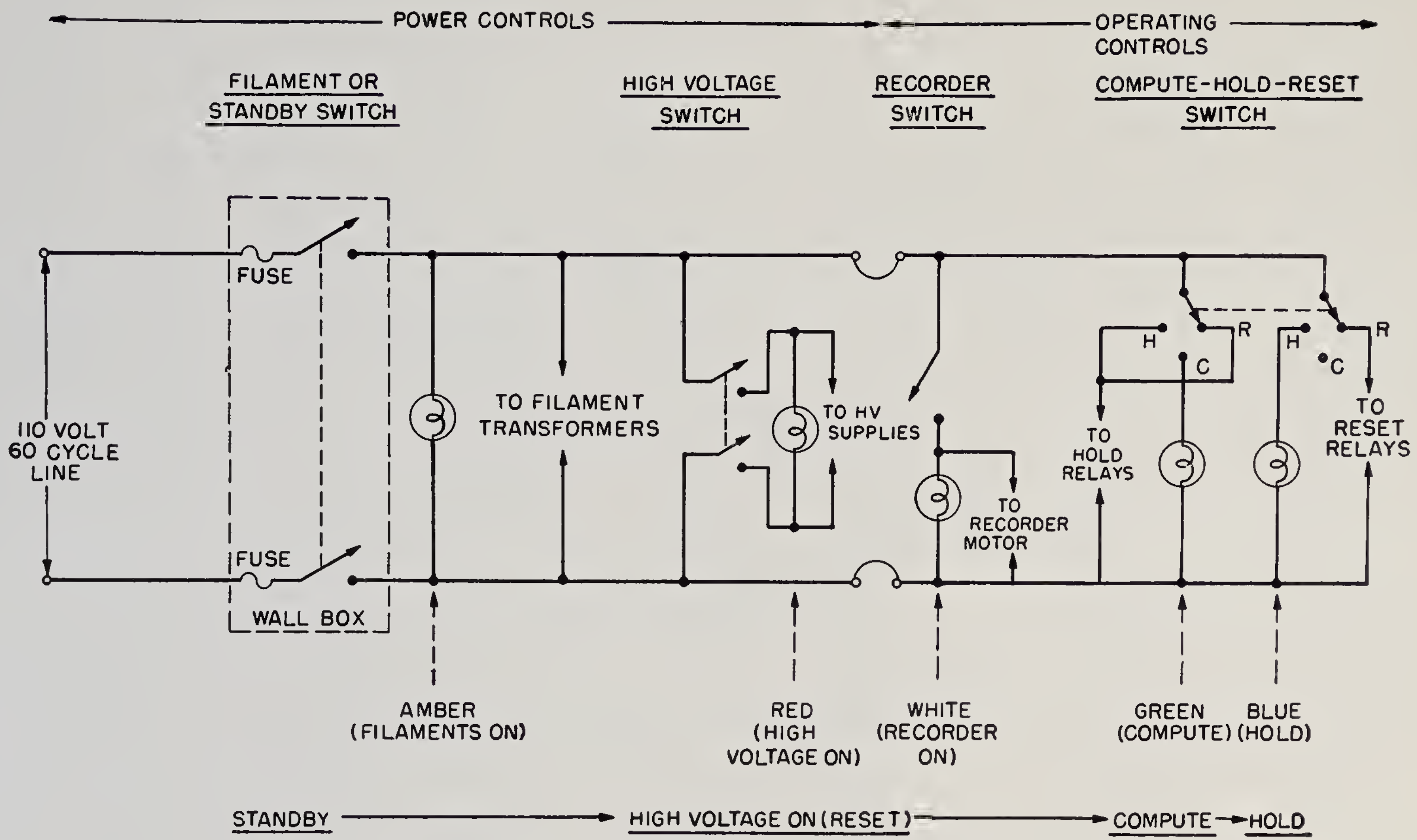

Fig. 7.5. Basic control system for a d-c analog computer.

4. The *compute-hold-reset switch* controls the reset and hold relays and thus serves to start and stop each computer run. Note that the reset and hold relays associated with each integrator are deenergized in the COMPUTE condition; this helps to reduce noise pickup in the integrators during the computation proper.

When the computer is to be turned off, the compute-hold-reset switch is turned to RESET and the recorder, high-voltage, and stand-by switches are turned off in that order. The computer can also be turned off at once by means of the stand-by switch, but the other switches should all be turned off before the computer is again put into operation.

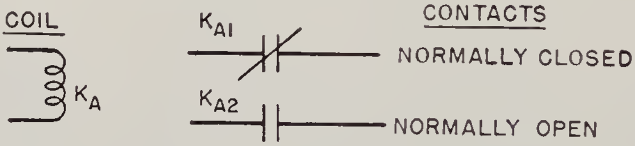

FIG. 7.6. Relay symbols. Normally open and closed contacts of the relay K_A anywhere in a circuit diagram are labeled K_{A1}, K_{A2}, etc.

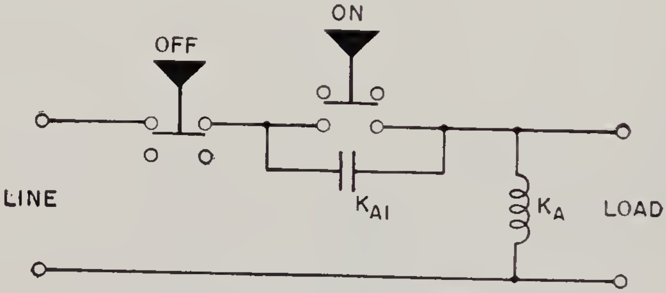

FIG. 7.7. A simple push-button circuit.

Relays and Push Buttons. More sophisticated control systems may be obtained through the use of push buttons and relays. Figure 7.6 shows the notation adopted for relays in the following diagrams. Figure 7.7 shows a simple push-button system. Each push button is a two-circuit switch. The load is energized by touching the ON button; this also actuates the relay K_A whose *holding contact* K_{A1} holds the circuit closed even after the ON button is released. The circuit is switched off by touching the OFF button momentarily.

Note that, with properly designed push-button control systems, *the holding relay is rarely used to initiate the control action proper; the control action is usually initiated by the push button itself.*

Push-button Control System. Figure 7.8 shows a modern *power-control* system. The stand-by switch is, again, a wall switch and incorporates a green *safety light* giving a positive indication that no power is applied to the computer proper. *A time-delay relay* K_T prevents the high-voltage supplies from being energized until the filaments have warmed up; an amber "ready" light indicates when the time-delay relay contact K_{T1} is

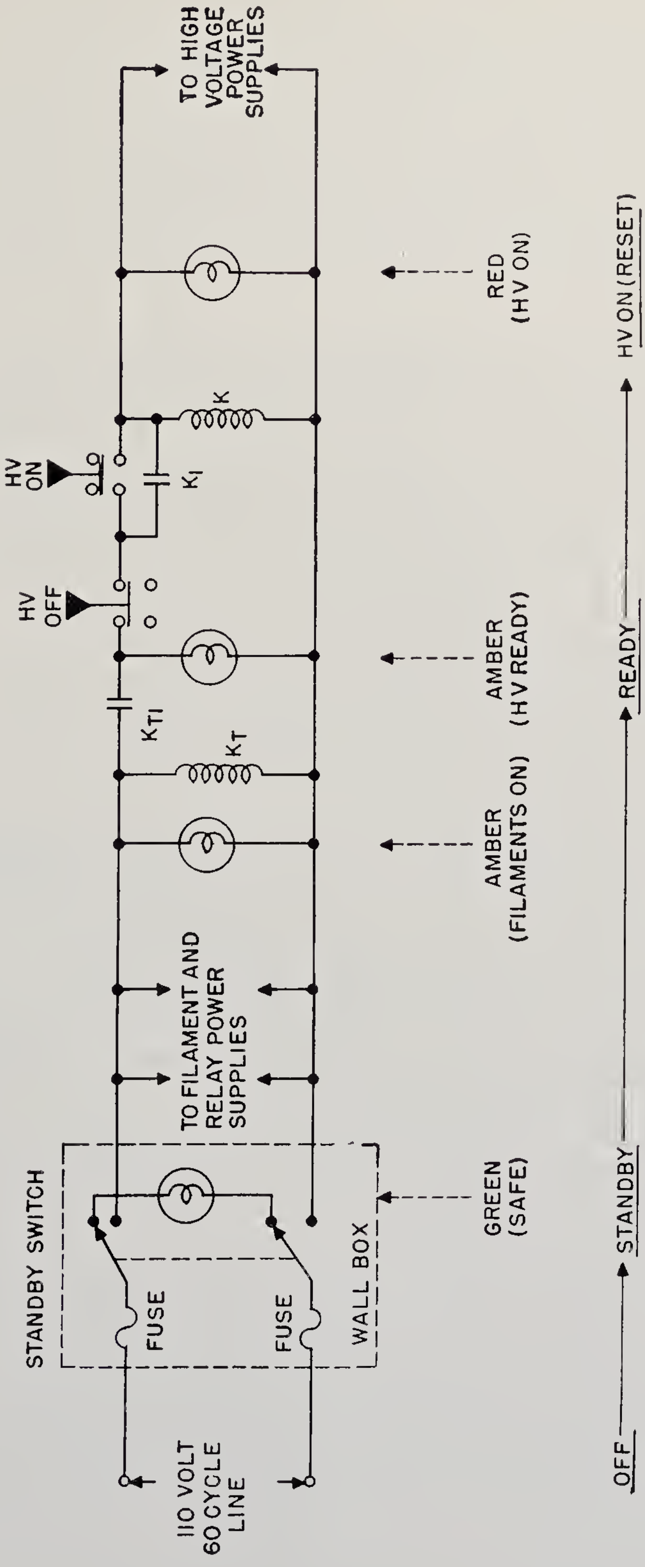

Fig. 7.8. A push-button power supply control system.

closed. A push-button system exactly like that of Fig. 7.7 is then used to energize the high-voltage supplies.

Figure 7.9 shows a convenient push-button *operating-control* system. Either alternating or direct current may be used to actuate the control relays. In the RESET condition, both the integrator reset relays and the hold relays are energized. The normal control sequence may be followed by reading the diagram from left to right; all switches and indicator lights are also shown.

1. The recorder paper feed should be started an instant before the actual computer run. The recorder motor is energized by the *recorder push button*, either directly or through a motor relay if direct current is used for the control-relay power supply. In order to avoid wasting recorder paper while the computer is not in the COMPUTE condition, the recorder push button has not been provided with an independent holding relay.
2. The *compute push button* deenergizes the integrator reset and hold relays and puts the computer in the COMPUTE condition. If the compute button alone is pushed, the computer will revert to the RESET condition as soon as the button is released. This may be useful for short test runs.
3. If the compute push button is operated while the recorder is made to run by means of the recorder push button, the computer is not only put in the COMPUTE condition but is held there by the four contacts K_{C1}, K_{C2}, K_{C3}, and K_{C4} of the *compute relay* K_C.

   K_{C1} and K_{C2} keep the recorder and the compute relay energized even after the recorder and compute buttons have been released. K_{C3} keeps the reset relays deenergized during the computer run, and K_{C4} permits relay K_H to be energized.
4. The compute relay stays energized in the COMPUTE and HOLD conditions. At the end of the computer run, it is deenergized by means of the *reset or stop push button.*
5. The *hold push button* is effective only in the COMPUTE condition. It serves to energize the hold relays and thus places the computer in the HOLD condition. The relay K_H keeps the computer in the HOLD condition after the hold button is released until K_H itself is deenergized by means of the compute or reset push buttons. The computer can thus be put in either the COMPUTE or RESET condition from the HOLD condition. The recorder is not usually stopped in the HOLD condition.

Many other control-circuit arrangements are possible. The greatest advantage of push-button control systems over switching systems is the

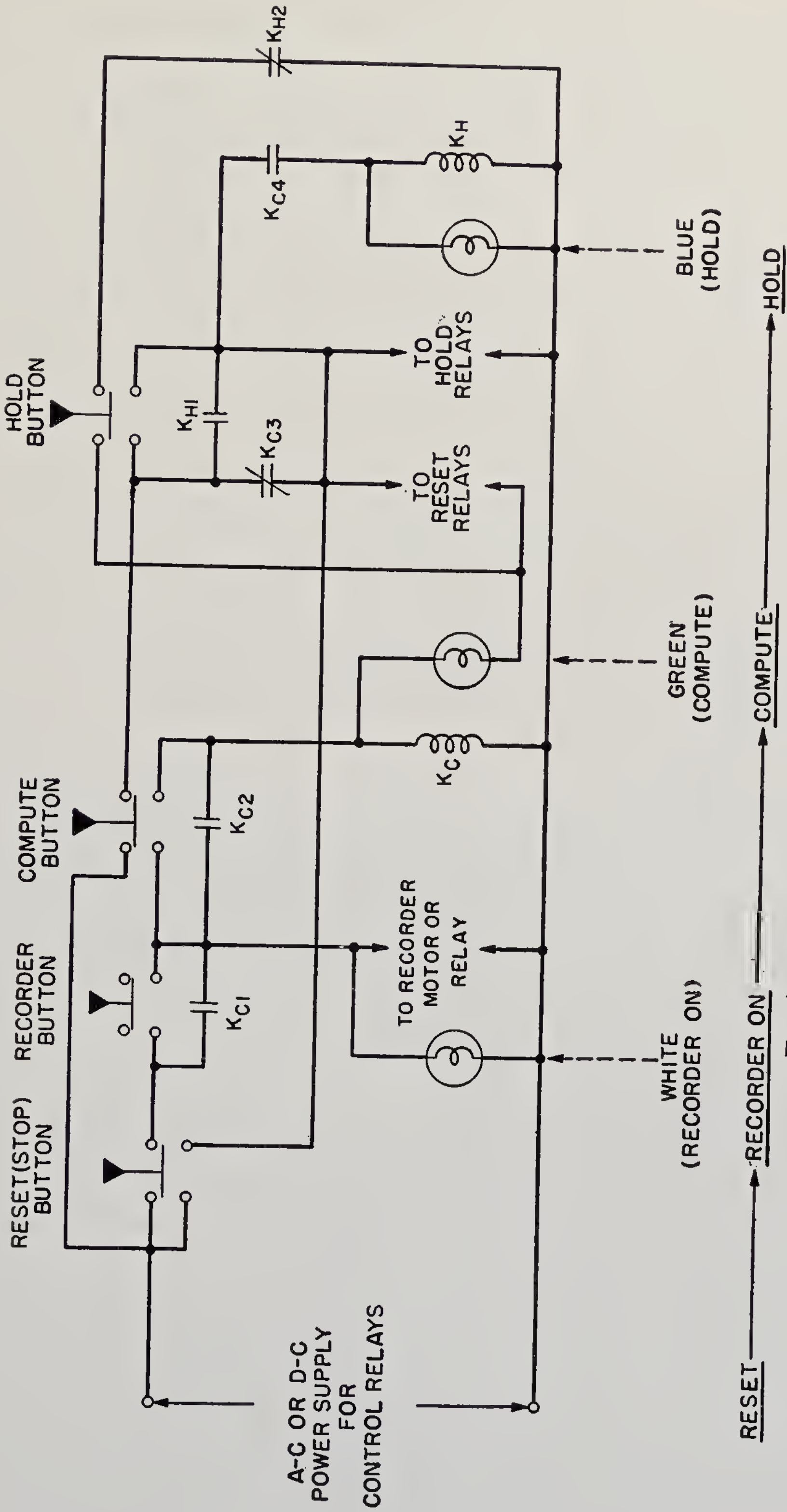

FIG. 7.9. A push-button operating-control system.

fact that with push buttons it becomes unnecessary to check all switch positions before starting operations.

An Operating-control System Employing Automatic Timing. An automatic interval timer can be used to start and stop the computer run at predetermined times and thus to produce computer runs of uniform length. A much more important application of automatic timing, however, is in computers employing potentiometers, driven by a synchronous motor, to generate functions of the machine time τ. In the latter case, it is necessary to have the time shaft set to the position corresponding to $\tau = 0$ at the time each computer run is initiated. Again, the computer run should be terminated when the limit of the potentiometer windings is reached.

Rather than having the motor start from rest at $\tau = 0$, it is preferable to let the motor come up to synchronous speed and to let a cam on the potentiometer shaft initiate the run by momentarily tripping a microswitch μ_1 at the correct instant. A second microswitch μ_2 can then be tripped to stop the run at the desired position of the potentiometer shaft. Figure 7.10 shows an operating-control system of this type.

The control sequence is as follows (all switches and indicator lights are shown in Fig. 7.10):

1. The *recorder-motor push button* energizes both the recorder paper feed and the time motor, either directly or through a relay if direct current is used for the control-relay power supply. This push button does not have an independent holding relay, since it is undesirable to have the recorder and time motors run unnecessarily.
2. To start a computer run, the recorder-motor push button *and* the *compute push button* are depressed simultaneously until the potentiometer shaft reaches the position corresponding to $\tau = 0$. The cam on this shaft will then trip the microswitch μ_1 momentarily; the latter will deenergize all reset and hold relays and will energize the *compute relay* K_C, thus placing the machine in the COMPUTE condition. The action of the compute relay K_C in this circuit is essentially the same as in the circuit of Fig. 7.9; the compute relay serves to keep the computer in the COMPUTE condition even after the recorder-motor and compute push buttons have been released.
3. The hold circuits also operate essentially as shown for the circuit of Fig. 7.9. In addition to energizing the hold relays, the *hold push button* serves also to stop the time motor (but not usually the recorder) in the HOLD condition. The contact K_{H2} of the relay K_H will keep the time motor stopped even after the hold push button has been released. Since the heavy time-motor armature has a tendency to coast, it may be fitted with a brake or mechanically disconnected by means of a solenoid clutch in the HOLD condition.

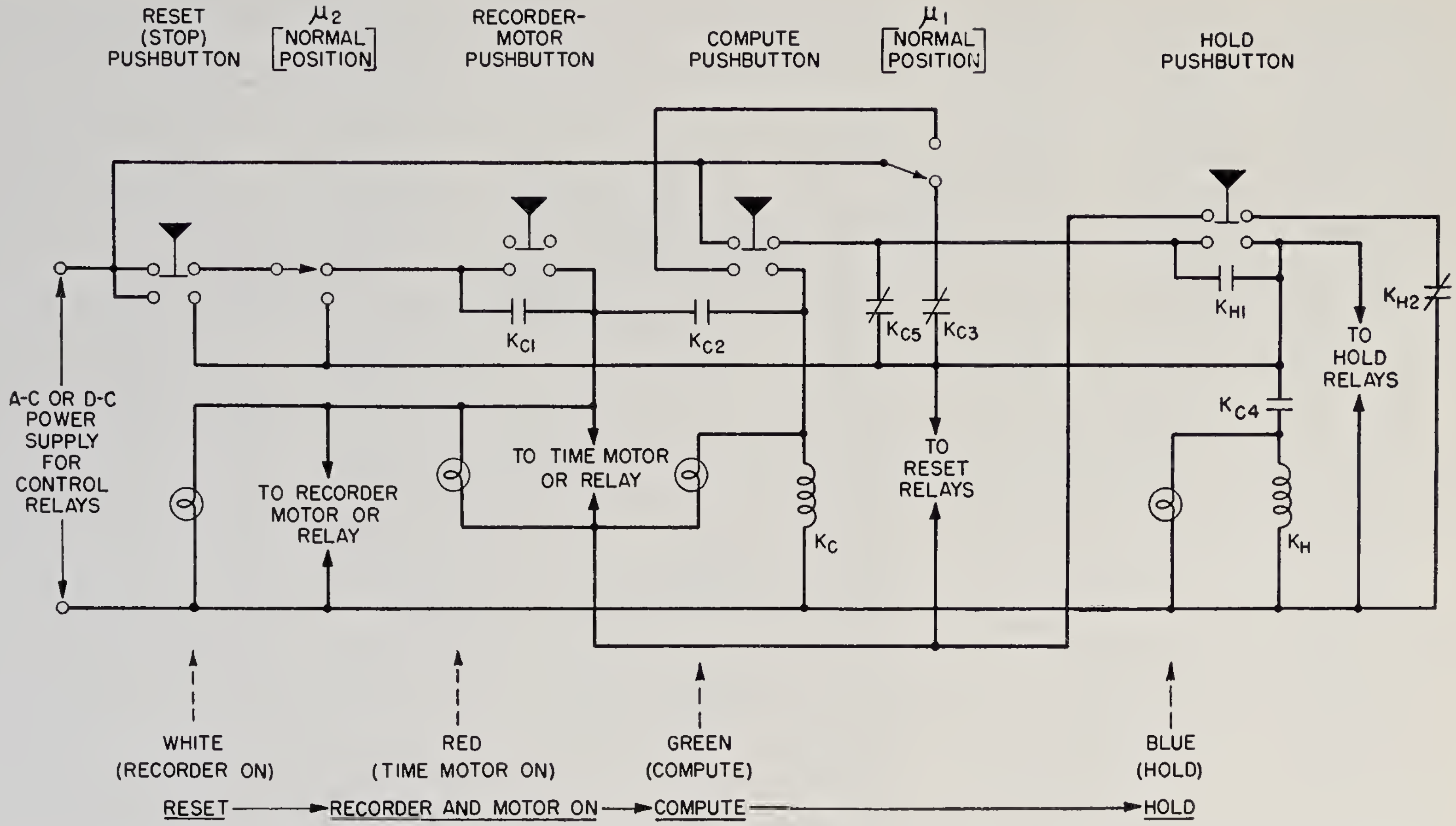

Fig. 7.10. Push-button control system with automatic timing. The HOLD button may be duplicated by a microswitch actuated at suitable times for digital readout (Sec. 7.7). Recorders capable of sufficiently fast starts and stops may be started and stopped together with the time motor.

4. The machine may be returned to the COMPUTE condition from the HOLD condition by depressing the compute push button. Under these circumstances, however, the time motor has to start from rest, and the computation may not be so accurate as is otherwise possible. It would be preferable to start the computation from the RESET condition with new initial conditions.
5. The computer run is stopped either at the end of a preset time interval by the cam-operated microswitch μ_2 or at any earlier time desired by pushing the *reset or stop push button.* The computer is thus returned to the RESET condition.

Remote Control. Many larger d-c analog computers require the operator to move along the various computer racks in order to make adjustments. In this connection, it is very useful to provide the operator with a *portable remote-control station.* This is simply a small utility box bearing duplicate operating controls and connected to the computer proper by a long cable (see Fig. 8.8). The operator can then run the computer while making adjustments, observing the recorder, etc., from any position in the computer room. As a rule, only the operating controls will be brought out to the remote-control station. It is also not necessary to have duplicate indicator lights and overload lights on the remote-control station. Figure 7.11*a* shows how to duplicate switches; it is seen that no definite switch positions for ON or OFF are possible with such arrangements. *Duplicate push buttons as illustrated in Fig.* 7.11*b are preferable in remote-control installations.*

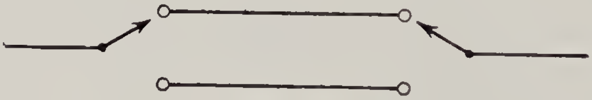

FIG. 7.11*a*. Duplication of switches. Definite switch positions for ON and OFF are lost; the circuit is closed if and only if both single-pole–double-throw switches contact the same channel.

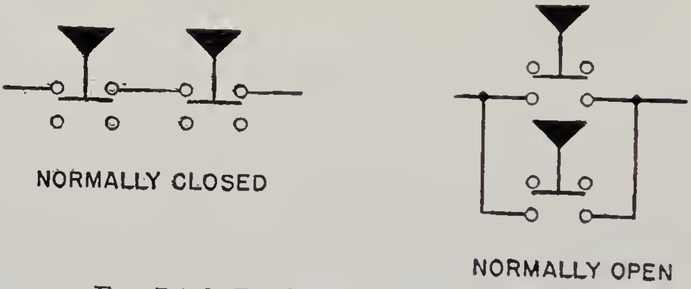

FIG. 7.11*b*. Duplication of push buttons.

Overload Indicators. Every computer should be designed so that overload voltages do not result in any physical damage, but voltages exceeding the normal operating ranges of the computing elements will still cause serious errors in the computation.

In order to warn the operator of overload voltages, the current flowing in a limiter diode (Sec. 6.7) may be used to trip a relay and actuate a warning light or buzzer. For overload voltages in excess of 70 volts, a better method is to connect a *neon bulb* across the amplifier or servomechanism output terminals as shown in the circuit of Fig. 7.12*a*. The neon bulb will first flash and then glow as the output voltage reaches an *absolute value* predetermined by the resistors r_1 and r_2.

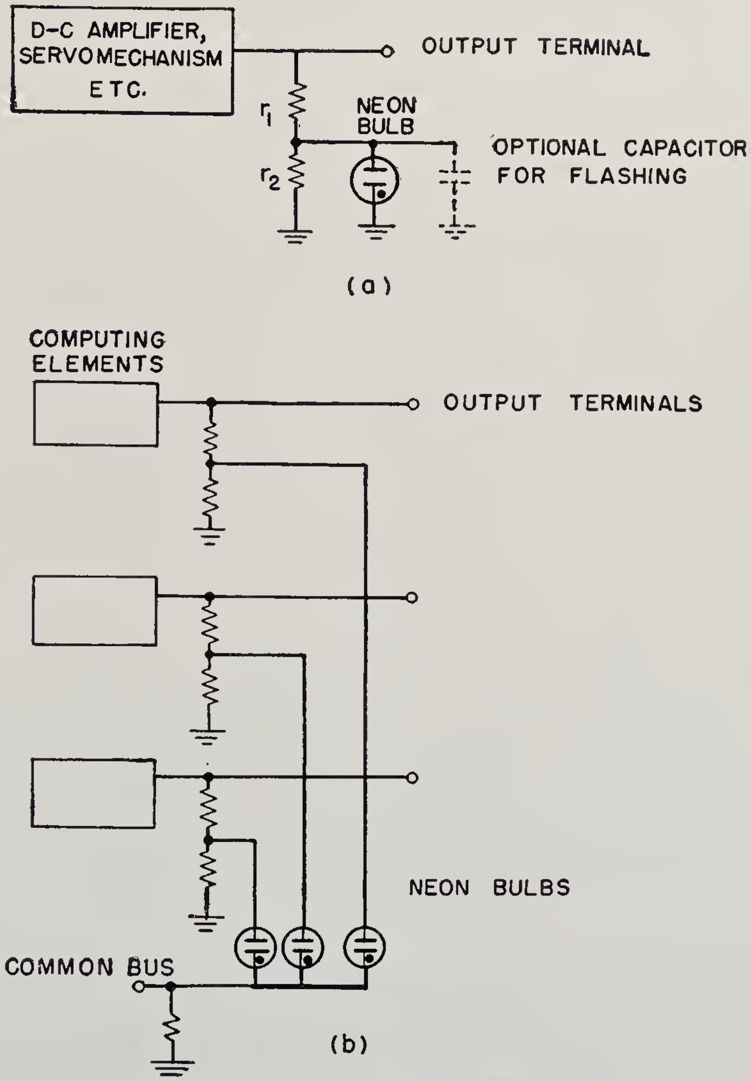

FIG. 7.12. Neon-bulb overload indicators.

The neon bulbs may also be made to actuate a relay.[1] In the circuit of Fig. 7.12*b*, the neon bulbs associated with each computing element are all returned to a common bus. If any of the neon bulbs should fire due to an overload, the voltage on the common bus is changed and serves to actuate a relay, usually through an amplifier circuit. This relay can then energize a buzzer or return the computer into the HOLD or RESET condition. The computer run in question must then be repeated with a changed setup or with new scale factors.

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[1] Garwin, *Rev. Sci. Instr.*, June, 1950.

The *special error-indicating circuits* for operational amplifiers employing automatic balancing circuits (described in Sec. 5.6) are also a species of overload-indicating devices. The error-indicating neon bulbs may be returned to a common bus in the manner described for overload lights.

7.3. RECORDING EQUIPMENT

Introduction: Recorder Scales. The recording equipment associated with d-c analog computers automatically plots graphs of one or more voltages (dependent machine variables) against the time τ as the independent variable. Some types of recorders also permit the plotting of one voltage (dependent variable) against another voltage or dependent machine variable.

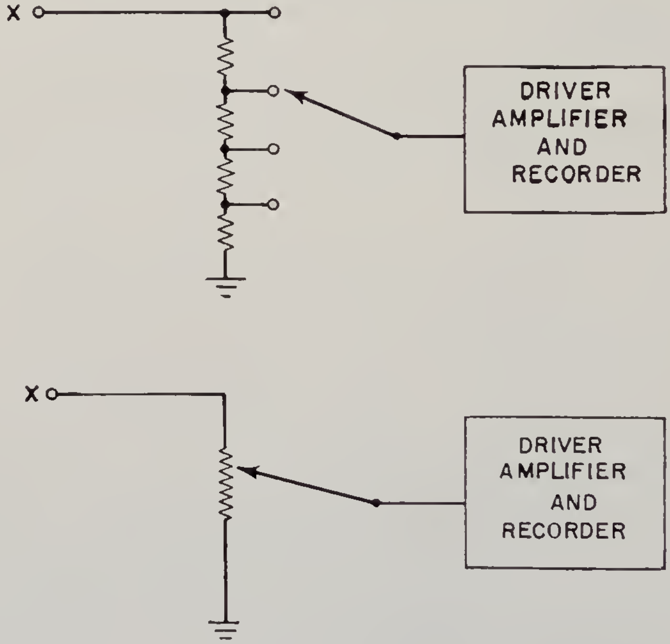

Fig 7.13. Two methods for adjusting recorder sensitivity.

All instruments used to plot d-c analog-computer solutions are, essentially, *recording voltmeters* whose deflections are proportional to variations of the machine variables. As a rule, the records are made on coordinate paper bearing scale divisions to indicate the values of voltage and time recorded. In some recorders, the number of scale divisions per machine unit or *recorder sensitivity* K is fixed for each of several *ranges* selected by means of a *range switch*. Figure 7.13*a* shows how different values of K may be obtained by means of a simple attenuator preceding the recording device. In other types, the *recorder scale factor* A_x for a problem variable x, or

$$\left.\begin{aligned} A_x &= \frac{\text{recorder scale divisions}}{\text{unit problem variable}} \\ &= \frac{\text{recorder scale divisions}}{\text{machine unit}} \; \frac{\text{machine unit}}{\text{unit problem variable}} \\ &= Ka_x \end{aligned}\right\} \tag{7.3}$$

[where a_x is the scale factor proper (see Sec. 2.2) relating the problem variable x to the corresponding machine variable X], is continuously adjustable by varying the recorder sensitivity K. With the latter arrangement, it is particularly easy to make the recorder scale divisions correspond to convenient values of the problem variables. The records will then be more useful and easier to read. Recorders of this type may have to be calibrated for each new problem, *e.g.*, by means of a potentiometer as shown in Fig. 7.13*b*. This recorder *sensitivity control* should be equipped with a positive shaft lock lest the setting be disturbed accidentally after calibration.

The following is a list of properties desirable in recording devices used for computing purposes (see also Sec. 7.7).

1. *Accuracy of calibration; frequency response.* The recorders should have an accuracy comparable to that of the computer in question (0.1 to 3 per cent) and should maintain this accuracy under all conditions of signal amplitude and frequency occurring during computation.
2. *Direct-recording feature.* The recorders should be capable of producing a record *immediately available* without processing, since computer records are frequently needed at once in order to determine the further course of the computation.
3. Records should be reasonably *permanent* and capable of being *reproduced* by standard photographic printing processes. Semitransparent recording paper is useful in this latter respect.
4. *Multichannel operation* makes it possible to record several machine variables on the same time scale. Records of this type are often particularly convenient for purposes of interpretation and comparison.

Survey of Recording Devices. The choice of a recorder for a given computer application depends chiefly on the accuracy required, on the frequency response demanded by the problem conditions, and on the time scale used. In the following paragraphs the most frequently used types of recording devices are listed together with some of their pertinent properties.

Cathode-ray Oscillographs. The frequency response of the familiar cathode-ray oscillograph exceeds that of all presently used computing elements.

The accuracy of cathode-ray oscillographs is low (5 to 10 per cent) if external scales are used for reading the recorded values. The accuracy may be increased to a remarkable extent (to about 0.5 per cent) by introducing self-calibrating features, such as marker pips and reference lines into the display.

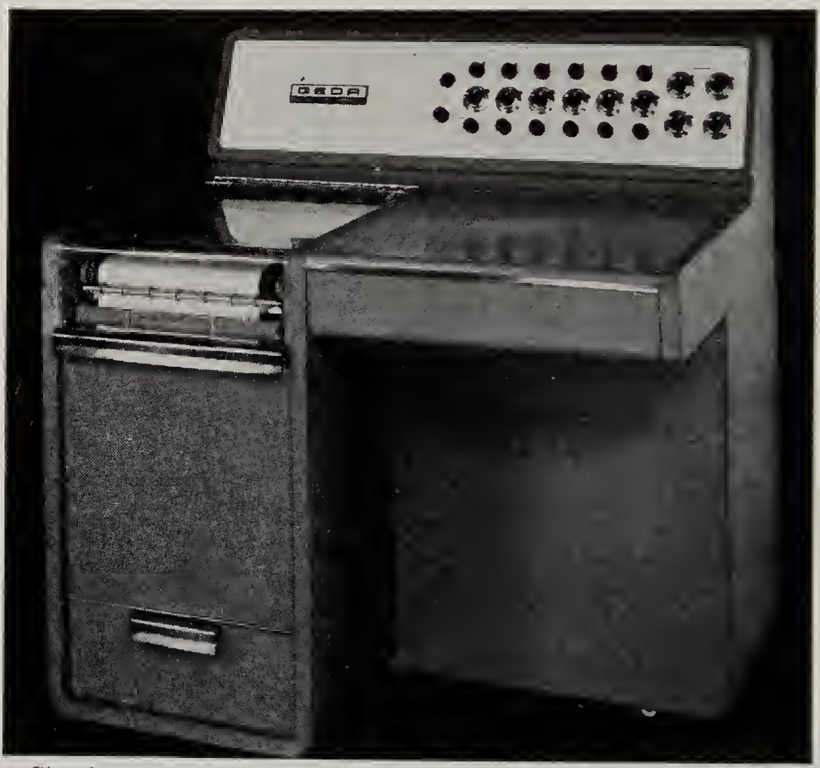

Fig. 7.14. Six-channel, direct-inking magnetic oscillograph and driver amplifiers mounted in a desk-type console. The entire computer may be controlled from this console (Goodyear Aircraft Corporation).

Whereas a cathode-ray oscillograph does not permit direct recording, it may be possible to watch the display before or even while it is being photographed. Cameras with self-contained rapid film-processing devices (Polaroid Land camera) make it possible to obtain the finished oscillograph record within 2 min after photographing the oscillograph screen.

Direct-writing Magnetic Oscillographs. Most of the computer records shown in this book were obtained on direct-writing magnetic recorders of the general type shown in Fig. 7.14. Such devices consist of a paper-feed mechanism and as many *pen motors* as there are recording channels. Each pen motor is essentially a d'Arsonval meter movement with a pointer bearing a lightweight writing device at its end. In the six-channel direct-inking recorder shown in Fig. 7.14, the meter pointers are hollow and feed ink onto the recording paper at the point of contact. In the perhaps more desirable arrangement shown in Fig. 7.15, the direct-inking pens are replaced by tiny heated wires which produce black traces on special heat-sensitive paper. Direct-inking magnetic oscillographs are capable of fair accuracy (1 to 5 per cent). The frequency response of some types may be made flat up to 100 cps through partial electrical compensation of the pen inertia. These devices may be made sufficiently small for convenient multichannel operation. To save recorder pens, each channel should be fitted with a current-limiting device which operates even if the recorder amplifier and/or one or more of its power supplies fail.

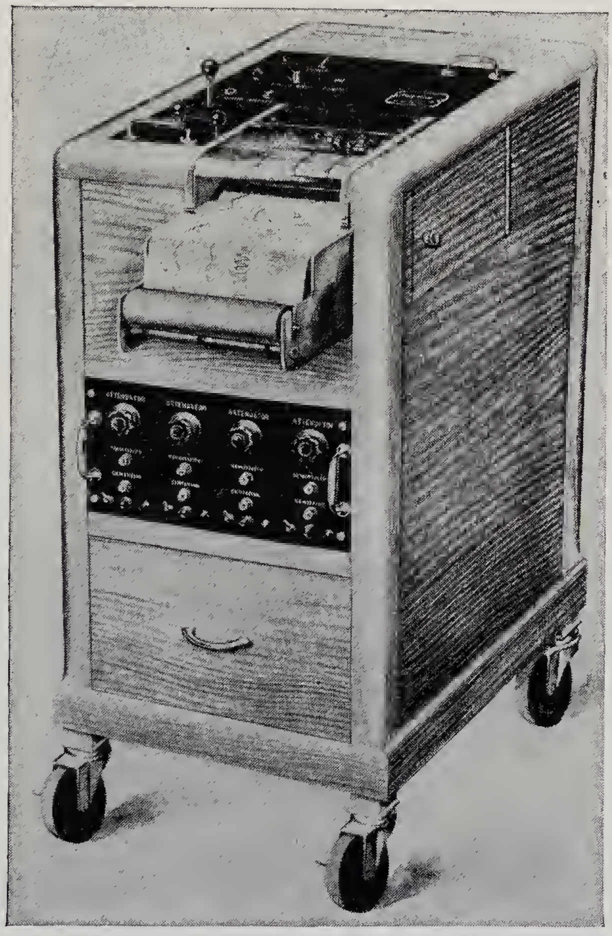

Fig. 7.15. Four-channel magnetic oscillograph for direct recording on heat-sensitive paper (Sanborn), used with a REAC computer (Reeves Instrument Corporation).

Recorder Driver Amplifiers. Magnetic oscillographs require currents of the order of 25 ma for full-scale deflection. Since such currents cannot be supplied from most computer amplifiers, current amplifiers (low-impedance drivers) are needed to drive the pen motors. Such driver amplifiers complete with power supplies may be purchased together with the recorders, but the prices of many commercial models are so very high that it may well pay to construct a set of amplifiers in the laboratory and to operate them from the main computer power supplies.

The principles of d-c amplifier design given in Chap. 5 apply to the design of recorder drivers, so that it will suffice here to present a number of typical circuits.

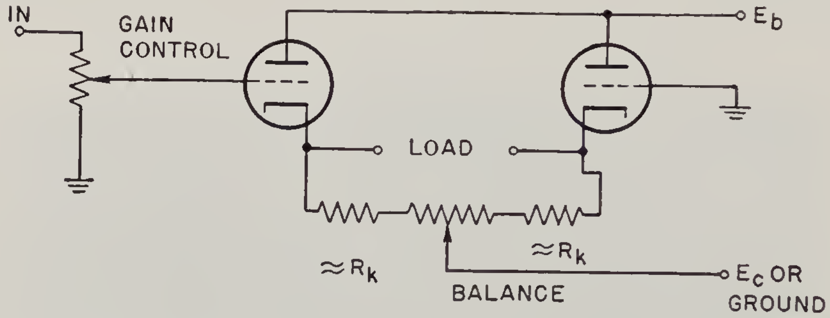

FIG. 7.16. Differential cathode follower. The current output may be increased if larger tubes and lower values of R_k are used. If several tubes are used in parallel, 1,000-ohm resistors should be inserted in series with each grid to prevent parasitic oscillations.

Typical Design Values
Two 6AS7G (two sections each side), R_k = 1,000 ohms, 50 watts
Four 6SN7 (four sections each side), R_k = 6,000–10,000 ohms, 20 watts

The *differential cathode-follower* circuit shown in Fig. 7.16 has better linearity than a simple cathode follower; besides, the bridge-balanced arrangement minimizes the unbalancing effects of all supply-voltage variations. The current gain of the circuit is

$$\frac{\text{Load current}}{\text{Input voltage}} = \frac{g_m}{2 + (\mu + 1)\dfrac{R_L}{r_p} + \dfrac{R_L}{R_k}} \tag{7.4}$$

where g_m and r_p refer to one of the two identical tubes. The operating point must be determined from the tube characteristics (see Sec. 5.2). The linearity of the circuit may be improved at the expense of the current gain by introducing a resistance in series with the load. Typical circuit values are shown in Fig. 7.16.

Figure 7.17 shows a differential amplifier with preamplification. Further refinements may include the use of limiters (Sec. 6.7) to prevent overloading the pen motor; the latter is endangered particularly if one of the differential cathode-follower tubes should burn out.

Figure 7.18 shows a more sophisticated low-impedance driver possessing high linearity and constant gain. This is essentially a high-gain d-c amplifier with cathode-follower output and current feedback; the output tube itself limits the output current in both directions through cutoff and grid current, respectively.

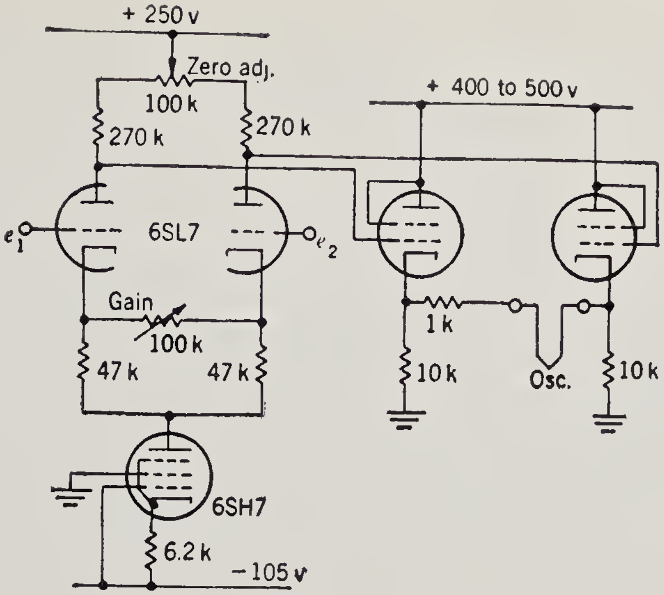

FIG. 7.17. Differential cathode follower with preamplification (G. E. Valley and H. Wallman, *Vacuum Tube Amplifiers,* MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948).

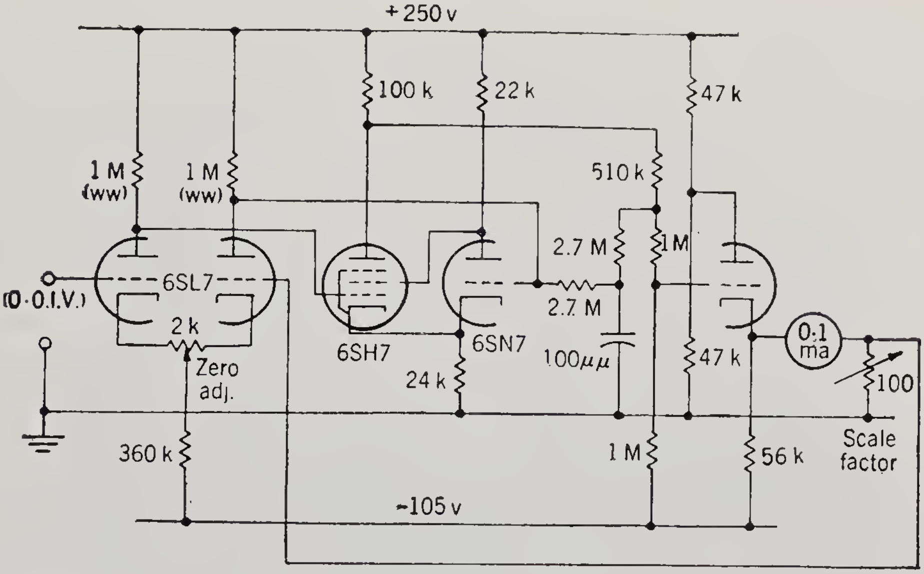

FIG. 7.18. A high-quality recorder driver amplifier (G. E. Valley and H. Wallman, *op. cit.*).

Figure 7.19 shows a simple equalizing network which may be inserted ahead of a driver amplifier to compensate in part for the poor high-frequency response of recorder pens. By means of such networks or equivalent feedback arrangements,[1] it is possible to keep the accuracy of certain magnetic recorders within 3 per cent at frequencies up to 100 cps.

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[1] Christian, D. R., *AIEE* Paper 47-226, presented at the AIEE meeting in Dayton, Ohio, Sept. 23–25, 1947. See also H. B. Shaper, Electrodynamic Direct-inking Pen, *Electronics*, March, 1946.

Other Electromechanical Oscillographs. A number of devices permit photographic recording by means of a light beam deflected by a mirror positioned through some current- or voltage-sensitive device (moving coil, string, crystal, etc., types of galvanometer). Such recorders are widely used for various commercial purposes. Some of these devices are accurate to better than 2 per cent and may have a frequency response flat up to 15,000 cps because of the low inertia of their moving parts. The inconvenience of photographic recording, however, constitutes a serious obstacle to the use of such recorders in many computer applications.

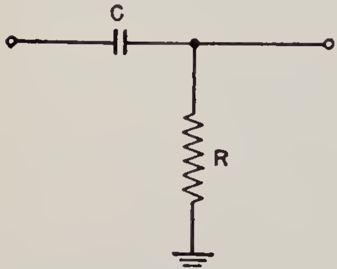

FIG. 7.19. Simple equalizing network inserted ahead of driver amplifier to improve the high-frequency response of a magnetic recorder.

Facsimile-type or sweep-balance recorders[1] may permit direct recording on various types of paper at frequencies up to 20 cps. Another interesting class of recorders employs current-sensitive paper and between 25 and 300 discrete styli, each actuated within a small range of input voltages.[2] Recorders of this type may be made self-calibrating.

Servo Tables. Voltage-actuated servomechanisms similar to the ones used to position computing potentiometers (see Sec. 6.8) may be used as reliable and accurate means for positioning recorder pens. Accuracies of 0.1 per cent of full-scale deflection are possible. The paper feed may be accomplished by a clock motor or by a second servomechanism; the latter arrangement makes it possible to plot one voltage with respect to another voltage. Periodic *timing marks* can indicate the time scale. The most serious limitation of servo recording devices is their poor high-frequency response; few if any are useful at signal frequencies above 5 cps.

In the servo recorder shown in Fig. 7.20*a*, one servomechanism positions the pen parallel to the axis of a cylindrical drum carrying the paper. The paper feed is accomplished by rotating the drum through a second servomechanism. This device permits a recording accuracy of about 0.2 per cent of full scale at signal frequencies below 2 cps and also serves as a curve-follower in the arrangement of Fig. 6.38.

In the servo recorder of Fig. 7.20*b*, pen and carriage servomechanisms permit the use of a flat paper surface with vacuum hold-down. The recording accuracy varies between 0.1 per cent and 0.2 per cent as the plotting speeds are increased to 15 in./sec; phase shift will not exceed 5 deg for 4-in. excursions at 1 cps or for 0.5-in. excursions at 5 cps for the pen (0.5 in. at 2 cps for the carriage). The acceleration limits of pen and carriage are approximately 750 in./sec^2 and 250 in./sec^2, respectively.

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[1] See, for instance, Keinath, *Instruments*, **17**: 200, 1946.

[2] Radiation, Inc., Melbourne, Fla.

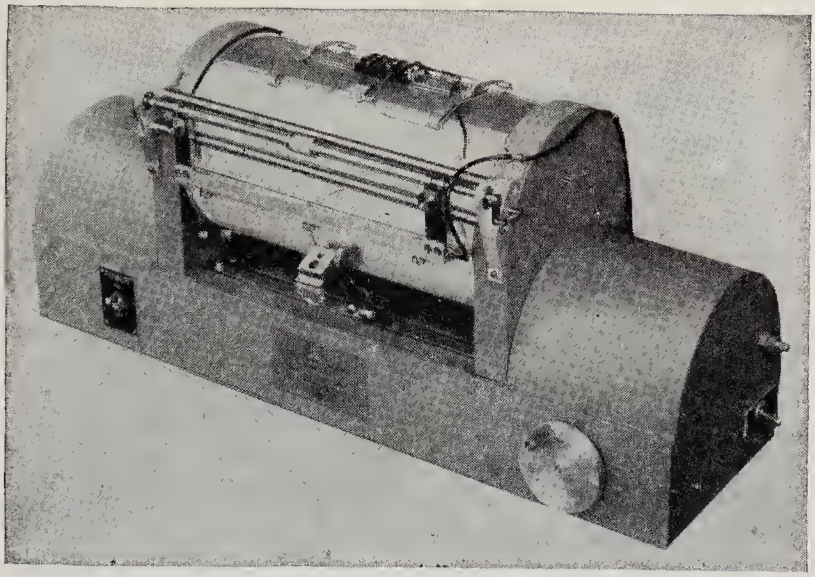

Fig. 7.20*a*. Servo recorder developed by the Reeves Instrument Corporation, New York City. This servo-driven drum also serves as a manually driven input table and as a universal function-generating potentiometer of the type shown schematically in Fig. 6.38.

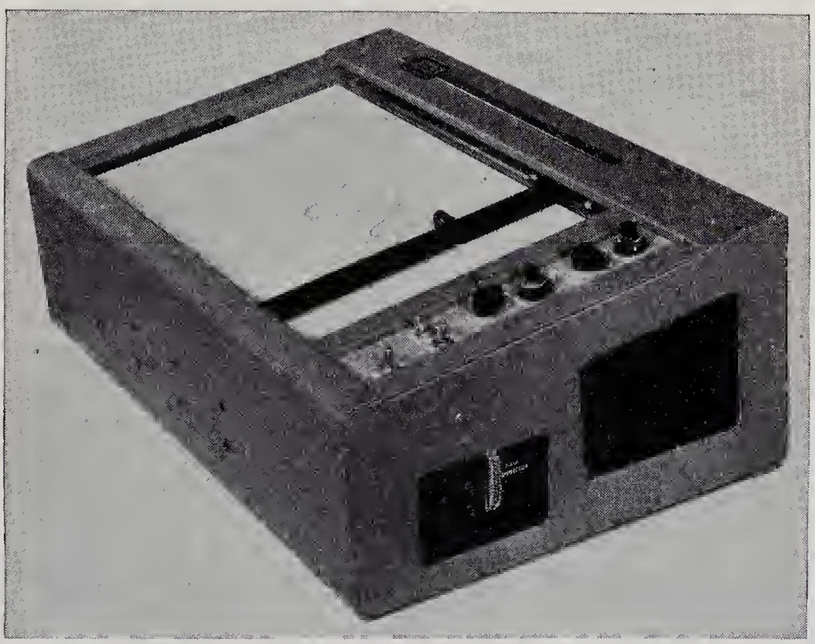

Fig. 7.20*b*. This servo recorder permits the use of a flat recording surface and contains all necessary amplifiers and power supplies (Electronic Associates, Inc., Long Branch N.J.).

An optional capacitive pickup permits the use of the recorder as a curve-follower (Sec. 6.9). A larger flat-surface servo plotting board is shown in Fig. 7.21 (see also Fig. 1.17). Such servo boards may incorporate two servo-driven pens for simultaneous plotting of two curves.

It is also possible to construct servo tables which make plots in polar coordinates instead of rectangular cartesian coordinates. Another interesting recording device is the so-called "crab" consisting of a recording pen traveling over the recording paper on a three-wheeled carriage. The crab is positioned by two servomechanisms. One of the latter determines the course angle by "steering" the carriage wheels, while the other one determines the distance traveled. Crabs are particularly useful for trajectory plotting. They are also convenient to use, since they will "walk" on any level plotting table and no special provisions for pen mounting or paper feed are required.

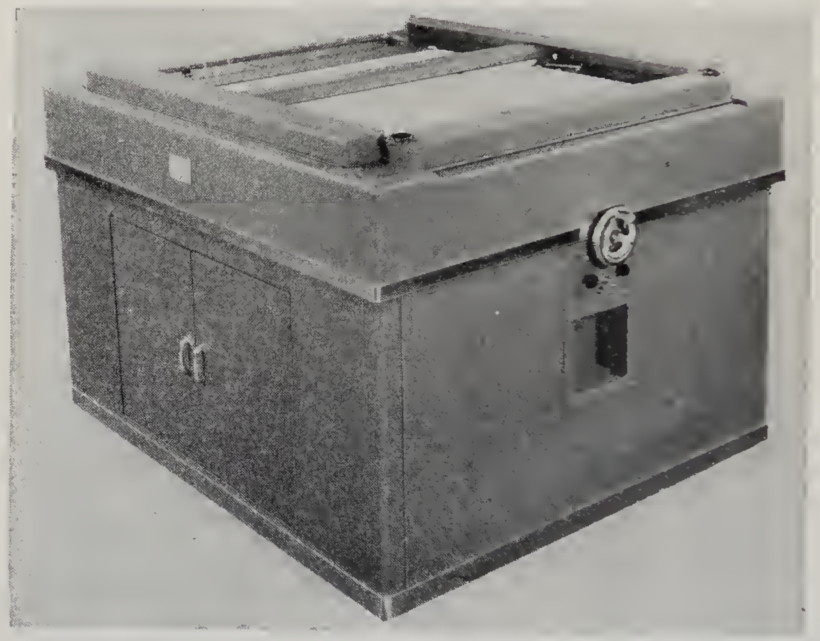

FIG. 7.21. Large servo plotting board (Reeves Instrument Corporation, New York City).

Dependence of Recorder Time Scales on Paper Speed. Since synchronous motors are used to drive the paper of most recording oscillographs, the paper speed will depend on the supply frequency. While the average frequency of a-c power lines is held constant within very small limits, instantaneous frequency deviations as large as 0.7 per cent are possible. Accordingly, one cannot rely on the power-line frequency for the recorder time scales in high-accuracy computers. This difficulty may be overcome in the following ways:

1. A record of the power-line frequency may be maintained and used to correct the recorder time scales.
2. A stable audio-frequency oscillator (tuning-fork oscillator) may be used to generate *timing marks* on the recording paper.
3. The output of a tuning-fork oscillator may be amplified and fed to the synchronous motors used to drive function generators in the computer (see Sec. 6.6) as well as to the paper-feed motor.

7.4. COMPUTER POWER SUPPLIES

Every electronic analog computer requires power supplies to supply the voltages needed for the operation of the computing elements and associated components. The power supplies necessary for the operation of a typical d-c analog computer are listed below:

1. *Filament power supplies* (1 to 24 volts alternating or direct current).
    - *a.* Unregulated alternating current is suitable for filament operation in power-output tubes and in the high-level stages of voltage amplifiers.
    - *b.* Regulated alternating current or preferably regulated direct current should be used to supply the filaments of amplifier input tubes in order to minimize the drift due to cathode-emission changes (see Sec. 5.1).
2. *Regulated high-voltage power supplies.*
    - *a.* Positive plate supplies.
    - *b.* Negative bias supplies.
    - *c.* Positive and negative *reference supplies* for computing voltages.
3. *Power supplies for servomechanisms* (sometimes self-contained in each servo amplifier).
4. *Low-voltage d-c power supplies* for various control relays, solenoids, clutches, motors, and other associated equipment.

The designer must specify the currents and voltages required from each of these power supplies as well as the accuracy with which the voltage must be regulated.

Primary Power Sources. The operating voltages for most d-c analog computers are derived from the 110-volt, 60-cycle a-c line by means of *transformers and/or rectifiers.* In very large installations as well as in certain mobile applications, *motor-generator sets* are used to supply high-voltage direct current. Especially in smaller installations, storage and dry *batteries* should not be disregarded as sources of computer operating power; the use of batteries frequently makes electronic regulation unnecessary.

Filament Power Supplies. Unregulated alternating current (6.3 volts) is generally used to supply the filaments of high-level stages in the equipment, and this type of filament operation is well known. In order to avoid hum resulting from filament emission, it is advisable to return the center tap of the filament transformers to a point which is positive with respect to the cathodes in question.

The filaments of low-level stages of d-c amplifiers are frequently supplied with regulated alternating or direct current; the latter is somewhat preferable because the use of direct current helps to minimize hum pickup. The regulators used with either a-c or d-c filament supplies usually employ *saturable reactors* which tend to change their impedance as the result of current changes. Such regulators are generally available in the form of complete commercial units and are, thus, not usually built in the laboratory.

High-voltage Power Supplies. High-voltage supplies include positive plate-voltage supplies, negative bias-voltage supplies, and positive and negative supplies for computing voltages. Practically identical circuits are suitable for all these supplies; the output voltage will be positive or negative with respect to ground, depending on which one of the output terminals is grounded.

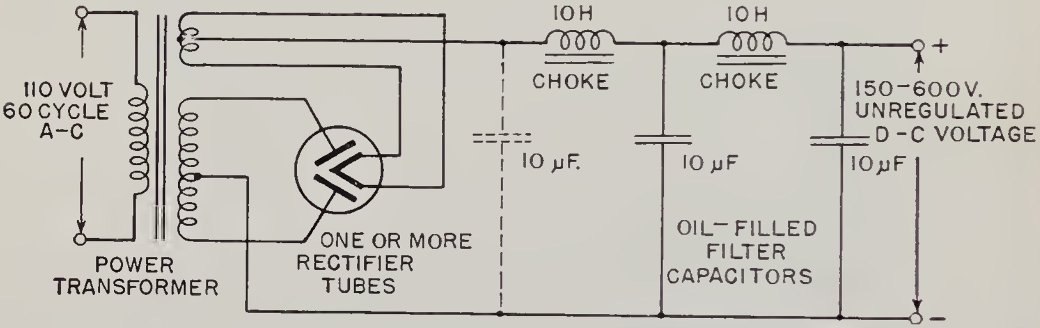

FIG. 7.22. Typical transformer-rectifier unit for a high-voltage power supply. The use of a capacitor-input filter (dash lines) permits a higher output voltage and better filtering but increases the internal impedance of the unit. Generally speaking, vacuum-type rectifier tubes will give less trouble than gas-type rectifiers.

Each high-voltage power supply will, in general, consist of a *transformer-rectifier unit* and a *regulator unit*. A typical *transformer-rectifier unit* is shown in Fig. 7.22. The design of such units for given output voltages and currents is adequately covered in many textbooks and handbooks on electronics.[1]

The purpose of the *regulator unit* associated with practically every computer high-voltage supply is twofold. Specifically,

1. The regulator unit must keep the supply voltage in question constant within specified limits in order to minimize d-c amplifier drift due to supply-voltage changes.

2. The regulator unit must keep the power-supply-output incremental impedance (see Sec. 5.3) below a specified minimum value, not only throughout the range of expected computer signal frequencies, but also

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[1] Reich, H. J., *Principles of Electron Tubes*, Chap. 11, McGraw-Hill, New York, 1941; Terman, F. E., *Radio Engineers' Handbook*, McGraw-Hill, New York, 1943.

throughout a range of higher frequencies. This is necessary in order to avoid undesirable coupling between different amplifier stages and computing elements in the machine. Feedback effects due to such coupling might otherwise lead to errors in the computation and even to uncontrolled oscillations.

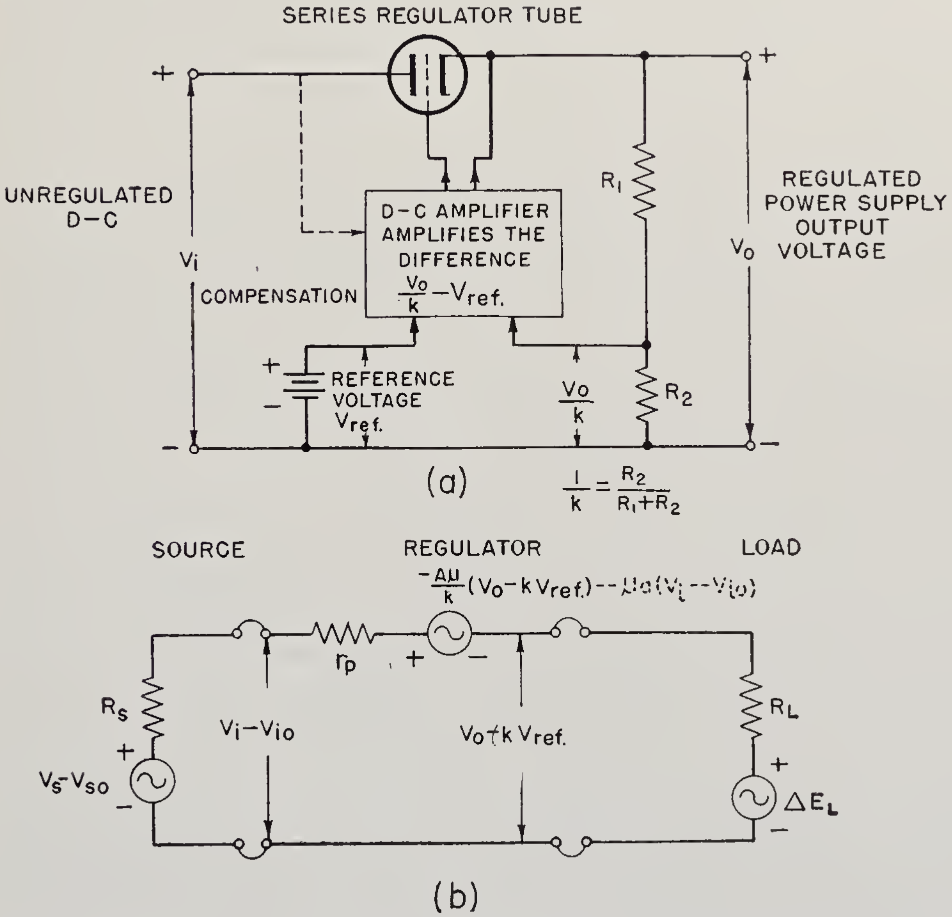
SERIES REGULATOR TUBE
UNREGULATED D-C
REGULATED POWER SUPPLY OUTPUT VOLTAGE
V_i
V_o
COMPENSATION
D-C AMPLIFIER AMPLIFIES THE DIFFERENCE $\frac{V_o}{k} - V_{ref.}$
R_1
R_2
REFERENCE VOLTAGE $V_{ref.}$
$\frac{V_o}{k}$
(a)
$\frac{1}{k} = \frac{R_2}{R_1+R_2}$
SOURCE
REGULATOR
LOAD
$-\frac{A\mu}{k}(V_o - kV_{ref.}) - -\mu a(V_i - -V_{io})$
R_s
$V_s - V_{so}$
r_p
$V_i - V_{io}$
$V_o - kV_{ref.}$
R_L
ΔE_L
(b)

FIG. 7.23. Block diagram and linear equivalent circuit of a degenerative electronic regulator unit. The grid voltage of the regulator tube is $\frac{A}{k}(V_o - kV_{\text{ref}})$ if no compensation is used. With compensation (dash lines) the grid voltage is $\frac{A}{k}(V_o - kV_{\text{ref}}) + a(V_i - V_{io})$.

The second function of the regulator unit is at least as important as the first one and is the principal reason for the almost exclusive use of *degenerative electronic regulators* in d-c analog computers. The solid lines in Fig. 7.23*a* show the block diagram of a degenerative electronic regulator. Such a unit is essentially a d-c power amplifier whose output plate current is taken as the supply current. Voltage feedback, through the resistors R_1 and R_2, tends to keep the power-supply output voltage V_o equal to a constant multiple kV_{ref} of the voltage V_{ref} where

$$\frac{1}{k} = \frac{R_2}{R_1 + R_2}$$

The voltage V_o/k is continuously compared with the reference voltage V_{ref}. The regulator amplifier tends to minimize the absolute value of the difference $(V_o/k) - V_{\text{ref}}$ by changing the effective resistance of the regulator tube, which is in series with the voltage source. The voltage feedback, will, moreover, tend to reduce the power-supply-output incremental impedance.

Analysis of a Degenerative Regulator. The performance of a degenerative regulator with respect to voltage changes in the voltage source and current changes in the load can be analyzed conveniently by means of a *linear equivalent circuit* of the type discussed in Sec. 5.2. The solid lines in Fig. 7.23*b* show such a linear equivalent circuit for d-c conditions. The unregulated d-c source is indicated at the left; it has a resistive source impedance R_s. The source voltage V_s is capable of undesirable voltage changes $(V_s - V_{s0})$ with respect to its reference level V_{s0}. As demonstrated in Sec. 5.2, the regulator tube may be considered to act like a voltage source with the voltage ($-\mu$ times the grid voltage applied to the vacuum tube) and the internal impedance r_p, where μ and r_p are the amplification factor and the plate resistance of the regulator tube, respectively. The grid voltage of the regulator tube is $(A/k)(V_o - kV_{\text{ref}})$ as indicated in the block diagram of Fig. 7.23*a*; here $A < 0$ is the voltage gain of the d-c amplifier preceding the regulator tube proper. The load is characterized by its resistance R_L, which will be taken to include the effect of the resistors R_1 and R_2. The effects of current changes in the load are conveniently described by including in the equivalent circuit for the load a voltage source of voltage ΔE_L.

From the nodal equations of the equivalent circuit shown in Fig. 7.23*b* one finds, for d-c conditions,

$$V_o - kV_{\text{ref}} = \frac{(V_s - V_{s0})R_L + \Delta E_L(R_s + r_p)}{R_s + r_p + R_L\left(1 - \dfrac{A\mu}{k}\right)} \tag{7.5}$$

The internal impedance Z_o of the regulated power supply, with respect to voltage changes ΔE_L in the load, is, also for d-c conditions,

$$Z_o = -\frac{\partial \Delta E_L}{\partial(i - i_0)} - R_L = \frac{R_s + r_p}{1 - \dfrac{A\mu}{k}} \tag{7.6}$$

where $(i - i_o)$ is the deviation of the load current from its reference value i_0. Equations (7.5) and (7.6) show *that the greater the value of* $|\mu A/k|$ *the* *more the regulator will reduce the effects of source-voltage changes as well as the internal impedance of the power supply.*

Compensation. The performance especially of low-gain regulators may be improved by adding a voltage proportional to the unregulated regulator input voltage V_i (see Fig. 7.23) to the regulator amplifier signal. Such a voltage is best introduced in the output stage of the regulator amplifier, as indicated by the dotted lines in Fig. 7.23*a*. The effective grid-voltage change of the regulator tube is now

$$\frac{A}{k}(V_o - kV_{\text{ref}}) + a(V_i - V_{i0})$$

where V_{i0} is the reference level of the regulator input voltage V_i. The quantity $a < 0$ is the voltage gain or transfer function of the compensation circuit from the regulator input terminals to the grid of the regulator tube.

Through the use of the linear equivalent circuit, it is possible to derive expressions for the power-supply output-voltage changes for d-c conditions

$$V_o - kV_{\text{ref}} = \frac{(V_s - V_{s0})(\mu a + 1)R_L + \Delta E_L[(\mu a + 1)R_s + r_p]}{(\mu a + 1)R_s + r_p + \left(1 - \frac{\mu A}{k}\right)R_L} \tag{7.7}$$

and for the power-supply output impedance (at direct current)

$$Z_o = -\frac{\partial \Delta E_L}{\partial(i - i_0)} - R_L = \frac{R_s(\mu a + 1) + r_p}{1 - \frac{\mu A}{k}} \tag{7.8}$$

for a regulator with compensation. It is seen that considerable improvements in regulator performance may be obtained by choosing values of the compensation gain a such that, approximately,

$$a = -\frac{1}{\mu} \tag{7.9}$$

The compensation gain a can be adjusted experimentally by observing the residual ripple on an oscilloscope connected to the regulator output terminals; the value of a is adjusted so as to minimize the ripple voltage. All regulators should be adjusted with the average rated load current flowing, so that all the tubes will have the correct operating voltages. The adjustment will have to be repeated whenever a tube in the regulator is replaced.

Practical Regulator Design. The application of these principles to practical circuits will be shown further below. *Much of the information given in Chap. 5 on the design, construction, and operation of computer d-c* *amplifiers will apply to the design, construction, and operation of the d-c amplifiers used in electronic regulators.* The input stages of high-quality regulator amplifiers should obtain their filament current from a regulated supply. The entire regulator constitutes a degenerative feedback loop, and its stability must be ensured by a judicious application of the principles presented in Sec. 5.4. Frequently, the gain of a regulator amplifier may be increased through the application of regenerative feedback in the manner discussed in Sec. 4.6. In applications where reduction of the long-time drift of the power-supply output voltage is of paramount importance, the use of self-balancing d-c amplifiers (see Sec. 5.6) will permit outstanding regulator performance. If all computer amplifiers are balanced frequently, however, or if self-balancing d-c amplifiers are used as computer amplifiers, long-time stability is not so important in a computer power supply as low output impedance and freedom from drift during the relatively short time of each computer run.

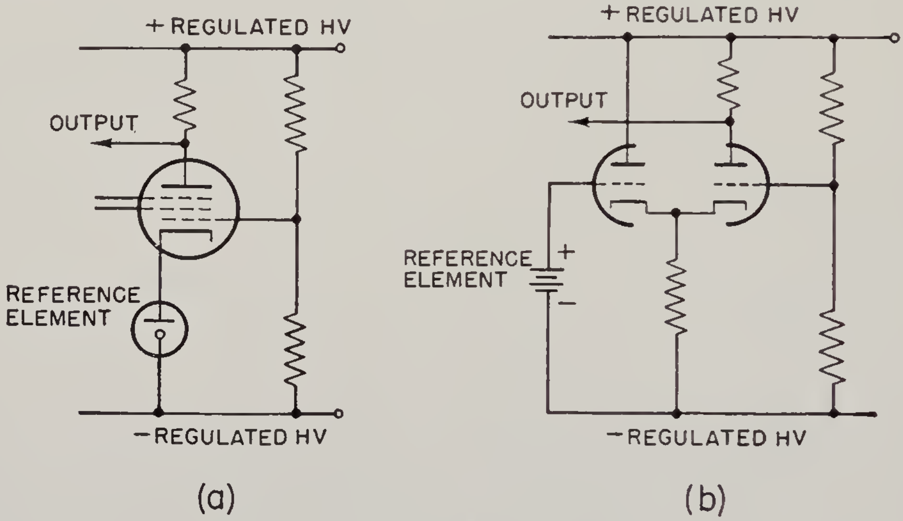

FIG. 7.24. Comparison circuits.

Reference Elements and Comparison Circuits. Gas-discharge tubes such as neon bulbs and voltage-regulator tubes are frequently used as reference elements, since they tend to maintain a fairly constant voltage across their terminals. This voltage does, however, tend to vary slowly as the tube ages and is also somewhat dependent on the ambient temperature and on the current through the tube. The effects of temperature changes can be minimized by operating each voltage-regulator tube at a preferred current (about 12 ma for a VR105[1]).

Batteries operated under conditions of low current drain are preferable to gas-discharge tubes as accurate reference elements. It is not necessary or desirable to use standard cells as reference elements for computer power supplies; small batteries of the type used as B batteries in portable radios are capable of giving a voltage reference accurate to 0.05 per cent for a period of several months if the current drain is low (the grid of a vacuum tube is usually the only load) and the ambient temperature is kept reasonably constant.

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[1] Greenwood, I. A., J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948.

Figure 7.24 shows two comparison circuits used in voltage regulators to obtain an amplified output voltage proportional to the difference between a fraction V_o/k of the regulator output voltage V_o and a given reference voltage. In the circuit of Fig. 7.24*a*, the reference voltage is applied to the cathode of a vacuum tube, whereas the feedback voltage V_o/k is applied to the control grid. The differential-amplifier circuit shown in Fig. 7.24*b* is preferable to the circuit of Fig. 7.24*a* because of its inherent compensation for the effects of variations in cathode emission and because the vacuum-tube plate current does not flow through the reference element.

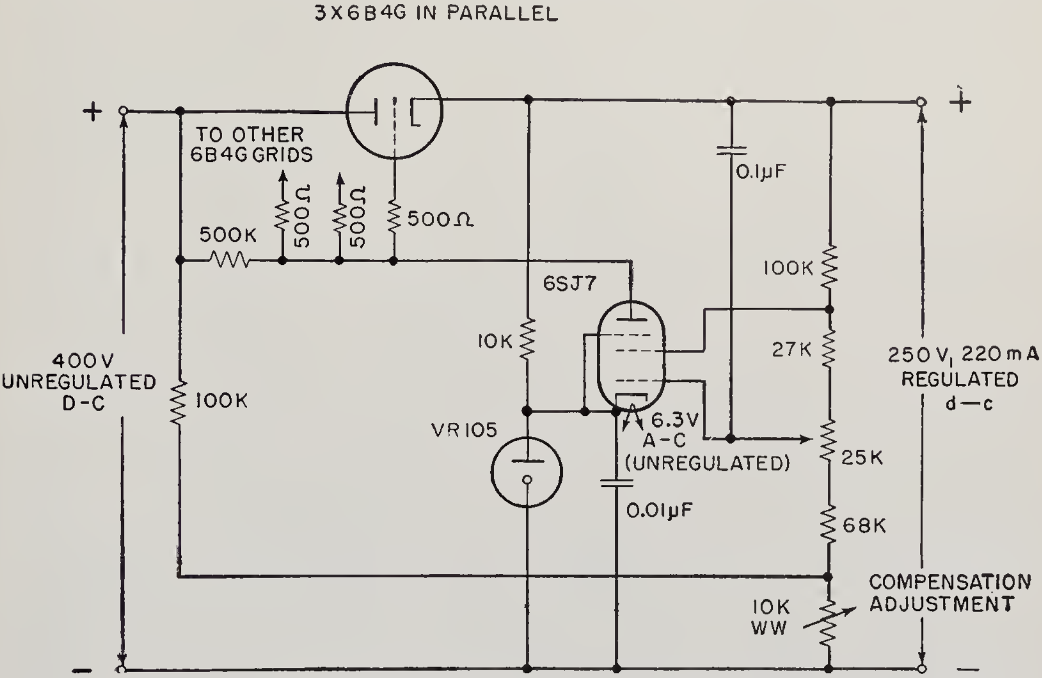

FIG. 7.25. Simple degenerative electronic regulator with compensation (I. A. Greenwood, J. V. Holdam, and D. MacRae, *Electronic Instruments*, MIT Radiation Laboratory Series, Vol. 21, McGraw-Hill, New York, 1948).

Examples of Regulator Design for Computer Applications. Figure 7.25 shows a simple degenerative regulator suitable for use in both the positive and negative power supplies of a low-cost d-c analog computer. This regulator was widely used at the MIT Radiation Laboratory[1] and employs

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[1] *Ibid.*

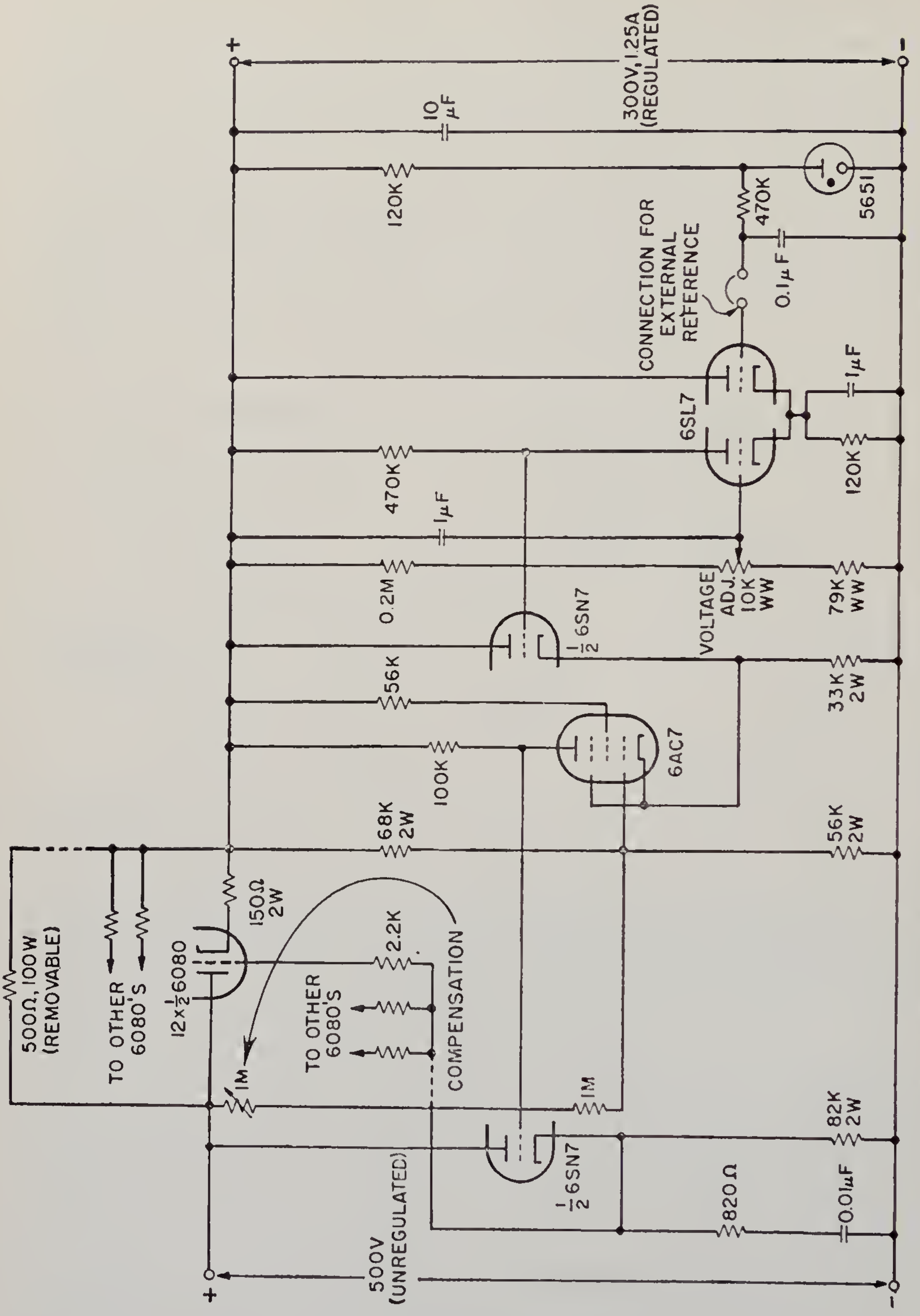

FIG. 7.26. A precision regulator circuit for a d-c analog computer (Electronic Associates, Inc.).

compensation to obtain an output impedance below 30 ohms at all frequencies from 0 to 100,000 cps. The output impedance for direct current is about 20 ohms with all reasonable transformer-rectifier units, and the regulation corresponds to a change in output voltage of less than 0.5 per cent for a 10 per cent change in line voltage.

Figure 7.26 shows a high-quality regulator circuit. The difference between a fraction $(1/k)V_o$ of the output voltage V_o and the reference voltage obtained from a voltage-regulator tube is amplified by two differential-amplifier stages; a compensation voltage is added in the second stage. The resulting regulating voltage drives 12 series-regulator grids through a cathode follower. The VR-tube reference may be replaced by an external reference if desired. In order to keep the regulator output impedance below 0.2 ohms up to 10,000 cps in spite of decreasing regulator loop gain, a 10 μf capacitor is connected across the output terminals. Circuits of this type can regulate the output voltage within 0.0002 per cent for line-voltage changes of 10 per cent if the d-c amplifier filaments are supplied with regulated a-c or d-c;[1] 0.01 to 0.05 per cent regulation for a 10 per cent line voltage change and from 50 per cent load to full load is ample for most computing purposes.

Computer Supplies with Common Reference Voltage. In d-c analog computers using several (positive and negative) high-voltage power supplies, it is advantageous to use a common reference element for all power supplies. The effects of changes in the reference voltage will then be at least partially canceled by the corresponding mutually opposing changes in the positive and negative supply voltages. Figure 7.27 illustrates a possible circuit for the use of a common reference element for a positive and a negative regulated power supply.

The Construction of Electronic Regulators. As mentioned above, degenerative electronic regulators are essentially d-c amplifiers, and all the information given in Sec. 5.9 regarding the physical construction of such units and the choice of components will therefore apply here. Since regulator amplifiers may have high gains, care should be taken to avoid microphonics by shock-mounting at least the input tubes. If at all possible, the regulators should be mounted on a chassis separate from the transformer-rectifier units in order to minimize the danger of introducing 60-cycle hum into the regulators. Again, hum pickup as well as spurious feedback will be minimized by shielding the grid leads in the regulator input stages.

Positive and Negative Power Supplies for Computing Voltages Every d-c analog computer must have provisions for supplying the original *input voltages* used in each computer setup. These voltages include the input voltages for function generators (some of these functions may be simply constant terms) and the initial-condition voltages applied to each integrator. It is customary to provide each d-c analog computer with terminals carrying voltages of plus and minus 1 machine unit (plus and minus 40 to 100 volts, as the case may be) for this purpose. These voltages may be obtained from other positive and negative high-voltage supplies having a common reference element only if the correct regulated voltages are available. No voltage dividers may be used, since the loads supplied by the +1 and −1 terminals will be different for each computer setup. For this reason, most multipurpose computers have special regulated power supplies or even battery sources for supplying these computing voltages. Ordinary well-regulated high-voltage supplies may be employed, or one may use the output of a chopper-stabilized d-c amplifier (phase inverter) which amplifies the voltage of a battery reference (standard cell) connected across its input terminals.[1] The drain on the

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[1] Miller, S. E., Sensitive D-c Amplifier with A-c Operation, *Electronics*, November, 1941.

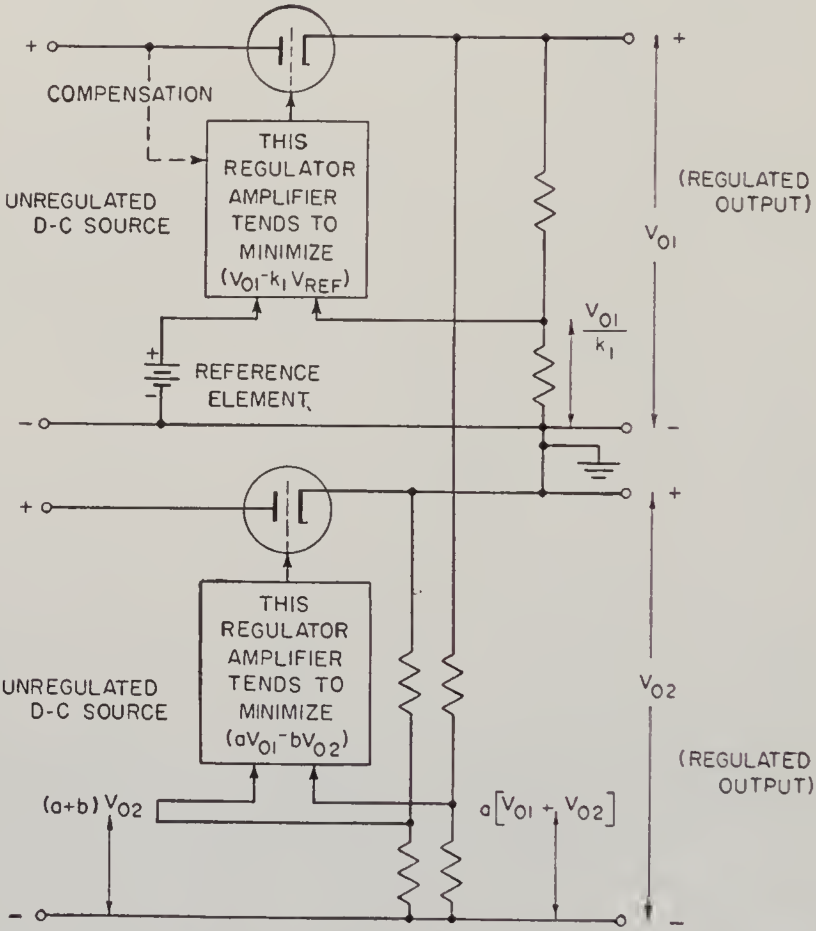

FIG. 7.27. Positive and negative computer power supplies regulated by reference to a single reference element.

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[1] Electronic Associates, Inc., Long Branch, N.J.

standard cell is minimized by a feedback circuit of the type shown in Fig. 6.40.

The output voltage of one such power supply will serve as a *reference voltage* for the computer. The value of every variable voltage in the entire computer will depend on the reference voltage; but *the exact value of the reference voltage will, in general, not be too important,*[1] *since the values of all machine variables can be measured in terms of the value of the reference voltage.* All calibrations of recorder and meter scales as well as limiters and other function generators will, therefore, always be done in terms of fractions of the reference voltage.

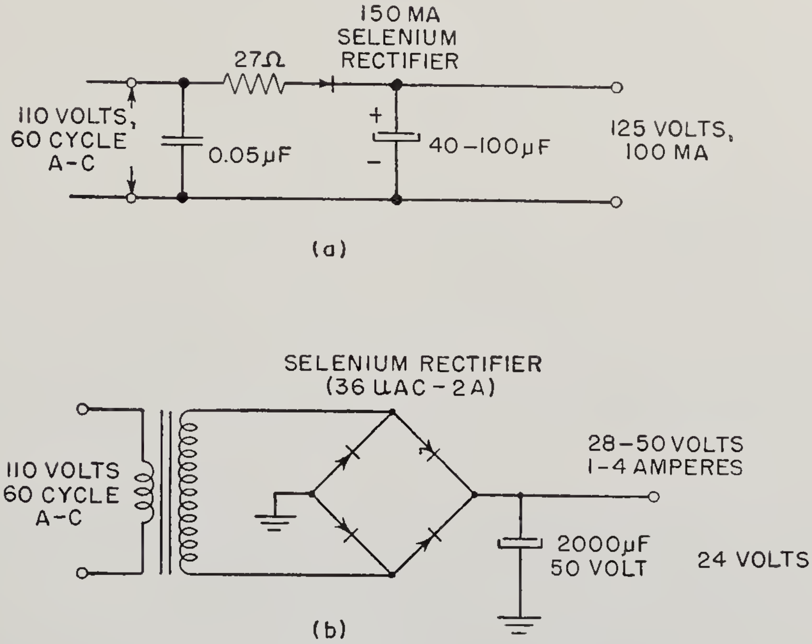

FIG. 7.28. Power supplies for d-c control relays.

Power Supplies for Operating Control Relays. Control relays (see Sec. 7.1), as well as solenoids used to brake motors or to operate clutches, are often operated on 24 to 28 volts direct current. Figure 7.28 shows two simple power supplies suitable for supplying direct current for control relays. Figure 7.28*a* shows a method of obtaining direct current directly from the 110-volt, 60-cycle a-c line by means of a high-voltage selenium rectifier. Figure 7.28*b* shows a transformer power supply with bridge-connected selenium rectifiers and a ripple filter.

If particularly fast relay operation is desirable, a relay may be fed an initial voltage surge of 3 to 10 times the rated voltage by means of the resistor-capacitor circuit shown in Fig. 7.29.

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[1] This is especially true in machines used only for the solution of sets of linear differential equations with constant coefficients.

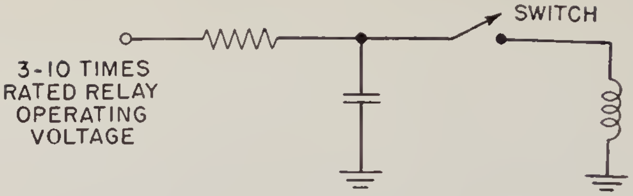

FIG. 7.29. Circuit for fast relay operation.

7.5. SUMMARY OF D-C ANALOG-COMPUTER OPERATING PROCEDURE

The operating procedure for different d-c analog computers will tend to vary depending on the design and function of the machine and especially on the design of the control circuits. Thus, in modern high-quality computers, many necessary control functions may be performed automatically and may not require the operator's attention at all (see Sec. 7.2).

In this section, a number of essential points common to the operating procedure of many d-c analog computers are summarized. This material refers mainly to flexible, multipurpose differential equation solvers of the "slow" d-c analog-computer type.

Placing a Computer in Operation. Before attempting to connect the computer to the power line by throwing the main power switch, computers employing simple switch-type control systems (see Sec. 7.2) require a check of the switch positions in both the power-supply and operating control systems. The operating manual should contain a *check list*, such as

"Before placing the computer in operation . . . check the following switch positions: Filament Switch OFF; High Voltage Switch OFF; Reset-Compute-Hold Switch on RESET; Recorder Switch OFF . . . ."

In the case of intelligently designed push-button control systems, no such check will, in general, be necessary.

The computer is connected to the power line by throwing the main power switch (usually a wall switch) to the ON position. The filament voltage is then applied directly or by subsequently throwing an additional filament switch. After a proper time delay (which may or may not be enforced by a time-delay relay), plate voltage may be applied to the computer. In some installations, the filaments run continuously.

Permit the computer to warm up for at least 30 *min with filament and plate voltage* ON *before beginning computation.* This time may be used to set the computer up for new problems, if desired. In the following, it will be assumed that all scale factors have been tentatively chosen, that a tentative computer setup has been made in accordance with a block diagram as outlined in Chap. 2, and that all potentiometers, initial conditions, and limiting levels have been set up according to the block diagram.

Preparation for a Computer Run. After the high-voltage power supplies have been turned on, the power-supply voltmeters may be checked for correct indications. If the computer has d-c amplifiers which require manual balancing, each d-c amplifier must be balanced in turn by switching it into the BALANCE condition and by adjusting the balance controls until zero output voltage is read for zero input voltage. This balancing process should be repeated every half-hour or, preferably, after every two or three computer runs. In the case of d-c amplifiers employing automatic balancing circuits (see Sec. 5.6), the auxiliary balancing controls may be touched up once a day or once a week.

After a final check of the connections (see Sec. 2.9), the computer is ready for the first test run. The computer is placed in the COMPUTE condition by operating the *compute* switch or push button. If the flashing of one or more of the overload or error-indicator lights indicates that the corresponding amplifiers are overloaded or not operating properly, changes in the computer setup or scale factors must be made to correct this situation.

When changing scale factors, one must be sure to record this fact on the block diagram. Do not forget to readjust all initial-condition potentiometers as well as all limiting levels and function-generator settings corresponding to the new scale factors.

Recorder Adjustment. Depending on the type of recorder used (see Sec. 7.3), the recorder scale factor may be set initially, or the recorder may require calibration in the RESET condition. The recorder zero levels also require checking (see also Sec. 7.7).

Computation. In general, each computer run is started by operating the *compute* switch or push button. The computer run may be terminated by placing the control switch in the RESET position or by actuating the *reset* or *stop* push button; or, again, the machine may return to the RESET condition through the action of an automatic timer. This procedure is repeated for each run. It may or may not be necessary to turn the recorders on and off separately for each run (see Sec. 7.2). In some computer laboratories, each run is duplicated in order to check precision and as an easy means for obtaining duplicate records.

7.6. CHECKING OF SOLUTIONS

Accuracy checks and estimates of the errors involved in analog-computer solutions are essential if the results obtained by the computing machine are to be relied on. The accuracy of the computed solution will depend intimately on the problem in question.[1]

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[1] Korn, G. A., The General Difference Analyzer, *Mathematical Tables and Other Aids to Computation*, January, 1952; Miller, K. S., and F. J. Murray, A Mathematical Basis for an Error Analysis of Differential Analyzers, *J. Math. and Physics*, **32**: 136, 1953.

It is frequently possible to compare a number of special cases of each set of analog-computer solutions with the corresponding correct solutions obtained numerically or analytically. Agreement of the analog-computer solutions for the special cases with the correct solutions will generally indicate a comparable accuracy in the case of the remaining analog-computer solutions unless the latter should be more sensitive to the effects of imperfections in the computing elements than the test solutions. This last question must be considered in the light of the nature of the solutions, and the special cases must be chosen judiciously.

Assuming, as will be most frequently the case, that the d-c analog computer is intrinsically capable of solving a given problem with the required accuracy and that the scale factors and the computer setup have been chosen so as to make the most of the capabilities of the computer, the validity of the solutions will depend only on the proper functioning of the machine. It will then be useful to have methods for checking the computing elements for proper individual and combined operation.

Some Checking Methods (see also Sec. 7.7). Gunning and Mengel[1] suggest the following simple checks:

1. A simple minimum check consists in introducing suitable test voltages as initial conditions in all integrators of the completed computer setup and checking the output voltages of all amplifiers and function generators against computed values, with the machine in the RESET condition (*static check*).
2. It is relatively easy to repeat a number of computer runs using a set of *new machine variables* (voltages) equal in value to those used in the original computation but having *opposite signs*. A comparison of the results of such check runs with the original results frequently tends to expose errors, particularly errors due to drift or saturation of d-c amplifiers in the computer.
3. Doubling the computing time by doubling the values of all integrating capacitances will tend to show up errors due to bad high-frequency response of computing elements.

Checking by Resubstitution. Satisfactory results from the aforementioned simple checks will constitute necessary but not sufficient conditions for correct solutions. For a rigorous accuracy check, it will be best to utilize numerical methods which involve a resubstitution of the analog-computer solutions into the given mathematical relations. Such a method is particularly useful for checking solutions of sets of linear simultaneous equations (see Sec. 2.11).

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[1] Gunning, W. F., and A. S. Mengel, *Report RM* 236, RAND Corporation, Santa Monica, Calif.

In the important case of problems involving the solution of sets of ordinary differential equations, Gunning and Mengel advise *numerical checks or improvements of the machine solutions based on Picard's process of successive approximations* whenever this is possible. One reduces the given differential equations to the form

$$px_i = f_i(x_1,x_2, \ . \ . \ . \ ,x_n,t) \qquad (i = 1, 2, \ . \ . \ . \ , n) \tag{7.10}$$

Equations (7.10) are used to compute the functions px_i of the independent variable t from the solutions $x_1, x_2, \ . \ . \ . \ , x_n$ obtained from the analog computer. Numerical integration[1] may then be used to compute each x_i from the corresponding function px_i; the new solutions may be used to *check* or *improve* the machine solutions. This method is useful also for problems so complicated that only an approximate d-c analog-computer solution is possible.

7.7. D-C ANALOG COMPUTERS WITH AUTOMATIC PROGRAMMING FEATURES

Some of the larger d-c analog computers incorporate semiautomatic features which facilitate coefficient settings and checking of computer setups; other circuits permit the machine to perform a planned series of computations with only nominal supervision.

Simplified Methods for Setting Constant Coefficients. One may set coefficient-setting potentiometers quickly and accurately by applying a reference voltage and reading the output voltage on a servo or digital voltmeter. No loading correction (Sec. 4.1) is needed if the load remains connected between the potentiometer arm and ground, or (if the load is a summing resistor) between the potentiometer arm and the summing junction of an operational amplifier. In order to avoid amplifier overloads in the latter case, one may provide special relays which ground all summing junctions during coefficient setting (POT SET condition in Electronic Associates computers).

The arm of each coefficient-setting potentiometer can also be *positioned automatically* by a servomechanism controlled by a central keyboard or punch-card reader (see also Fig. 8.5). A single servomechanism can set as many as 50 potentiometers selected by small remote-controlled magnetic clutches.

Automatic Recorder Calibration. Oscillograph-type recording devices are easily fitted with relay circuits which successively (1) ground the recorder input terminals to indicate the reference line (which may be displaced as desired), (2) apply a calibrating voltage to each channel, and

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[1] See A. S. Householder, *Principles of Numerical Analysis*, McGraw-Hill, New York, 1953, for various practical methods of numerical integration.

(3) actuate a small printer which prints an identifying number on each record, before or after each computer run.

Preset Automatic Programming. A series of d-c analog-computer runs involving *preset successive changes in coefficient settings or computer setup* may be performed automatically with the aid of timer-controlled switches or stepping relays.[1] Such switching devices can successively initiate and stop the first computer run (see also Sec. 7.2), actuate relays which change parameter values or computer setup, start the second computer run, etc. Arrangements of this type are particularly useful for studies requiring investigation of many parameter values.

Special Applications of the HOLD Condition. An integrator in the HOLD condition constitutes an analog *storage device* which "remembers" its last-computed output voltage. A number of interesting applications of the HOLD condition are listed below.

1. *Digital Readout.* The computer may be placed in the HOLD condition periodically or at predetermined time intervals to permit an analog-to-digital converter[2] to "read" the values of selected machine variables for checking, printing, card-punching, or data transmission.
2. *Scale-factor Changes.* Scale-factor changes or other setup changes may be introduced in the HOLD condition after the computer has run for a predetermined time interval. Similar action may be initiated automatically whenever selected machine variables exceed or fall below predetermined values: comparator circuits (Table 6) can actuate relays which place the computer in the HOLD condition, introduce suitable setup changes, and restart the computation.[3]

Sequential Automatic Programming. Even more sophisticated computing methods, similar to those used with digital computers, become available if the hold and/or reset relays of individual integrators or groups of integrators can be controlled *separately.* This is easily done if the relay connections are brought out to the computer patch bay. An integrator can be used to store a given variable even though its derivative does not appear in the computer setup (Fig. 7.30).[4]

Selected integrators remaining in the HOLD condition during the RESET portion of an automatic programming cycle will introduce stored results of the preceding computer run as initial conditions into each new computer run. Such integrators can also be connected to actuate relays or position servomechanisms during the RESET portion of the cycle, so that computer setup and/or parameter values are made dependent upon past results. Such circuits permit automatic step-by-step solution of difference equations[1] as well as the use of automatic *iteration methods* for the solution of equations and of eigenvalue and boundary-value problems[2] (see also Sec. 3.10). Computer setups of this kind will be most useful when many similar problems of a suitable type must be solved, such as in special-purpose computers. Practical iteration methods can often be developed from those used in numerical analysis.

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[1] Hansen, G. R., The Time-sequence Controller for Automatic Operation of the Electronic Differential Analyzer, *Memo* 20-81, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif., Mar. 2., 1953.

[2] Burke, H. E., A Survey of Analog-to-digital Converters, *Proc. IRE*, **41**: 1455, 1953.

[3] Automatic Scale-change Attachment for GEDA Computers, *GEDA Bulletin* 54-3 (*GER*-5839) of May 13, 1954, Goodyear Aircraft Corporation, Akron, Ohio.

[4] Wadel, L. B., An Electronic Differential Analyzer as a Difference Analyzer, *J. Assoc. for Computing Machinery*, July, 1954; see also G. A. Korn, *op. cit.*, and Gunning and Mengel, *op. cit.*

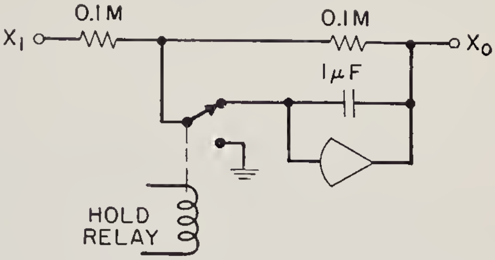

FIG. 7.30. Separately controlled "storage integrators" can be *reset* (Fig. 7.4*a*) with the rest of the computer in the HOLD condition and *hold* (Fig. 7.4*c*) their output voltages in the COMPUTE or RESET condition. Thus results of earlier computer runs can be introduced into subsequent computations.

Automatic Checking Circuits. The simplest automatic method for checking the computer setup and operation of computing elements involves successive digital readout and printing of all potentiometer and amplifier output voltages in the RESET condition. This method corresponds to the "static check" described in Sec. 7.6 and checks potentiometer settings, summing amplifiers, and initial-condition settings.

Improved static checking systems also permit one to switch each integrator summing junction into a summing-amplifier circuit whose output is read out to indicate the *initial rate of change* of the integrated variable. Furthermore, since ordinary static checks cannot properly check points whose initial voltage is zero, one may introduce *artificial initial conditions* during a preliminary static check followed by the ordinary static check with the true initial conditions.[3] Similar tests may also be applied in mid-problem with the computer in the HOLD condition.

An alternative static-checking scheme ("Static Problem Check" in Reeves computers) places a feedback resistor across each integrating capacitor and simultaneously substitutes a test voltage for each integrator output voltage.[4]

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[1] *Ibid.*

[2] Wadel, L. B., Automatic Iteration on an Electronic Analog Computer, lecture presented at the Western Electronics Show and Convention, Los Angeles, Calif., 1954.

[3] Problem Checks, *Memo* dated Apr. 20, 1955, Electronics Associates, Inc., Long Branch, N.J.

[4] McCoy, R. D., and B. D. Loveman, Problem Checker Checks Computer, Too, *Control Eng.*, July, 1955.

Integrator operation, which cannot be checked in the RESET condition, can be tested by switching circuits which permit each integrator to integrate a known voltage for a known time ("dynamic" check). The hold relays are then actuated, and the integrator output voltages are read out successively.

Automatic checking systems produce the results of periodic checks on computer setups and component failures in the form of printed records. Such methods can be very useful, particularly if the computations in question are extensive. One should, however, remember that the checks described are all essentially static checks; they cannot reveal instability, phase shift, or large effects of small computing errors *during actual computation.* Automatic checking methods are not intended as substitutes for numerical checks of analog-computer solutions.

CHAPTER 8

THE DESIGN OF COMPLETE D-C ANALOG-COMPUTER INSTALLATIONS

8.1. COMPLETE D-C ANALOG-COMPUTER INSTALLATIONS: COMPOSITE DESIGN

Before the design of a computing machine is attempted, it will be necessary to note, preferably in writing, the purposes for which the proposed computer installation is to be utilized. This will help to determine

1. The time scale or scales to be used
2. The accuracy and dynamic range needed for each mathematical quantity
3. The type and number of mathematical operations to be performed
4. The types of associated equipment which might be used in connection with the computer; also special requirements as to power supply, size, weight, etc.
5. The amount of money, time, and effort which can be spent economically on the computer installation, considering the type and number of computations to be performed

These considerations may or may not point to the use of a d-c analog computer. In the first case, they will form the basis of a rough choice of the d-c analog-computing elements to be built.

As the design takes shape, various combinations of computing elements will be considered from the point of view of the stated requirements and reconsidered with respect to the possible choices of power supplies and other associated equipment. After a number of such "go-arounds," the composite design will have progressed to a stage where the design of individual components can be considered in more detail.

The range of working frequencies, as determined by the types of expected solutions and the time scales used, together with considerations of cost and accuracy, will determine the choice of *multipliers* and *function generators* to be used.

The design of the *operational amplifiers* will constitute a significant portion of the over-all computer design. Decisions on the following questions, among others, will be required:

1. *D-c amplifier gain, frequency response, drift, and output requirements.*
2. *What types of networks are needed?* Will they be permanently installed or will interchangeable plug-in components be used? In the case of a multipurpose computer, the most desirable combinations of summing and feedback resistors, integrating capacitors, etc., must be determined. Coefficient-setting potentiometers are convenient but increase the amplifier output requirements.
3. *Will precalibrated precision network components be used, or will the computer incorporate a calibrating device?* The latter arrangement combines flexibility and low cost with a slight reduction in operating convenience.
4. *How will the computing networks be arranged physically with respect to the d-c amplifiers and the computer front panel?*

Consideration of these questions will usually result in compromises between cost and accuracy. The designer is now in a position to chose *power supplies* (Sec. 7.4), *control circuits* (Secs. 7.1, 7.2, and 7.7), and *recording equipment* (Sec. 7.3). Finally, it will be necessary to specify the *physical construction* of the computer (Sec. 8.2) and the *front-panel layout* (Sec. 8.3) with a view to convenience of operation and maintenance.

Accuracy Requirements. Solutions of about slide-rule accuracy are sufficient for many engineering applications. Better accuracy is required for trajectory computations (Sec. 3.5), especially if coordinate conversions are involved (Sec. 6.11). In other applications, such as in studies of nonlinear oscillations, small errors may cause radical changes in the *type* of solution (mode changes). Although accurate computing elements are helpful in such situations, *computer accuracy is not a substitute for an investigator's professional knowledge.* Thus it is much better to test the effects of variations in the critical parameters than to trust blindly in computer results. The more significant results of analog-computer studies may be checked by numerical computation.

Since it is not a simple matter to predict errors due to the combined operation of several computing elements, the accuracy of a proposed or existing analog computer is usually specified in terms of the *component accuracies* of its individual computing elements. Component accuracy is expressed in volts, machine units, or per cent. Percentage accuracy is usually referred to the full output range of the computing element in question.[1] High-quality d-c analog computers have component accuracies between 0.005 and 0.5 per cent of full scale (see also Chaps. 4 to 6).

The accuracy of the final results obtainable with an analog computer having given component accuracies depends radically on the specific problem, on the computer setup, and on the scale factors used. In particular, solutions of linear differential equations with constant coefficients will be most accurate if the solutions involve only well-damped modes (Sec. 4.7); for this reason, d-c analog computers of relatively low cost are suitable for linear servo analysis.[1]

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[1] "Full scale" equals 200 volts for computing elements having a plus and minus 100-volt operating range.

Computer *precision* relates to the repeatability of solutions. Percentage differences between repeated solutions indicate the effects of component changes, noise, and drift.

Miscellaneous Design Considerations. Component accuracy is only one of several factors of interest in d-c analog-computer design. The essential purpose of many d-c analog computers is to aid the thinking processes of fairly high-priced engineering personnel, rather than just to produce numerical results. Such machines should, therefore, be designed *so that the user can concentrate on his problem rather than on the computer.* The computing elements should require as few critical adjustments as possible, and the machine should give positive indication of overloads or failures. Modern control circuits (Secs. 7.1, 7.2, and 7.7), convenient patch bays (Sec. 8.3), and automatic checking features (Sec. 7.7) account for an appreciable part of the cost of larger d-c analog-computer installations.

A design emphasizing *reliability* and *maintainability* will reduce downtime and maintenance costs. The various components of a computing machine must be carefully chosen and modified so as to work together as parts of an integrated system. Nevertheless, individual chassis units should be capable of operating independently of other components as far as possible. If this rule is observed, it will be easier to test and service individual subassemblies, to expand the computer installation, and to make design changes in individual components.

8.2. CONSTRUCTION, WIRING, AND INSTALLATION

The mechanical construction of any computer should emphasize ruggedness and accessibility for easy maintenance. D-c analog computers should be constructed, whenever possible, of commercially available components. Standard enclosed relay-rack construction is, in general, a good choice for laboratory-built installations; factory-built computers may use specially designed cabinets or consoles. Compact arrangements of electronic components must provide for good ventilation. In larger computers, it is good practice to separate power supplies and a-c motors physically from the electronic circuits in order to minimize hum pickup. Lightweight mobile installations may employ aircraft-type racks and chassis and miniaturized electronic circuitry.

[1] McDonald, D., Analog Computers for Servo Problems, *Rev. Sci. Instr.*, **21**: 154, 1950.

The choice of component parts as well as the construction of specific subassemblies is treated in the appropriate sections of Chaps. 5 to 7 (see also Sec. 8.3).[1]

Wiring Practice. No specialized wiring techniques are required for the construction of d-c analog computers. Some emphasis should be placed on rugged wiring, so that slight shocks and vibrations cannot change circuit constants. As a matter of course, reasonable safety precautions should be observed, and all wiring and fusing should be done according to the Underwriter's Laboratories' specifications.

In order to save time and labor, it is well to give some thought to "production engineering" techniques when a number of identical amplifiers or other computing elements are to be wired. Such techniques include the use of subpanel terminal strips for mounting resistors and of printed-circuit subassemblies.

Interunit Wiring. All wiring interconnecting the various component units of a d-c analog computer usually falls into one of the following general classifications:

1. *Ground straps* (*Sec.* 5.9). *It is not good practice to rely on chassis grounds in d-c analog computers.*
2. Filament and high-voltage *power-supply wiring* for electronic components. High-voltage power-supply wiring in common to more than one amplifier should preferably have an impedance of less than 1 or 2 ohms in order to minimize coupling between amplifiers.
3. *Control wiring* for making connections to the reset and hold relays on integrators and to other relays controlled by a common timer, etc., as well as wires leading to special overload indicators.
4. *Metering system wiring* may be necessary in computers possessing a central system for checking, balancing, and calibration.
5. "*Signal*" *wiring* carrying the computer voltages (machine variables) proper between the computing elements and a central patch bay. The use of shielded d-c amplifier input leads is recommended (see also Sec. 8.3).

All connections between the chassis units and interunit wiring should be made by means of high-quality plug-in connectors which must permit easy removal of individual chassis as well as positive connections.

Installation. While the spirit of research may be dominant in a computer laboratory, it is sometimes wise to remember that a finished computing device is no longer a breadboard. It is absolutely necessary to observe all safety regulations specified by the Underwriter's Laboratories with respect to mechanical as well as electrical installation of the computing equipment. The floor beams of some office buildings may not be able to support the concentrated weight of relay racks containing heavy power supplies. In such cases it is best to place the computer cabinets on a low wooden platform which will distribute their weight. Interconnections between different computer cabinets can be very conveniently arranged underneath such a platform.

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[1] See also J. F. Blackburn, *Components Handbook*, MIT Radiation Laboratory Series, Vol. 17, McGraw-Hill, New York, 1949.

All power connections to the computer except possibly for service outlets should be through a main wall switch containing the necessary fuses and circuit breakers. Instead of relying on the high-voltage indicator lights for the protection of maintenance personnel, it may be useful to provide a green safety light which is on only when the main wall switch is *off*.

Sufficient space should be allowed in front of the relay-rack units for unhampered operation and in back of these units for convenient maintenance. If at all possible, the room containing the computer should be devoted to the computer and associated equipment alone, so that the operating personnel will not be disturbed unnecessarily in their somewhat complex task. Sound insulation of the ceiling also helps in this respect. The computing room might also contain a desk, blackboard, drawing board, and a bulletin board. A large table on which to unroll and examine computer records is helpful. It may be desirable in some installations to place the recorder on a rubber-tired cart which can be rolled into position wherever the operator happens to be working.

A large computer installation will generate an appreciable amount of heat, so that some attention will have to be given to the air conditioning of the computer room. The accuracy of high-quality installations will be improved if the ambient temperature is kept reasonably constant (within about 10°C) by a thermostat controlling the air-conditioning system; low humidity is also desirable.

Aside from purely technical considerations, the careful planning of a pleasing and efficient electronic computer installation will be repaid in terms of more intangible assets. A computer installation may be made a show place and, as such, may prove to be of considerable sales value to the organization operating it.

8.3. FRONT-PANEL LAYOUT AND PATCH-BAY DESIGN

The components on the front panel of a d-c analog computer serve three functions: computer adjustment, computer setup, and computer operation. An elegant and well-thought-out front-panel layout permitting the operating personnel to perform these functions easily and conveniently is a very important design consideration. It is usually a good plan to combine all operating controls and indicator lights on a special control panel. Overload lights can be mounted on the same panels as the corresponding d-c amplifiers or servomechanisms. Some designers prefer to separate power controls and operating controls.

It is very helpful to duplicate the computer start, compute, hold, and reset controls on a hand-held *remote-control box* (see also Sec. 7.2 and Fig. 8.3) which enables the operator to start and stop the computer while he is changing the setup, adjusting parameters, checking the recording equipment, or making calculations on the blackboard. The indicator lights on the computer front panel should be sufficiently large to be readily seen from any position in the laboratory, and an audible overload indication may be provided.

Adjustment controls, such as d-c amplifier balancing controls, servo gain and damping controls, and power-supply voltage adjustments, should be equipped with shaft locks or recessed underneath protective covers, so that they cannot be accidentally disturbed. The same precautions apply to the adjustment controls of diode function generators or tapped function potentiometers.

Coefficient-setting potentiometers should be arranged within easy reach of the operator. Potentiometer dials must be easily readable; each dial must have a clearly visible label associating the potentiometer with its number in the block diagram.

Patch-bay Design. In small multipurpose d-c analog computers, patch-cord connections can be made directly to jacks on the front panels of the various computing elements. Although such a design combines low cost with freedom from cross talk and leakage between the different terminals, most modern multipurpose machines employ a central *patch bay*, which offers greater operating convenience and permits computing elements to be serviced or removed without interference with the computer setup.

Most patch bays incorporate *removable problem boards* (*prepatch panels*). Problems are patched on these problem boards rather than on the computer, which is thus freed for other work. The problem boards are plugged into the computer for computation; the boards can be stored to preserve computer setups for later reuse. Since much of the total time spent in using any computer is devoted not to actual computation but to the evaluation of solutions and to computer setups for new problems, multiple problem boards can increase the utility of a computer installation very appreciably.

D-c analog-computer problem boards can accept plug-in network components as well as patch cords (Fig. 8.1). The better types of problem boards do not incorporate actual jacks but merely hold the various plugs so that their tips protrude through the board. When the problem board is inserted into the machine, a locking lever raises all plug tips against corresponding contact springs with a self-cleaning wiper action (see, for instance, Fig. 8.1*a*).

The design of d-c analog-computer patch bays and the associated wiring must minimize leakage and cross talk. Leakage from any ungrounded patch-bay terminal into an integrator summing point acts like an additional input connection, so that leakage resistances must exceed 10^5 megohms under all conditions of temperature and humidity. Since reasonable leakage to ground is not objectionable (see also Secs. 4.5 and 5.9), leakage effects can be eliminated by metal problem boards, grounded metal barriers between patch-bay terminals, or grounded printed silver grids on plastic patch-bay bases and problem boards. Suitable plastic masks will prevent printed grids from shorting "hot" patch-cord tips.

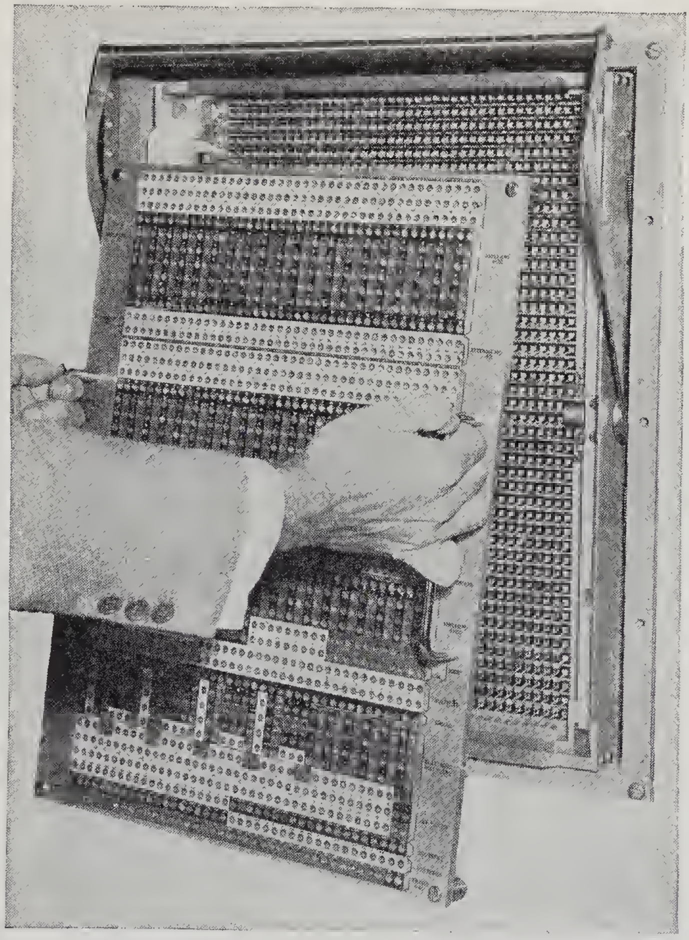

FIG. 8.1*a*. This metal prepatch panel grounds the shields of the coaxial patch cords inserted into appropriate holes. The plugs themselves protrude through the panel; the large locking lever raises them against corresponding contact springs with a self-wiping action. To protect the problem board against accidental contacts with "hot" patch-cord tips, the operator removes all voltages from the patch cords by depressing the locking bar while changing connections. As an added safety feature, all potentiometer taps are protected by fuses (Electronic Associates, Inc.).

Computers with metal problem boards or coaxial jacks permit the use of shielded patch cords to minimize cross talk, but the problem board must be protected against accidental contacts by "hot" patch-cord tips (see also Figs. 8.1*a* and 8.5*b*).

In the writers' opinion, many existing arrangements of patch-bay terminals leave much to be desired. As a rule, the terminals of all amplifiers, all potentiometers, etc., are grouped together so as to conserve panel space. A survey of actual computer setups (Chaps. 2 and 3) suggests improved arrangements approximating the sequence of computing elements in typical computations. Such arrangements could do away with many long patch-cord connections; succeeding computing elements would be connected by short patch cords or, preferably, by rigid links (dual plugs). Intelligently placed buses (multiples) similar to those used in the Boeing analog computer (Sec. 8.4) would further reduce the number of long patch-cord connections. Computer setups of this type would be far easier to work with than the often bewildering maze of patch cords typical of existing installations. Since several modern computers permit patch-bay terminals to be assigned at will, each user might well choose a terminal arrangement particularly suited to his own requirements. The writers also suggest the use of special multiple-plug assemblies or "prepatch panels" for frequently used circuits, such as special operational amplifiers, limiters, comparators, etc.

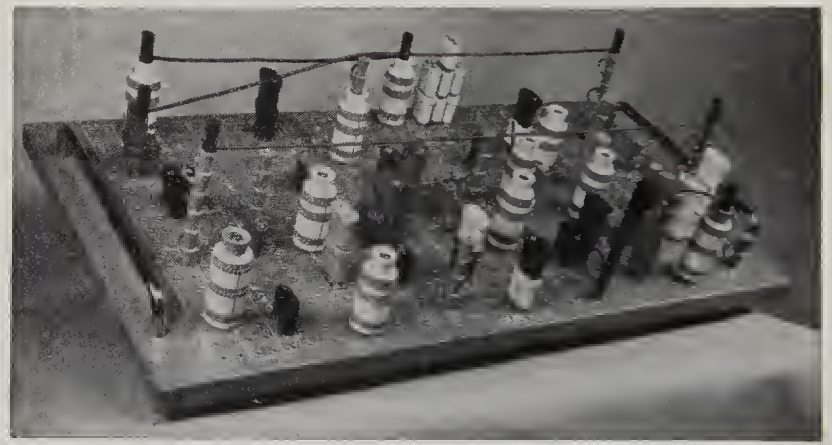

FIG. 8.1*b*. GEDA problem board with stacked plug-in network components for operational amplifiers. Note the plug-in decade resistors (Goodyear Aircraft Corporation).

Programming systems controlled by *switches* and/or by *punched cards* have been moderately successful in some d-c analog-computer installations. In particular, keyboard-set potentiometers (Sec. 7.7) or decade attenuators are easily controlled by card-reading apparatus. Such equipment can justify its high initial cost in applications requiring many coefficient values to be stored for frequent reuse. Thus, in large flight simulators, a single card may insert an entire set of aircraft characteristics into the computer. The experimental Dynamic Systems Synthesizer (DSS) developed by the RCA Laboratories Division[1] for the United States Air Force permits the programming of entire computer setups by specially designed card readers, or by card readers used in conjunction with accessory patch bays.

8.4. EXAMPLES OF COMPLETE D-C ANALOG COMPUTERS

Commercially available multipurpose d-c analog computers built in the United States[2] have a number of common features which it is well to survey in order to avoid unnecessary repetition later on. Most purchasers of such equipment begin by buying a *basic computer unit* in the form of a cabinet or console containing *operational amplifiers, coefficient-setting potentiometers, diodes* suitable for use in fabricated diode circuits, *power supplies, control circuits*, and the *computer patch bay.* This basic computer installation can be *expanded* according to each user's special requirements. Additional equipment is mounted first in the main cabinet and, eventually, in accessory cabinets connected to the central patch bay (Fig. 1.1).

The d-c amplifiers of American-built machines permit the use of direct-analog methods (Sec. 1.5) as well as the classical differential-analyzer technique, although individual manufacturers may emphasize one or the other approach. Wire-wound computing resistors are used almost exclusively. Resistance changes due to ambient-temperature variations are minimized through the use of special resistance wire and/or through

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[1] Princeton, N.J.

[2] The examples of Sec. 8.4 describe commercially available equipment, but it is only fair to point out that much of the development of d-c analog computers in this country can be attributed to projects supported by the United States Government. Some of the obscure beginnings of d-c analog techniques can be traced to wartime radar and fire-control circuitry developed in the United Kingdom as well as in this country. Many early computer d-c amplifier designs were based on circuits used in the Bell Telephone Laboratories' M-9 fire-control computer. The original Goodyear and Reeves computers were, respectively, developed under contracts with the United States Air Force and the United States Navy; the Navy's CYCLONE computer center, operated by the Reeves Instrument Corporation, has long been a proving ground for d-c analog techniques. The RCA Laboratories' TYPHOON flight-simulator project, developed under a Navy contract, has been a veritable fountainhead of improved computing elements (Secs. 5.6, 6.2, 6.3, and 6.4), and the same organization's Dynamic Systems Synthesizer (DSS), an Air Force project, involves component developments of similar importance (Secs. 5.8, 6.4, and 8.3). The RAND Corporation's imaginative redesign of an early commercial d-c analog computer and the Curtiss-Wright computer components built for the Applied Physics Laboratory, Johns Hopkins University (Figs. 8.2 and 8.3), were other government-supported projects which have contributed to the development of computing elements and control circuits.

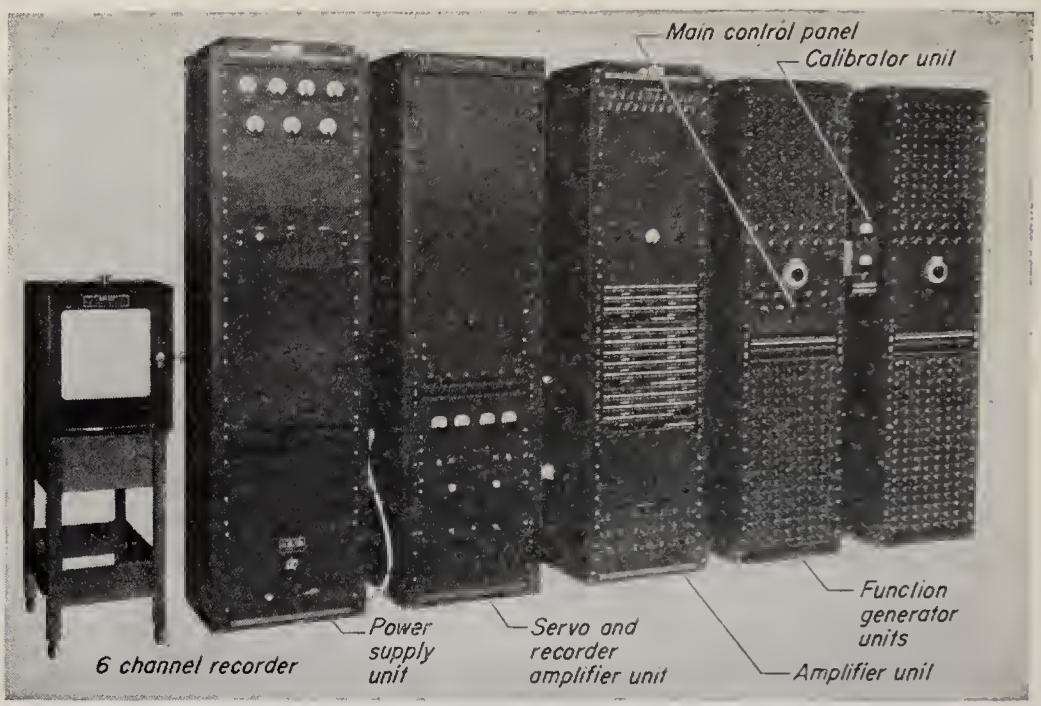

FIG. 8.2. The Curtiss-Wright analog computer (CURTIAC).

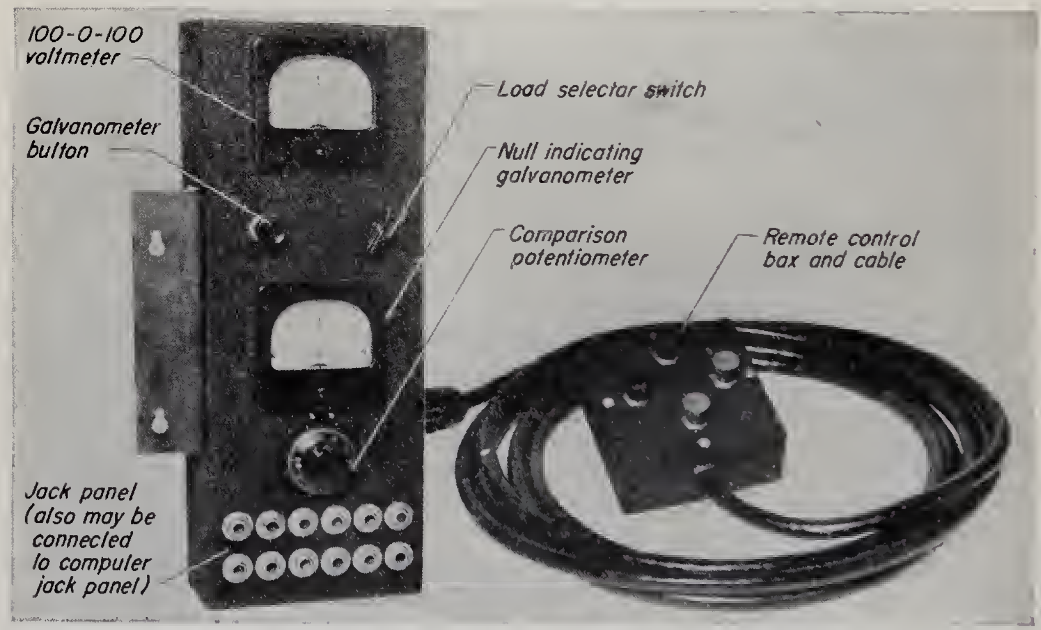
100-0-100 voltmeter
Galvanometer button
Load selector switch
Null indicating galvanometer
Comparison potentiometer
Remote control box and cable
Jack panel (also may be connected to computer jack panel)

FIG. 8.3. Curtiss-Wright CURTIAC calibrator unit and remote-control station.

encapsulation of resistors in plastic; several manufacturers employ constant-temperature ovens. All machines using plug-in resistors and capacitors provide an accessory calibration bridge capable of measuring transfer functions and/or impedances.

With the exception of the smallest table-top units, all American-built d-c analog computers can be supplied with *computer servomechanisms* for multiplication and function generation by means of ganged potentiometers. All manufacturers provide *electronic multipliers* (usually based on time-division circuits, Sec. 6.4), *electronic function generators* (usually diode function generators), *reference power supplies,* and *operational relays* as standard or optional accessories. Most commercially available machines have *patch bays with removable problem boards* (Sec. 8.3); the larger and more elaborate d-c analog computers can be furnished with automatic programming features such as *keyboard-set potentiometers, digital readout,* and *automatic checking circuits* (Sec. 7.7) as optional equipment.

The following descriptions of individual commercially available d-c analog computers will indicate how different designers have dealt with the design problems previously discussed.

Small Computer Installations. Low-cost d-c analog computers may merely be sets of individual two- or three-tube d-c amplifiers operated from a common power supply and interconnected by patch cords.[1] A preferable arrangement employs plug-in amplifiers grouped on a larger chassis whose front panel serves as a simple patch bay[2] (see also Fig. 5.31). An excellent report by Hagelbarger, Howe, and Howe[3] shows page after page of remarkably accurate results from dozens of elegant applications of a simple table-top d-c analog computer (see also Chaps. 2 and 3).

The *Donner*[4] *Model* 30 *analog computer* shown in Fig. 1.2*a* comprises 10 operational amplifiers (Fig. 5.29*c*), all necessary power supplies and controls, and removable problem boards for either plug-in or solder-in networks and connections. The small machine has reset and hold circuits for 5 integrators; two or more computers can be interconnected to operate together. An accessory cyclic-reset timer permits repetitive operation. Other optional accessories include coefficient-setting potentiometers, a calibration bridge, a time-division multiplier (Fig. 6.16), a simple logarithmic multiplier (Sec. 6.5), a diode function generator (Fig. 6.26*c*), and a low-frequency sine-wave generator. These accessories are self-contained and can be used with other analog computers. Specified component accuracies vary between 0.1 per cent of full scale (±100 volts) for operational amplifiers operated under the most favorable conditions and about 2 per cent for the logarithmic multiplier.

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[1] Raggazini, J. R., R. H. Randall, and F. A. Russell, Analysis of Problems in Dynamics by Electric Circuits, *Proc. IRE,* May, 1947, pp. 444–452.

[2] MacDonald, D., *op. cit.*

[3] Hagelbarger, D. W., C. E. Howe, and R. M. Howe, Investigation of the Utility of an Electronic Analog Computer in Engineering, *Report UMM*-28, Aeronautical Research Center, University of Michigan, Ann Arbor, Apr. 1, 1949.

[4] Donner Scientific Company, Berkeley, Calif.

The substantially more expensive *table-top REAC* shown in Fig. 8.4 is a compact version of the REAC described on p. 406 and uses similar high-quality d-c amplifiers.

The *Heath*[1] *table-top analog computer* (Fig. 1.2*b*) is a complete flexible computer unit distributed as a do-it-yourself kit for less than $700. The machine employs plug-in feedback networks and features one of the best patch-bay layouts yet designed for this purpose. Resistors and capacitors can be plugged directly between successive d-c amplifiers, so that relatively few long patch-cord connections are needed (see also Sec. 8.3).

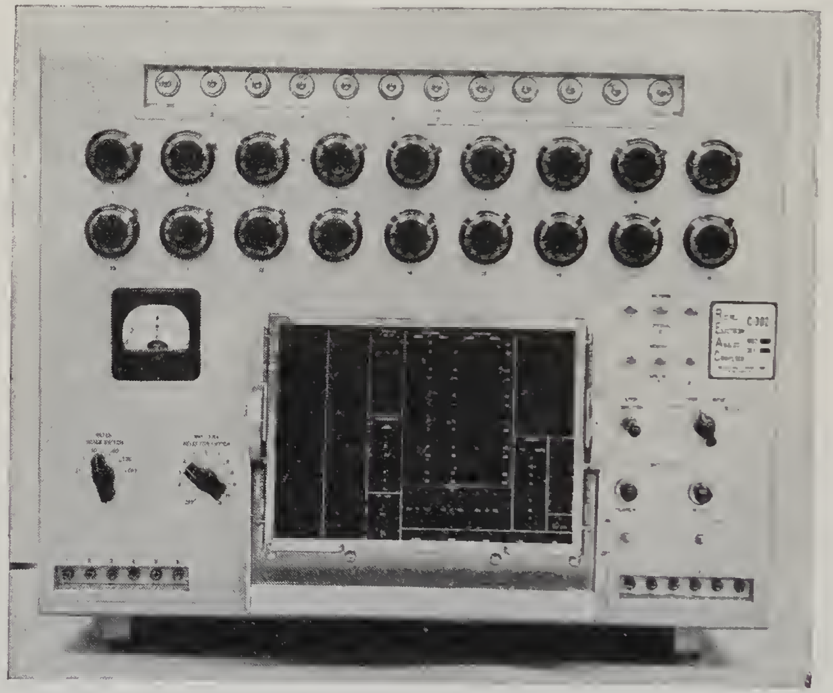

FIG. 8.4. This compact table-top REAC comprises 12 chopper-stabilized d-c amplifiers, 18 coefficient-setting potentiometers, 8 diodes, power supplies, all control circuits, and a patch bay with removable problem board.

The basic unit comprises 15 d-c amplifiers, a number of diodes and multipurpose relays, and 30 coefficient-setting potentiometers. A diode function generator is available as an accessory kit.

The low-cost potentiometers used in the Heath computer are set, with the load connected, by the null method of Fig. 4.6 with the aid of a switch-set comparison voltage divider and a meter. The same null method permits one to measure the gain of an operational amplifier or to set six small bias supplies used for initial conditions and diode bias. The meter is also available for amplifier balancing and output measurements.

The Heath three-tube printed-circuit d-c amplifier adds a voltage-amplifier stage and a series-balanced output stage to a regenerative stage similar to those shown in Figs. 5.29*c*, 5.30*b*, and 5.30*c*. This arrangement provides high gain (over 50,000) and an output voltage range in excess of plus and minus 100 volts at 10 ma but requires three power supplies and a relatively large number of parts.

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[1] Heath Company (Division of the Daystrom Corporation), Benton Harbor, Mich.

The Boeing[1] Electronic Analog Computer. The design of Boeing's basic computer unit strikes a particularly fortunate balance between component accuracy, operating convenience, and low cost (the price of a basic unit is of the order of $5,000). This compromise has been achieved through the use of an elegant regenerative d-c amplifier circuit (Fig. 5.32), through restriction of the operating range to plus and minus 50 volts, and through very simple mechanical construction with a clever front-panel layout. Multiple installations of Boeing computers, improved by the addition of precision feedback networks and either chopper or commutator stabilization, are capable of competing quite seriously with much more expensive machines.

Fig. 1.16*a* shows a front-panel view of the Boeing computer. A standard relay rack has been converted into a "skeleton rack" bearing the plus and minus 300-volt regulated power supplies, a central vertical interconnecting panel, and a control panel. The computing elements are mounted on small subpanels to the left and right of the central interconnecting panel.

The basic unit includes 20 d-c amplifiers, of which 8 can be used only as summing amplifiers or phase inverters, 20 helical potentiometers (0.5 per cent), 12 diodes with bias supplies, 5 reference-voltage sources, and a calibration bridge permitting resistance-ratio settings to within 0.1 per cent. All but 8 amplifiers have reset relays; hold relays are optional. The machine is capable of repetitive operation with reduced accuracy.

Each amplifier panel bears the input jacks of a summing network comprising five 1-megohm resistors permanently connected to the amplifier input terminal. The amplifier input and output terminals are also brought out to jacks, so that various types of operational amplifiers can be set up with the aid of stackable plug-in resistors and capacitors (Figs. 1.16*b* and *c*). Patch cords terminate in ordinary banana plugs.

The central interconnecting panel shown in Fig. 1.16*a* is not permanently connected to the computing elements but contains a number of buses running from top to bottom. These buses, rather than long patch cords, serve to interconnect computing elements located at different levels on the front panel. The buses of several racks may be interconnected. This practical and extraordinarily simple patching system may be supplemented by an optional desk-type patch bay with removable problem boards.

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[1] Boeing Airplane Company, Seattle, Wash.

Besides the basic computer unit, the company furnishes electronic multipliers, diode function generators, and computer servomechanisms employing d-c amplifiers and d-c motors. Each servomechanism permits addition of up to 5 input voltages and can be used as an integrator with d-c tachometer feedback; chopper stabilization is optional.

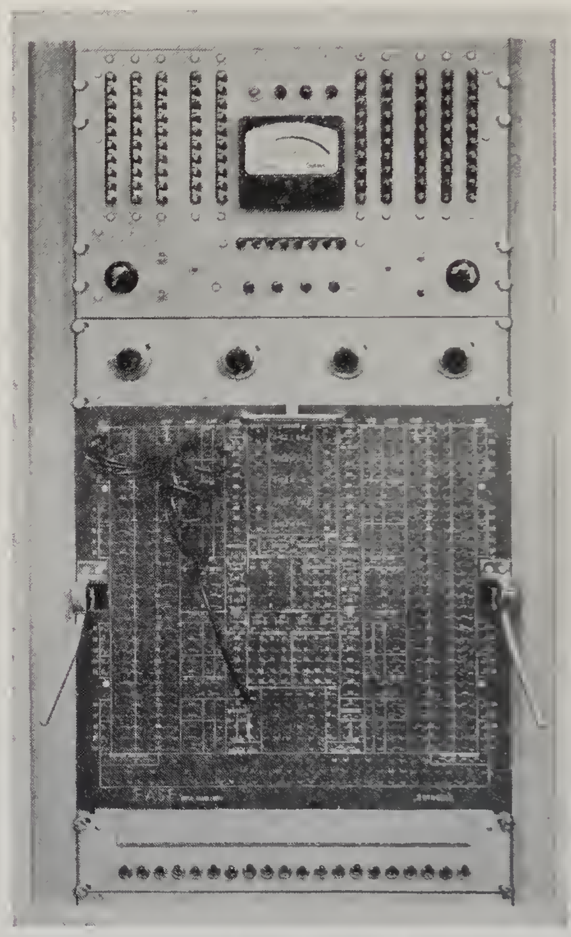

FIG. 8.5*a*. Push-button control panel and patch bay of a Berkeley EASE computer. The control panel comprises a keyboard for automatic setting of coefficient-setting potentiometers. Four manually adjusted potentiometers are mounted on a plug-in panel (just above the removable problem board) which can be stored and reused when problems are changed (Berkeley Division, Beckman Instruments, Inc.).

The Berkeley[1] Electronic Analog and Simulation Equipment (EASE).[2] The catalog of EASE components is somewhat unique in that it includes low-cost operational amplifiers and low-cost time-division multipliers as well as more accurate and expensive computing elements and up-to-date control circuits. An EASE installation offers, thus, interesting possibilities for expansion. EASE components, such as groups of plug-in amplifiers, fit into standard relay-rack cabinets. Plug-in computing resistors and capacitors are used in the smaller installations.

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[1] Berkeley Division, Beckman Instruments, Inc., Richmond, Calif.

[2] The larger commercially available d-c analog computers are described in alphabetical order. In view of the continuing changes in detailed computer specifications, this material should be regarded as a series of examples illustrating modern design practice rather than as a comparative survey of existing equipment.

The simple d-c amplifier and multiplier circuits employed in low-cost EASE installations are very similar to the corresponding circuits shown in Figs. 5.29*c* and 6.16; the higher-priced components incorporate chopper stabilization. The circuit of the EASE diode function generator is essentially that of Fig. 6.26*c*. Other optional accessories include various precision servomechanisms, a digital voltmeter with printed readout, and a computation-timer circuit.

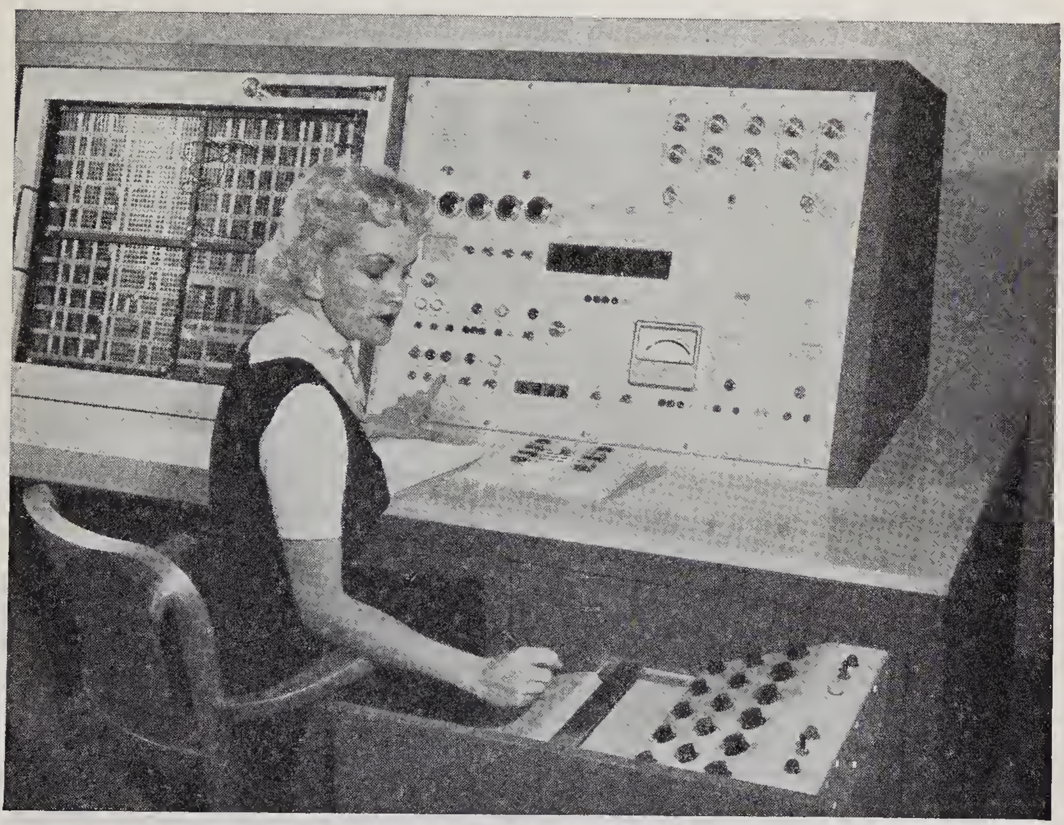

FIG. 8.5*b*. This EASE 1200-series control console provides complete programming and control facilities for a large computer installation. The 3600-terminal patch bay employs metal problem boards ("insulation with metal," Secs. 5.9 and 8.3) of advanced design; special nylon inserts permit the use of shielded patch cords and still prevent accidental grounding of hot patch-cord tips. Note the drawer-mounted recorder.

The push-button control panel of the EASE computer is an interesting example of advanced design (Fig. 8.5*a*). Push buttons permit the operator to select the RESET, COMPUTE, and HOLD conditions (Sec. 7.2) as well as repetitive operation at reduced accuracy. Additional push buttons set the scale of the built-in voltmeter and select amplifier outputs for metering or readout. Finally, the panel permits remote control of selected coefficient-setting potentiometers by means of a keyboard (see also Sec. 7.7). Optional additional equipment permits similar remote settings of the diode function generator.

In larger computer installations, the EASE 1200-series control console (Fig. 8.5*b*) is capable of controlling at least 100 operational amplifiers, 16 computer servos, 30 electronic multipliers, and 350 potentiometers set either manually by means of a keyboard or automatically by a punched-tape reader. The console houses overload-warning and problem-checking circuits (Chap. 7), a digital voltmeter, a drawer-mounted six-channel recorder, and a constant-temperature oven for the computing networks.

The Electronic Associates[1] Precision Analog Computing Equipment (PACE). The elaborate and accurate Electronic Associates computer incorporates a number of desirably conservative design features. The careful and definitely not cost-conscious design of PACE components is exemplified by the d-c amplifier circuit of Fig. 5.40.

The computing resistors and capacitors are trimmed and padded within 0.01 per cent of their nominal values; they are maintained at 100°F (plus or minus 1 deg) in a simple oven located directly behind the patch bay.

All d-c amplifier input connections are fully shielded; the multiple connectors at the rear of each dual amplifier chassis incorporate coaxial terminals for this purpose. The shielded amplifier connections, together with metal problem boards, shielded patch cords, and metal barriers between patch-bay terminals, minimize cross talk and positively confine all leakage into integrator terminals to the integrating capacitors (see also Secs. 5.9 and 8.3, and Fig. 8.1*a*).

Each d-c amplifier can serve either as a summing amplifier or as a summing integrator, with inputs corresponding to multiplication by 1, 1, 1, 5, 5, 10, and 10.[2] The function of each amplifier is determined by the position of a four-terminal "bottle plug" which exposes the letters SUM or INT on the problem board and changes the equalizing networks as well as the feedback (see also Fig. 5.40). Removal of the bottle plug opens the feedback connections and exposes the d-c amplifier input terminal for direct-analog applications.

The switch-type control panels used in earlier models have been replaced by push-button controls. The control circuits include a very simple problem-check circuit and a special POTENTIOMETER SET condition permitting all potentiometers to be set with their loads connected (Sec. 7.7).

Figure 8.6*a* shows a basic PACE computer unit. The fully expanded installation shown in Fig. 1.1 indicates the various accessory units. Some of the larger PACE installations have dual control panels and patch bays; such a machine can be used either as a single large computer or as two separate computers (Fig. 8.6*b*). The company also furnishes large and small servo plotting boards (Figs. 1.17 and 7.20*b*) suitable for use as automatic curve followers.

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[1] Electronic Associates, Inc., Long Branch, N.J.

[2] Strong, J. D., A Practical Approach to Analog Computers, *Instruments and Automation*, **28**: 602, April, 1955.

The Goodyear[1] Electronic Differential Analyzer (GEDA). A typical GEDA installation consists of a neat castered console containing the basic computer unit and a set of accessory racks which hold additional equipment packaged in modular units. The machine can be controlled from the main console and from an accessory recorder console (Fig. 7.14).

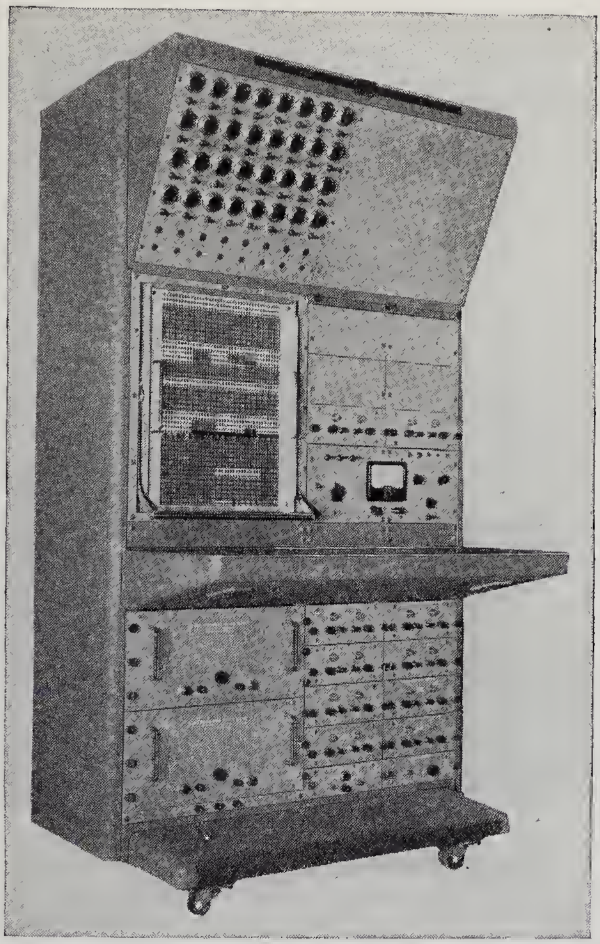

FIG. 8.6*a*. Basic PACE unit (Electronic Associates, Inc.).

GEDA problem boards permit stacking of plug-in network components (Fig. 8.1*b*). The machine is particularly well suited for applications involving direct-analog operational amplifiers and built-up diode circuits; Goodyear engineers have contributed very significantly to the development of such techniques (refer to the examples of Chap. 3).

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[1] Goodyear Aircraft Corporation, Akron, Ohio.

GEDA d-c amplifiers employ commutator stabilization (Fig. 5.42). The high-quality time-division multiplier and the diode function generator used in the GEDA have been discussed in Chap. 6 (Figs. 6.17*b* and 6.26*b*). The conventional 60-cps computer servomechanisms incorporate tapped function potentiometers with plug-in adjustment networks (Sec. 6.10). Resolver servomechanisms, noise generators, recorder amplifiers, and servo recorder curve-followers are available as optional accessories.

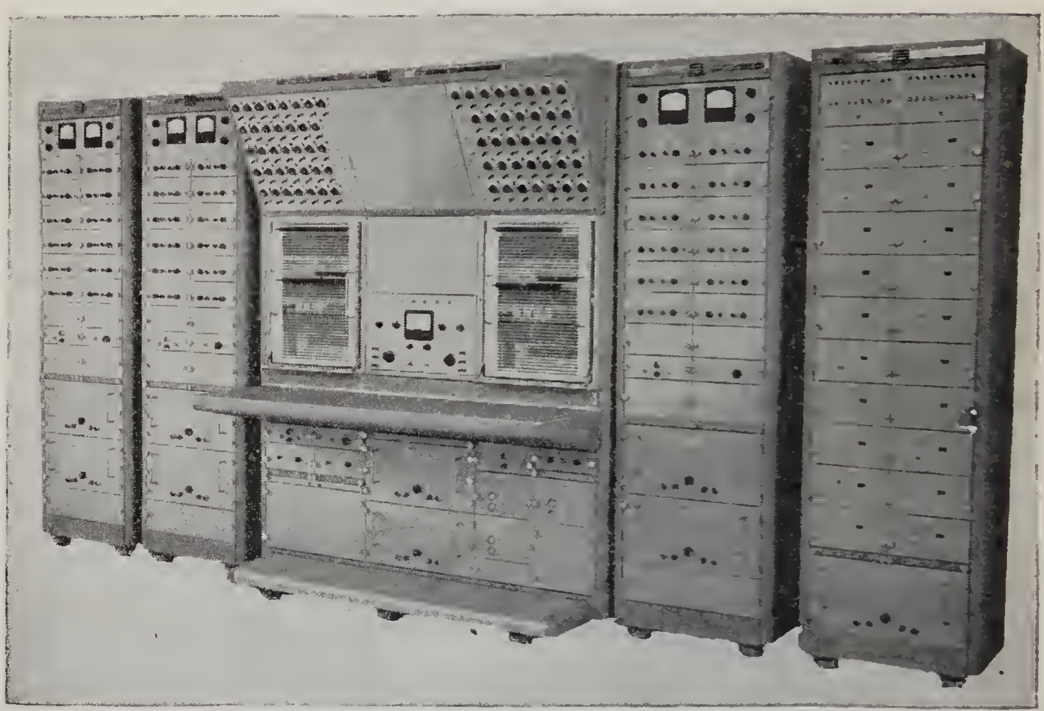


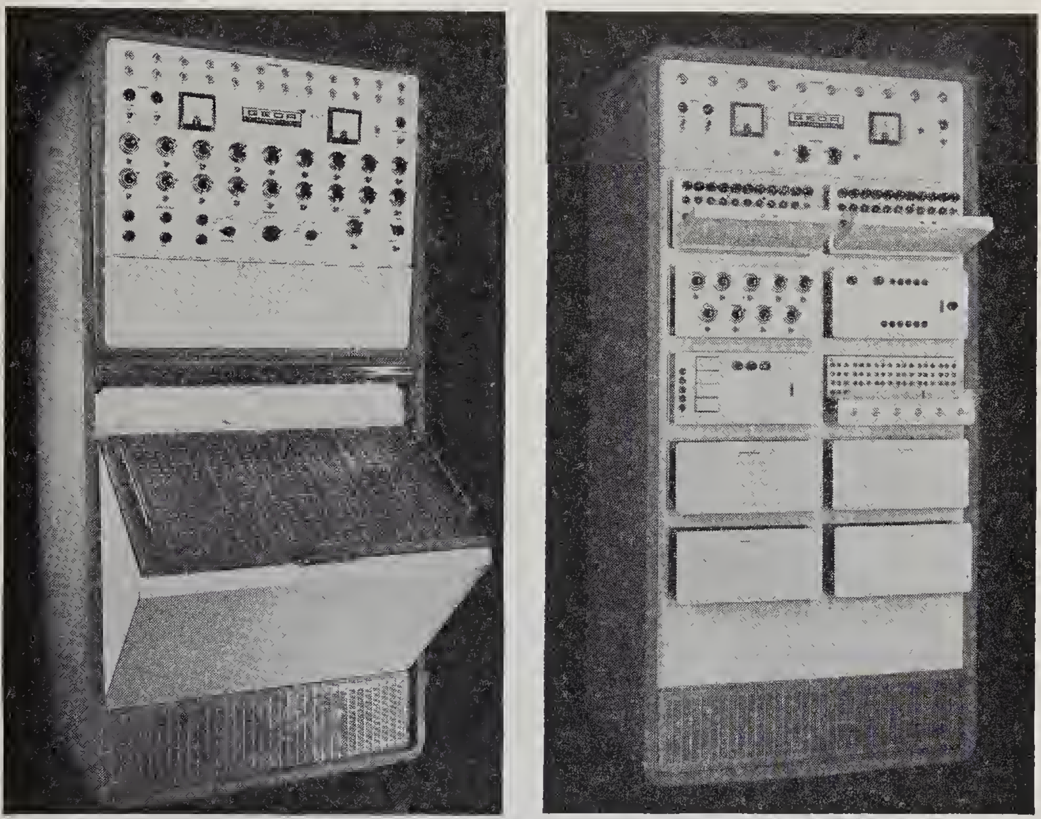
FIG. 8.6*b*. In this versatile analog-computer installation, problems involving up to 72 operational amplifiers, 72 coefficient-setting potentiometers, and 20 computer servomechanisms may be set up on two removable prepatch panels. The dual-control console permits use of the machine either as one large computer or as two complete separate computers (Electronic Associates, Inc.).

Larger GEDA installations are equipped with elaborate dual control consoles similar to that shown in Fig. 8.6*b* and comprising advanced semiautomatic control and problem-checking circuits (Sec. 7.7). Precision resistors and capacitors associated with pre-assembled summing amplifiers and integrators are located in constant-temperature ovens; in addition, individual amplifier racks still permit flexible plug-in connections of direct-analog-type feed-back networks and built-up diode circuits.

The Mid-Century[1] Analog Computers. The MC-500 d-c analog computer, as well as its smaller desk-side counterpart, the MC-400, incorporate conventional chopper-stabilized d-c amplifiers, coefficient-setting potentiometers, diode circuits, and computer servomechanisms. A six-channel photoformer unit permits electronic function generation; a spe-

[1] Mid-Century Instrumatic Corporation, New York City.

cial camera assembly assists in the preparation of photoformer masks. The Mid-Century AM-FM electronic multiplier (see also Sec. 6.3) has a static accuracy of 0.1 per cent of full scale (plus and minus 100 volts) and a better high-frequency response (flat to 400 cps with less than 1-deg phase shift at 100 cps) than most time-division multipliers of comparable accuracy.



(*a*) (*b*)

FIG. 8.7. The basic GEDA unit (*a*) is easily expanded by additional computing equipment housed in matching accessory cabinets (*b*). Larger GEDA installations employ a dual control console similar to that shown in Fig. 8.6*b* (Goodyear Aircraft Corporation).

The Reeves[1] Electronic Analog Computer (**REAC**®). The 400-series REAC offers a choice of accurate computing elements specifically designed so that they can be used as building blocks of very large computer installations (several hundred amplifiers) as well as for compact medium-size d-c analog computers. Computing elements, power supplies, and accessories are packaged as modular units which fit into an expansion console made up of cabinets with removable side skins. Figure 8.8 shows how REAC cabinets of various widths are bolted together to form a continuous console. Interconnecting cables are simply dropped into troughs at

[1] Reeves Instrument Corporation, New York City.

the rear of the cabinets and terminate into solderless taper pins,[1] so that it is easy to assemble different combinations of equipment. In particular, connections to the rear of the REAC patch bay are made by means of taper pins, so that the patch bay can be laid out to suit the user's special requirements. Grounded printed silver grids minimize patch-bay leakage; similar grids on the plastic problem boards are protected against short circuits by plastic masks imprinted with the terminal designations (Sec. 8.3).

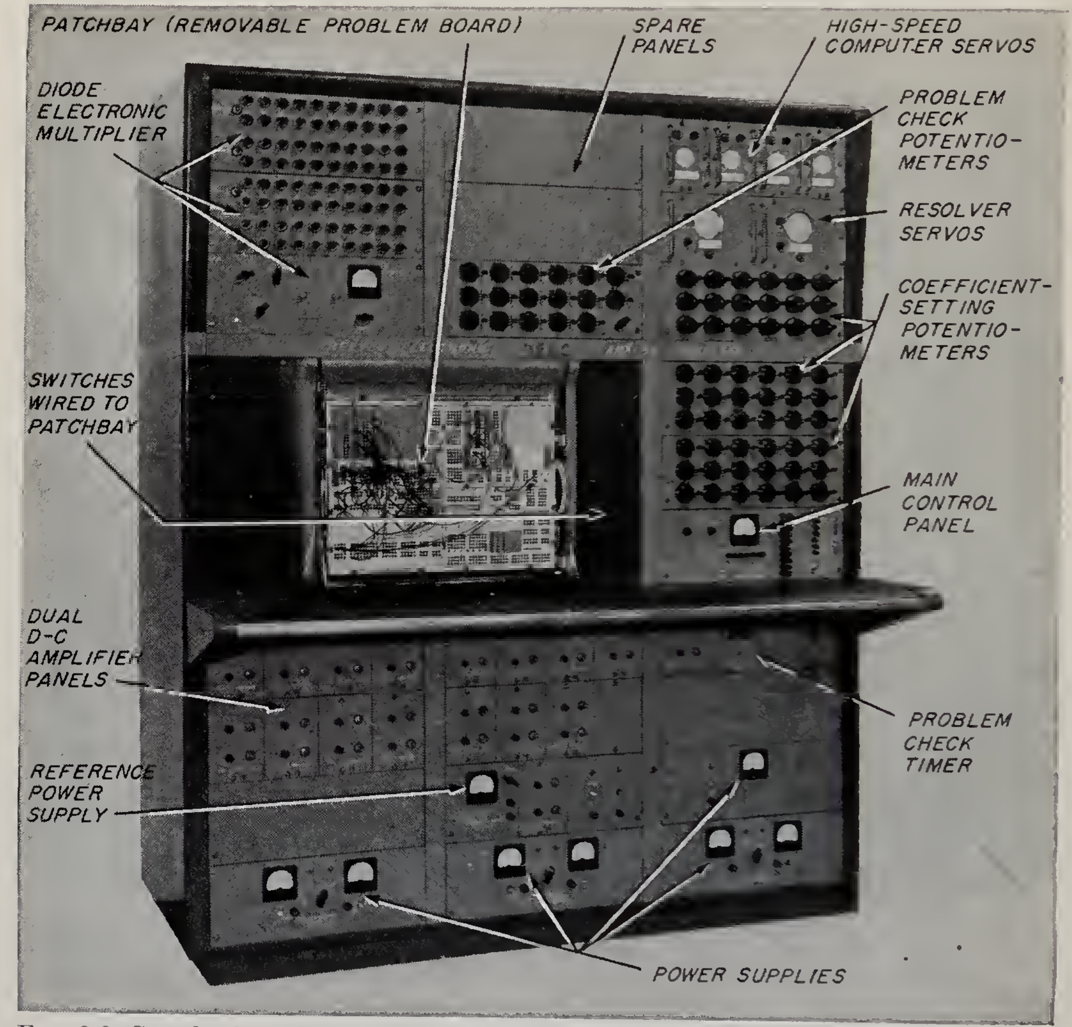

PATCHBAY (REMOVABLE PROBLEM BOARD)
SPARE PANELS
HIGH-SPEED COMPUTER SERVOS
DIODE ELECTRONIC MULTIPLIER
PROBLEM CHECK POTENTIOMETERS
RESOLVER SERVOS
COEFFICIENT-SETTING POTENTIOMETERS
SWITCHES WIRED TO PATCHBAY
MAIN CONTROL PANEL
DUAL D-C AMPLIFIER PANELS
PROBLEM CHECK TIMER
REFERENCE POWER SUPPLY
POWER SUPPLIES

FIG. 8.8. Standard modular units may be combined into a tailor-made REAC computing installation by adding cabinet frames between the removable side skins. The modular construction is repeated in the interchangeable sections of the prepatch panel. Note the push-button control panel (Reeves Instrument Corporation).

REAC design philosophy emphasizes reduction of human errors, particularly in computer installations requiring frequent problem changes. Switching operations changing the function of d-c amplifiers, servomultipliers, and resolver servomechanisms are accomplished by patch-cord connections on the removable problem boards (*patch-bay switching*), rather than by manually operated switches associated with the computing elements themselves. Removal of a problem board opens connections or deenergizes relays and thus annuls the switching; it is, then, impossible to forget such switching operations when problems are changed.

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[1] A product of Aircraft-Marine Products, Inc., Harrisburg, Pa.

The REAC *problem-check system* (static and "dynamic" problem check) described in Sec. 7.7 is a further aid to rapid problem preparation and is particularly useful when frequent problem changes are necessary. This feature enables operators to ascertain quickly whether the computing elements are connected correctly and whether they are performing properly. The problem-check circuits are also available as kits for modernization of older REACS.

The printed-circuit *d-c amplifiers* (Fig. 5.41) are ordinarily set up as phase inverters, summing amplifiers, or summing integrators. Summing-network inputs correspond to multiplication by 1, 1, 1, 4, 4, 10, and 10, so that any integral coefficient between 1 and 31 can be obtained directly. The REAC uses encapsulated 0.01 per cent computing resistors and 0.05 per cent polystyrene capacitors. Patch-bay switching converts any phase inverter or integrator into a high-gain amplifier.

An exceptionally wide choice of *computer servomechanisms* includes 400-cps servomechanisms of remarkably good high-frequency response (Fig. 5.35), high-speed resolver servos, a 400-cps continuous-rotation servomechanism with rate feedback useful, *e.g.*, as a resolver servo (Sec. 6.8), as well as conventional 400-cps and 60-cps computer servomechanisms. All these servomechanisms can drive tapped function potentiometers.

Electronic multipliers include a diode electronic multiplier based on the quarter-square principle (Sec. 6.5) and usable at frequencies up to 0.5 kc, as well as electronic multipliers of the time-division type described in Sec. 6.4. The Reeves *diode function generator* (Fig. 6.26*a*) is an accurate and conveniently adjustable series-diode type adaptable to the generation of certain functions of two variables (Sec. 6.13). Since functions of different complexities are encountered in the solution of problems, the limiter channels of REAC diode function generators can be allocated in different combinations. A *photoformer function generator* using a 3-in. cathode-ray tube is also available and provides an accuracy of about 0.5 volt up to 30 cps.

8.5. REPETITIVE COMPUTERS

The principles of the repetitive analog computer have already been discussed in Sec. 1.1. Such computers operate on accelerated time scales and contain provisions for resetting the machine variables to their given initial values periodically at repetition rates between 10 and 100 cps. Solutions of differential equations are thus repeated periodically and can be displayed on a cathode-ray oscillograph whose linear sweep is synchronized with the initial-condition-setting circuits. One can also display *parametric plots* or graphs of one machine variable against another by connecting the horizontal and vertical input terminals of the oscillograph to the proper points in the computer setup. These display methods permit instantaneous observation of the effects of changes in parameters and initial conditions; permanent records can be obtained quickly through the use of a Polaroid Land camera.

Applications of repetitive analog computers have run the gamut of analog-computer applications (Sec. 3.12) permitting the use of fast time scales and relatively low accuracy. The peculiar advantages of repetitive analog computers—very high computing speed and low cost—are particularly well realized in the following applications:

1. Rapid exploratory studies, to conserve computing time on more accurate analog or digital computers, or in connection with manual computations; in particular, rapid approximations of optimum system parameters for control systems (see also Secs. 3.1 and 3.4), and determination of regions of stability
2. Approximate computation and display of integral transforms, such as Fourier integrals, superposition integrals, and correlation functions (Sec. 3.12)
3. Approximate solution of boundary-value problems and eigenvalue problems (Sec. 3.10) by visual matching of parameters or initial conditions to the given boundary conditions

Some "slow" d-c analog computers have provisions for repetitive operation at reduced accuracy (Sec. 8.4).

Computing Elements for Repetitive Analog Computers. The computing elements employed in repetitive analog computers are essentially similar to those used in "slow" d-c analog computers, but the working frequencies required for repetitive operation are higher by a factor of 10^3 to 10^4. The resulting high-frequency response requirements crowd amplifier bandwidths and limit the forward gain permissible for stable feedback-amplifier operation, so that operational amplifiers may be subject to considerable phase distortion (Secs. 4.2 and 4.7). Although this situation could be somewhat improved through better amplifier design and special compensating networks (Sec. 4.7), fast repetitive operation also restricts the choice of nonlinear computing elements. The only applicable types are diode and photoformer function generators, quarter-square, logarithmic, and amplitude-selection multipliers, and crossed-fields electron-beam multipliers (Chap. 6). It appears, therefore, unreasonable to specify component accuracies better than 0.5 per cent of full scale for multipurpose repetitive analog computers.

Design of Repetitive Analog Computers: The Philbrick[1] Computer. Philbrick computing elements are mounted in individual small utility boxes connected to a "central unit" cabinet which contains the power supplies, the control circuits, and various signal generators.

Philbrick *operational amplifiers* (Fig. 8.9*a*) have permanently wired feedback networks; there are variable-gain amplifiers (coefficient-setting units), summing amplifiers, integrators, differentiators, and various direct-analog operational amplifiers. The *nonlinear computing elements* include the ingenious amplitude-selection multiplier described in Sec. 6.5, a diode function generator capable of generating functions of two variables (Sec. 6.13), and permanently wired special diode function generators, such as limiters, inert-zone units, and backlash or hysteresis units (see also Table 6). Every Philbrick computing element incorporates a phase inverter (Fig. 8.9*a*), so that both positive and negative output voltages can be obtained. All machine variables vary between minus 50 volts and plus 50 volts; neon overload indicators are provided.

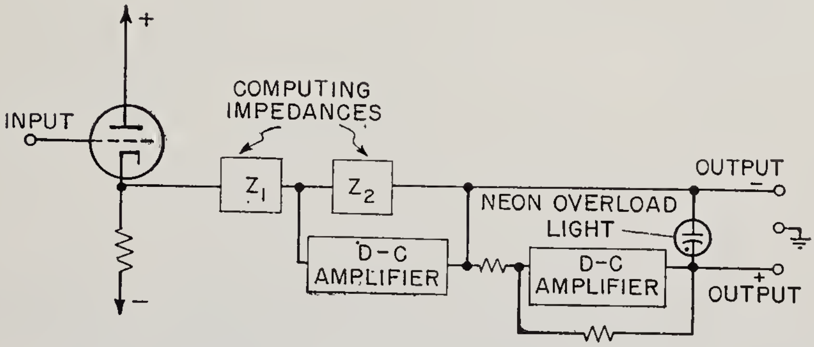

FIG. 8.9*a*. This Philbrick operational amplifier incorporates a phase inverter; two plug-in a-c amplifiers of the type shown in Figs. 5.30*b* and 5.31*a* are used. The feedback networks are permanently installed but permit adjustment of one coefficient. The cathode-follower isolation stage tends to reduce accuracy.

The small plug-in d-c amplifiers (Figs. 5.30 and 5.31) used in the Philbrick computer can be purchased separately and permit the construction of special-purpose computing elements.

The Philbrick analog computer can generate solutions periodically at a normal repetition rate of 20 cps. In many problems involving damped oscillations, all machine values return to the value zero in the course of the solution, so that all integrating capacitors will be discharged at the end of a computer "run." One can then introduce initial conditions by simply adding a step-function voltage equal in value to the desired initial condition to the corresponding machine variable at the beginning of each computer run (Fig. 7.2). In the Philbrick analog computer, this is achieved by means of the ordinary summing amplifiers, the step functions being obtained from a square-wave generator in the central unit. If, on the other hand, one or more machine variables do not return to zero at the end of a solution, it is possible to discharge the integrating capacitors at the end of each computer run through electronic switches controlled from the central unit.

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[1] George A. Philbrick Researches, Inc., Boston, Mass. The company publishes an excellent collection of papers (reprints) and lectures dealing mainly with repetitive analog computers under the title *A Palimpsest on the Electronic Analog Art*, edited by H. M. Paynter, 1955.

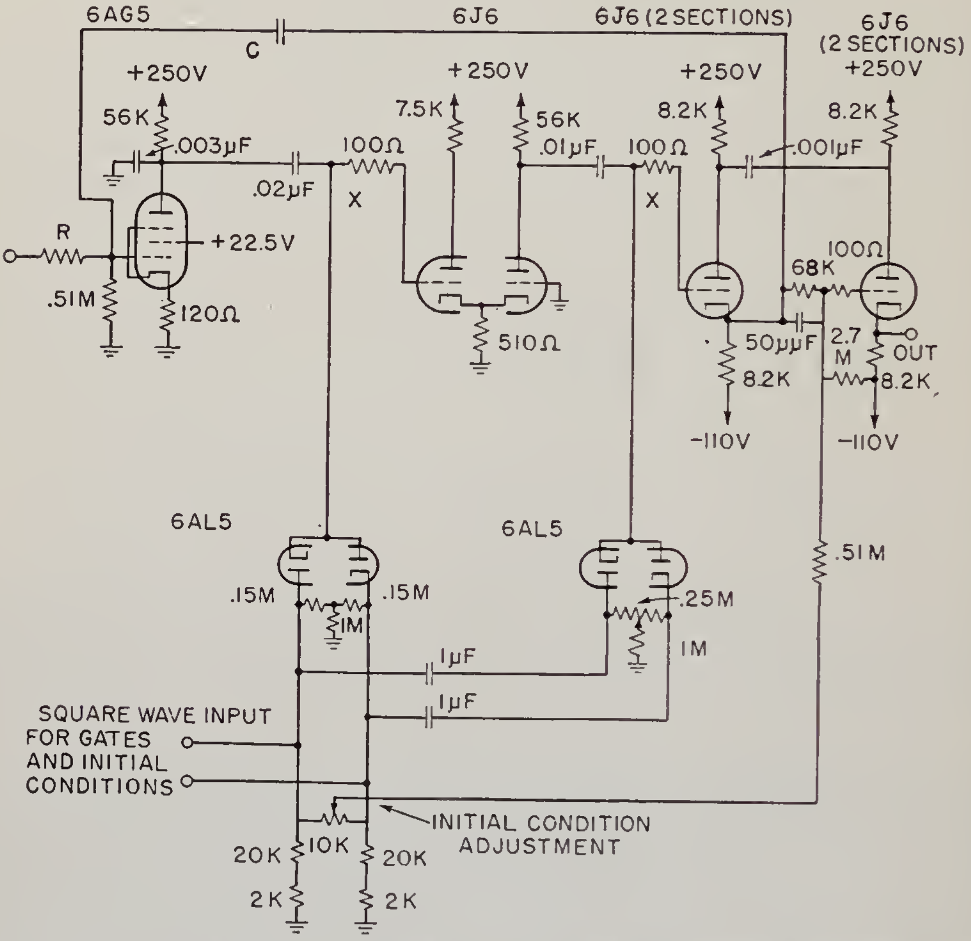

FIG. 8.9*b*. Circuit diagram of an integrator from the MIT repetitive computer. The open-loop gain is 1,500, and the open-loop half-power points are at 0.16 cps and 100 kc. Output range is plus and minus 20 volts at about 1 ma with less than 3 per cent distortion. Note the clamping circuits, used to discharge the integrating capacitor, and the summing circuit, used to add initial-condition voltages to the integrator output voltage. The initial-condition voltage is *not* used to charge the integrating capacitor directly in this arrangement. The same amplifier circuit is also employed in summing amplifiers with gains up to at least 150.

The *central unit* of the Philbrick analog computer contains plus and minus 300-volt power supplies, a control panel, and various plug-in multivibrators which supply square-wave voltages used to introduce initial conditions and step functions, to drive sweep circuits, etc. An accessory television-type raster generator permits simultaneous display of up to 8 variables on a single oscillograph screen calibrated by permanently displayed electronic marker lines ("electronic graph paper").

The specified accuracy of Philbrick operational amplifiers and multipliers is better than 1 per cent of full scale, presumably at frequencies near the repetition frequency. The specified accuracy of the various diode function generators is 2 per cent. Accuracy is somewhat affected by d-c amplifier drift, even though repetitive solutions need not actually contain a d-c component. The more recently designed MIT repetitive differential analyzer represents an interesting improvement in this respect.

The MIT Repetitive Differential Analyzer. An improved repetitive analog computer was developed under the direction of Dr. A. B. MacNee[1] at the Research Laboratory of Electronics of the Massachusetts Institute of Technology.[2] This machine comprises parallel-feedback integrators and summing amplifiers as well as function generators of the photoformer type (Sec. 6.6) and the crossed-fields electron-beam multiplier (Sec. 6.2), newly developed by the same group. Initial conditions are again introduced by resetting all machine variables to zero at the end of each computer run and by adding step-function voltages corresponding to the initial conditions to each integrator output at the beginning of each computer run. The accuracy of the initial model is comparable to that of the Philbrick computer.

The significant improvement in the new analyzer is the use of a-c amplifiers in all operational-amplifier circuits and other computing elements. Figure 8.9*b* shows a simplified diagram of an a-c amplifier used in the MIT computer. At the end of each computer run, all machine variables are reset to zero and all charges are removed from the coupling capacitors by means of electronic switches based on the clamping circuits commonly used in television sets. Positive and negative *gating pulses* applied at the appropriate points shown in Fig. 8.9*b* permit the diodes to conduct and will reduce the output voltage of each amplifier stage (measured at the points x) to zero if the relative magnitudes of the positive and negative gating pulses are adjusted correctly by means of the balancing controls shown.[3]

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[1] MacNee, A. B., An Electronic Differential Analyzer, *Proc. IRE,* **37**: 1315, 1949.

[2] Cambridge, Mass.

[3] See also N. T. Van der Walt, An Electronic Servo Simulator, *Elec. Eng.* (*London*), February,1952.

These clamping circuits assure the return of the machine variables to the correct initial values for all types of solutions and also tend to simplify the computer design. Repetitive analyzers, such as the Philbrick computer, which make use of d-c amplifiers still require regulated power supplies in order to avoid drift in the reference level of the solutions. In repetitive analyzers using a-c amplifiers power-supply regulation is not an essential requirement. Interaction between computing elements through the power supplies of repetitive computers can be eliminated by the use of conventional decoupling filters. Furthermore, the a-c amplifiers used in the MacNee type of repetitive analyzer could be operated with a single power supply, although this is not the case in the original design. The component accuracy of the MacNee machine is between 0.5 and 1 per cent of full scale (plus and minus 25 volts); repeated solutions usually agree within 0.1 per cent.

Possible Future Developments. Although the accuracy of repetitive analog computers is not sufficient to produce acceptable final results in many applications, improved compact machines of this type could be invaluable for rapid exploratory studies and especially for instruction purposes.

The design of new repetitive analog computers should pay increased attention to improved patch bays and flexible control circuits. The high computing speeds of repetitive analog computers offer fascinating possibilities in connection with simple automatic programming schemes (see also Sec. 7.7). Suitable feedback circuits could, for instance, permit automatic adjustments of system parameters so as to optimize some measure of effectiveness averaged over several computing cycles. Special-purpose repetitive analog computers of this type should be useful as "super-controllers" for continuous optimization of system parameters in actual automatic control systems (see also Sec. 3.4).

TABLES OF SPECIAL COMPUTER SETUPS

Table 1: Examples of Parallel-feedback-type Operational Amplifiers for a Number of Transfer Functions $\frac{X_o}{X_1} = -\frac{Z_o(P)}{Z_1(P)}$ (Sec. 1.5)

Additional circuits may be designed with the aid of Table 3.

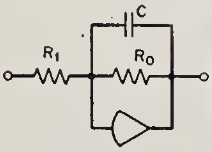
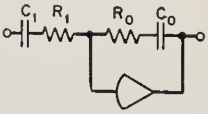
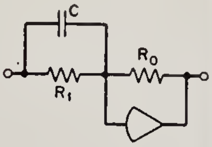
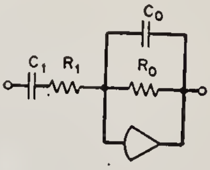
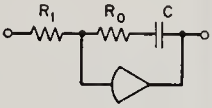
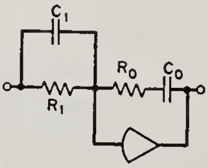
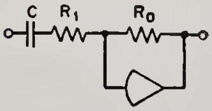
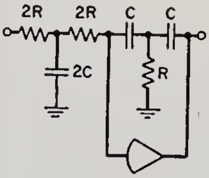
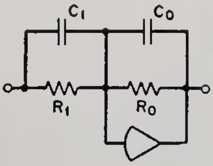
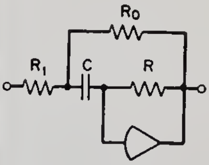
	Transfer function X_o/X_1	Circuit		Transfer function X_o/X_1	Circuit
1.1	$-\frac{R_o}{R_1}\frac{1}{R_oCP+1}$		1.6	$-\frac{C_1}{C_o}\frac{R_oC_oP+1}{R_1C_1P+1}$	
1.2	$-\frac{R_o}{R_1}(R_1CP+1)$		1.7	$-\frac{R_oC_1P}{(R_oC_oP+1)(R_1C_1P+1)}$	
1.3	$-\frac{R_oCP+1}{R_1CP}$		1.8	$-\frac{(R_oC_oP+1)(R_1C_1P+1)}{R_1C_oP}$	
1.4	$-\frac{R_oCP}{R_1CP+1}$		1.9	$-\frac{1}{4(RC)^2P^2}$	
1.5	$-\frac{R_o}{R_1}\frac{R_1C_1P+1}{R_oC_oP+1}$		1.10	$-\frac{R_oRCP}{(R_o+R)R_1CP+R_1+R_o}$	

Table 2: Additional Operational-amplifier Circuits (Sec. 1.5)

These circuits may simulate components of dynamical or electrical systems (Sec. 3.4) and are useful for the solution of partial differential equations (Sec. 3.10). Each circuit involves a simple integration and yields $-RCPX_o$ as well as X_o. The load resistance r_L is usually an input resistor of another operational amplifier. These circuits are most useful if frequent parameter changes are not required.

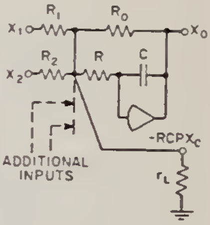
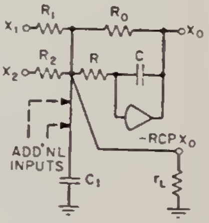
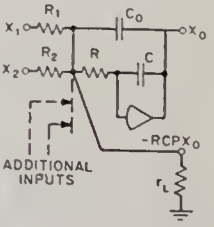
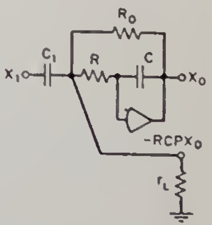
	Machine equation	Circuit
2.1	$X_o = -\left[\dfrac{R_o}{\left(\dfrac{1}{R}+\dfrac{1}{r_L}+\dfrac{1}{R_o}+\dfrac{1}{R_1}+\dfrac{1}{R_2}+\cdots\right)R_oRCP+1}\right] \times \left(\dfrac{X_1}{R_1}+\dfrac{X_2}{R_2}+\cdots\right)$	 R1 R0 X1 X0 R2 R C X2 -RCPXC ADDITIONAL INPUTS rL
2.2	$X_o = -\left[\dfrac{R_o}{R_oC_1RCP^2+\left(\dfrac{1}{R}+\dfrac{1}{r_L}+\dfrac{1}{R_o}+\dfrac{1}{R_1}+\dfrac{1}{R_2}+\cdots\right)R_oRCP+1}\right] \times \left(\dfrac{X_1}{R_1}+\dfrac{X_2}{R_2}+\cdots\right)$	 R1 R0 X1 X0 R2 R C X2 -RCPX0 ADD'NL INPUTS C1 rL
2.3	$X_o = -\left[\dfrac{1}{RCP\left(C_oP+\dfrac{C_o}{RC}+\dfrac{1}{R}+\dfrac{1}{r_L}+\dfrac{1}{R_1}+\dfrac{1}{R_2}+\cdots\right)}\right] \times \left(\dfrac{X_1}{R_1}+\dfrac{X_2}{R_2}+\cdots\right)$	 R1 C0 X1 X0 R2 R C X2 -RCPX0 ADDITIONAL INPUTS rL
2.4	$X_o = -\dfrac{R_oC_1P}{R_oC_1RCP^2+\left(\dfrac{1}{R}+\dfrac{1}{r_L}+\dfrac{1}{R_o}\right)R_oRCP+1}X_1$	

Table 3: Transfer Impedances for Operational Amplifiers* (Sec. 1.5)

This table is used to find networks whose transfer impedances $Z_o(P)$, $Z_1(P)$, $Z_2(P)$, . . . , match the desired performance equation

$$X_o = -\frac{Z_o(P)}{Z_1(P)} X_1 \qquad \text{or} \qquad X_o = -Z_o(P)\left[\frac{X_1}{Z_1(P)} + \frac{X_2}{Z_2(P)} + \cdots\right]$$

for an operational amplifier of one of the types shown in Figs. 1.14*a*, *b*, *c*, *d*. See Fig. 1.15*b* for an example illustrating the use of the table.

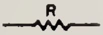
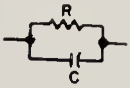
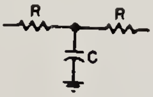
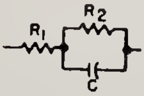
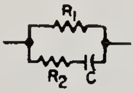
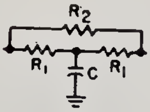
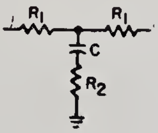
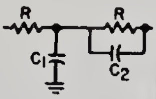
Transfer-impedance function	Network	Relations	Inverse relations
A		$A = R$	$R = A$
$\dfrac{A}{1 + PT}$		$A = R$ $T = RC$	$R = A$ $C = \dfrac{T}{A}$
$A(1 + PT)$		$A = 2R$ $T = \dfrac{RC}{2}$	$R = \dfrac{A}{2}$ $C = \dfrac{4T}{A}$
$A\left(\dfrac{1 + P\theta T}{1 + PT}\right)$ $\theta < 1$		$A = R_1 + R_2$ $T = R_2C$ $\theta = \dfrac{R_1}{R_1 + R_2}$	$R_1 = A\theta$ $R_2 = A(1 - \theta)$ $C = \dfrac{T}{A(1 - \theta)}$
		$A = R_1$ $T = (R_1 + R_2)C$ $\theta = \dfrac{R_2}{R_1 + R_2}$	$R_1 = A$ $R_2 = \dfrac{A\theta}{1 - \theta}$ $C = \dfrac{T(1 - \theta)}{A}$
$A\left(\dfrac{1 + PT}{1 + P\theta T}\right)$ $\theta < 1$		$A = \dfrac{2R_1R_2}{2R_1 + R_2}$ $T = \dfrac{R_1C}{2}$ $\theta = \dfrac{2R_1}{2R_1 + R_2}$	$R_1 = \dfrac{A}{2(1 - \theta)}$ $R_2 = \dfrac{A}{\theta}$ $C = \dfrac{4T(1 - \theta)}{A}$
		$A = 2R_1$ $T = \left(R_2 + \dfrac{R_1}{2}\right) C$ $\theta = \dfrac{2R_2}{2R_2 + R_1}$	$R_1 = \dfrac{A}{2}$ $R_2 = \dfrac{A\theta}{4(1 - \theta)}$ $C = \dfrac{4T(1 - \theta)}{A}$
		$A = 2R$ $T = \dfrac{R}{2}(C_1 + C_2)$ $\theta = \dfrac{2C_2}{C_1 + C_2}$	$R = \dfrac{A}{2}$ $C_1 = \dfrac{2T(2 - \theta)}{A}$ $C_2 = \dfrac{2T\theta}{A}$

Table 3: Transfer Impedances for Operational Amplifiers* (Sec. 1.5) (*Continued*)

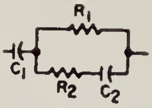
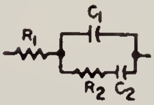
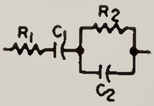
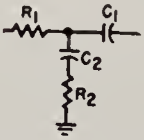
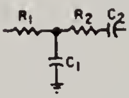
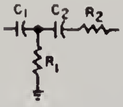
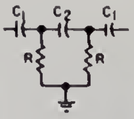
Transfer-impedance function	Network	Relations	Inverse relations
$\frac{1}{PB}\left[\frac{(1+PT_1)(1+PT_3)}{1+PT_2}\right]$ $T_1 < T_2 < T_3$		$B = C_1$ $T_2 = (R_1 + R_2)C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_1C_2$	$R_1 = \frac{T_1 + T_3 - T_2}{B}$ $R_2 = \frac{T_1T_3(T_1 + T_3 - T_2)}{B(T_3 - T_2)(T_2 - T_1)}$ $C_1 = B$ $C_2 = \frac{B(T_3 - T_2)(T_2 - T_1)}{(T_1 + T_3 - T_2)^2}$
		$B = C_1 + C_2$ $T_2 = R_2\left(\frac{C_1C_2}{C_1 + C_2}\right)$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_1C_2$	$R_1 = \frac{T_1T_3}{BT_2}$ $R_2 = \frac{(T_1T_2 + T_2T_3 - T_1T_3)^2}{BT_2(T_3 - T_2)(T_2 - T_1)}$ $C_1 = \frac{BT_2^2}{T_1T_2 + T_2T_3 - T_1T_3}$ $C_2 = \frac{B(T_3 - T_2)(T_2 - T_1)}{T_1T_2 + T_2T_3 - T_1T_3}$
		$B = C_1$ $T_2 = R_2C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = \frac{T_1T_3}{BT_2}$ $R_2 = \frac{(T_3 - T_2)(T_2 - T_1)}{BT_2}$ $C_1 = B$ $C_2 = \frac{BT_2^2}{(T_3 - T_2)(T_2 - T_1)}$
		$B = C_1$ $T_2 = R_2C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_1C_2$	$R_1 = \frac{T_1T_3}{BT_2}$ $R_2 = \frac{T_1T_2T_3}{B(T_3 - T_2)(T_2 - T_1)}$ $C_1 = B$ $C_2 = \frac{B(T_3 - T_2)(T_2 - T_1)}{T_1T_3}$
$\frac{1}{PB}(1 + PT_1)(1 + PT_2)$ $T_1 \neq T_2$		$B = C_2$ $T_1T_2 = R_1R_2C_1C_2$ $T_1 + T_2 = R_1C_1 + R_2C_2 + R_1C_2$	$R_1 = \frac{(\sqrt{T_1} - \sqrt{T_2})^2}{B}$ $R_2 = \frac{\sqrt{T_1T_2}}{B}$ $C_1 = \frac{B\sqrt{T_1T_2}}{(\sqrt{T_1} - \sqrt{T_2})^2}$ $C_2 = B$
$\frac{1}{PB}\left[\frac{(1+PT_1)(1+PT_2)}{P\sqrt{T_1T_2}}\right]$ $T_1 \neq T_2$		$B = C_2$ $T_1T_2 = R_1R_2C_1C_2$ $T_1 + T_2 = R_1C_1 + R_2C_2 + R_1C_2$	$R = \frac{(\sqrt{T_1} - \sqrt{T_2})^2}{B}$ $R_2 = \frac{\sqrt{T_1T_2}}{B}$ $C_1 = \frac{B\sqrt{T_1T_2}}{(\sqrt{T_1} - \sqrt{T_2})^2}$ $C_2 = B$
$\frac{1}{PB}\left[\frac{(1+PT_1)(1+PT_2)}{P^2T_1T_2}\right]$ $T_1 < T_2$		$B = \frac{C_1C_2}{C_1 + 2C_2}$ $T_1 = RC_1$ $T_2 = R(C_1 + 2C_2)$	$R = \frac{T_1(T_2 - T_1)}{2BT_2}$ $C_1 = \frac{2BT_2}{T_2 - T_1}$ $C_2 = \frac{BT_2}{T_1}$

Table 3: Transfer Impedances for Operational Amplifiers* (Sec. 1.5) (*Continued*)

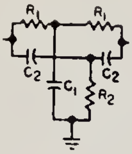
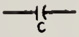
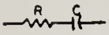
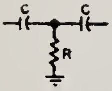
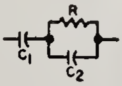
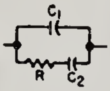
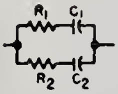
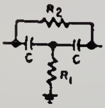
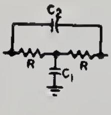
Transfer-impedance function	Network	Relations	Inverse relations
$A\left(\frac{1+PT_1}{1+P^2T_1T_2}\right)$	 $R_1C_1 = 4R_2C_2$	$A = 2R_1$ $T_1 = \frac{R_1C_1}{2} = 2R_2C_2$ $T_2 = R_1C_2$	$R_1 = \frac{A}{2}$ $R_2 = \frac{AT_1}{4T_2}$ $C_1 = \frac{4T_1}{A}$ $C_2 = \frac{2T_2}{A}$
$\frac{1}{PB}$		$B = C$	$C = B$
$\frac{1}{PB}(1+PT)$		$B = C$ $T = RC$	$R = \frac{T}{B}$ $C = B$
$\frac{1}{PB}\left(\frac{1+PT}{PT}\right)$		$B = \frac{C}{2}$ $T = 2RC$	$R = \frac{T}{4B}$ $C = 2B$
$\frac{1}{PB}\left(\frac{1+PT}{1+P\theta T}\right)$ $\theta < 1$		$B = C_1$ $T = R(C_1 + C_2)$ $\theta = \frac{C_2}{C_1 + C_2}$	$R = \frac{T(1-\theta)}{B}$ $C_1 = B$ $C_2 = \frac{B\theta}{1-\theta}$
		$B = C_1 + C_2$ $T = RC_2$ $\theta = \frac{C_1}{C_1 + C_2}$	$R = \frac{T}{B(1-\theta)}$ $C_1 = B\theta$ $C_2 = B(1-\theta)$
$\frac{1}{PB}\left[\frac{(1+PT_1)(1+PT_3)}{1+PT_2}\right]$ $T_1 < T_2 < T_3$		$B = C_1 + C_2$ $T_1 = R_1C_1$ $T_2 = (R_1 + R_2)\left(\frac{C_1C_2}{C_1 + C_2}\right)$ $T_3 = R_2C_2$	$R_1 = \frac{T_1(T_3 - T_1)}{B(T_2 - T_1)}$ $R_2 = \frac{T_3(T_3 - T_1)}{B(T_3 - T_2)}$ $C_1 = \frac{B(T_2 - T_1)}{T_3 - T_1}$ $C_2 = \frac{B(T_3 - T_2)}{T_3 - T_1}$
$A\left(\frac{1+PT_1}{1+PT_1+P^2T_1T_2}\right)$		$A = R_2$ $T_1 = 2R_1C$ $T_2 = \frac{R_2C}{2}$	$R_1 = \frac{AT_1}{4T_2}$ $R_2 = A$ $C = \frac{2T_2}{A}$
$A\left(\frac{1+PT_2}{1+PT_1+P^2T_1T_2}\right)$		$A = 2R$ $T_1 = 2RC_2$ $T_2 = \frac{RC_1}{2}$	$R = \frac{A}{2}$ $C_1 = \frac{4T_2}{A}$ $C_2 = \frac{T_1}{A}$

Table 3: Transfer Impedances for Operational Amplifiers* (Sec. 1.5) (*Continued*)

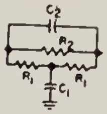
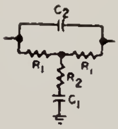
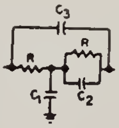
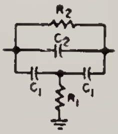
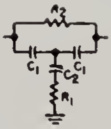
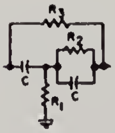
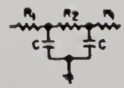
Transfer-impedance function	Network	Relations	Inverse relations
$A\left[\dfrac{1+PT_3}{1+PT_1+P^2T_1T_2}\right]$ $T_2 > \dfrac{T_1}{4}$ (complex roots) $T_3 > T_2$		$A = \dfrac{2R_1R_2}{(2R_1+R_2)}$ $T_1 = \dfrac{R_1(R_1C_1+2R_2C_2)}{2R_1+R_2}$ $T_2 = \dfrac{R_1R_2C_1C_2}{R_1C_1+2R_2C_2}$ $T_3 = \dfrac{R_1C_1}{2}$	$R_1 = \dfrac{AT_3^2}{2[T_3^2-T_1(T_3-T_2)]}$ $R_2 = \dfrac{AT_3^2}{T_1(T_3-T_2)}$ $C_1 = \dfrac{4[T_3^2-T_1(T_3-T_2)}{AT_3}$ $C_2 = \dfrac{T_1T_2}{AT_3}$
		$A = 2R_1$ $T_1 = R_2C_1 + 2R_1C_2$ $T_2 = \dfrac{R_1(R_1+2R_2)C_1C_2}{R_2C_1+2R_1C_2}$ $T_3 = \left(R_2+\dfrac{R_1}{2}\right)C_1$	$R_1 = \dfrac{A}{2}$ $R_2 = \dfrac{AT_1(T_3-T_2)}{4[T_3^2-T_1(T_3-T_2)]}$ $C_1 = \dfrac{4[T_3^2-T_1(T_3-T_2)]}{AT_3}$ $C_2 = \dfrac{T_1T_2}{AT_3}$
		$A = 2R$ $T_1 = R(C_2 + 2C_3)$ $T_2 = \dfrac{RC_3(C_1+C_2)}{C_2+2C_3}$ $T_3 = \dfrac{R}{2}(C_1+C_2)$	$R = \dfrac{A}{2}$ $C_1 = \dfrac{2[2T_3^2-T_1(T_3-T_2)]}{AT_3}$ $C_2 = \dfrac{2T_1(T_3-T_2)}{AT_3}$ $C_3 = \dfrac{T_1T_2}{AT_3}$
$A\left[\dfrac{1+PT_3}{1+PT_1+P^2T_1T_2}\right]$ $T_2 > \dfrac{T_1}{4}$ (complex roots) $T_3 < T_1$		$A = R_2$ $T_1 = 2R_1C_1 + R_2C_2$ $T_2 = \dfrac{R_1R_2C_1(C_1+2C_2)}{2R_1C_1+R_2C_2}$ $T_3 = 2R_1C_1$	$R_1 = \dfrac{AT_3^2}{4[T_1T_2-T_3(T_1-T_3)]}$ $R_2 = A$ $C_1 = \dfrac{2[T_1T_2-T_3(T_1-T_3)]}{AT_3}$ $C_2 = \dfrac{(T_1-T_3)}{A}$
		$A = R_2$ $T_1 = \dfrac{C_1(2R_1C_2+R_2C_1)}{2C_1+C_2}$ $T_2 = \dfrac{R_1R_2C_1C_2}{2R_1C_2+R_2C_1}$ $T_3 = \dfrac{2R_1C_1C_2}{(2C_1+C_2)}$	$R_1 = \dfrac{AT_3^2}{4[T_1T_2-T_3(T_1-T_3)]}$ $R_2 = A$ $C_1 = \dfrac{2T_1T_2}{AT_3}$ $C_2 = \dfrac{4T_1T_2[T_1T_2-T_3(T_1-T_3)]}{AT_3^2(T_1-T_3)}$
		$A = R_3$ $T_1 = \dfrac{R_1(2R_2+R_3)C}{R_1+R_2}$ $T_2 = \dfrac{R_2R_3C}{(2R_2+R_3)}$ $T_3 = \dfrac{2R_1R_2C}{(R_1+R_2)}$	$R_1 = \dfrac{AT_3^2}{2[2T_1T_2-T_3(T_1-T_3)]}$ $R_2 = \dfrac{AT_3}{2(T_1-T_3)}$ $R_3 = A$ $C = \dfrac{2T_1T_2}{AT_3}$
$A(1+PT_1)(1+PT_2)$ $T_1 < T_2$		$A = 2R_1 + R_2$ $T_1 = \left(\dfrac{R_1R_2}{2R_1+R_2}\right)C$ $T_2 = R_1C$	$R_1 = A\left(\dfrac{T_2-T_1}{2T_2}\right)$ $R_2 = A\dfrac{T_1}{T_2}$ $C = \dfrac{2T_2^2}{A(T_2-T_1)}$

Table 3: Transfer Impedances for Operational Amplifiers* (Sec. 1.5) (*Continued*)

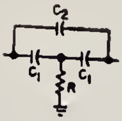
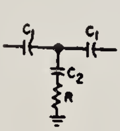
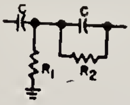
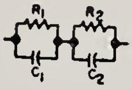
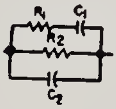
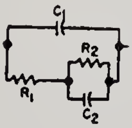
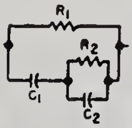
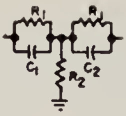
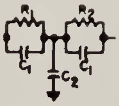
Transfer-impedance function	Network	Relations	Inverse relations
$\frac{1}{PB}\left(\frac{1 + P\theta T}{1 + PT}\right)$ $\theta < 1$		$B = C_2$ $T = RC_1\left(\frac{2C_2 + C_1}{C_2}\right)$ $\theta = \frac{2C_2}{2C_2 + C_1}$	$R = \frac{T\theta^2}{4B(1 - \theta)}$ $C_1 = \frac{2B(1 - \theta)}{\theta}$ $C_2 = B$
		$B = \frac{C_1^2}{2C_1 + C_2}$ $T = RC_2$ $\theta = \frac{2C_1}{2C_1 + C_2}$	$R = \frac{T\theta^2}{4B(1 - \theta)}$ $C_1 = \frac{2B}{\theta}$ $C_2 = \frac{4B(1 - \theta)}{\theta^2}$
		$B = \left(\frac{R_1}{R_1 + R_2}\right) C$ $T = R_2C$ $\theta = \frac{2R_1}{R_1 + R_2}$	$R_1 = \frac{T\theta^2}{2B(2 - \theta)}$ $R_2 = \frac{T\theta}{2B}$ $C = \frac{2B}{\theta}$
$A\left[\frac{1 + PT_2}{(1 + PT_1)(1 + PT_3)}\right]$ $T_1 < T_2 < T_3$		$A = R_1 + R_2$ $T_1 = R_1C_1$ $T_2 = \left(\frac{R_1R_2}{R_1+R_2}\right)(C_1+C_2)$ $T_3 = R_2C_2$	$R_1 = \frac{A(T_2 - T_1)}{T_3 - T_1}$ $R_2 = \frac{A(T_3 - T_2)}{T_3 - T_1}$ $C_1 = \frac{T_1(T_3 - T_1)}{A(T_2 - T_1)}$ $C_2 = \frac{T_3(T_3 - T_1)}{A(T_3 - T_2)}$
		$A = R_2$ $T_2 = R_1C_1$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = \frac{AT_2^2}{(T_3 - T_2)(T_2 - T_1)}$ $R_2 = A$ $C_1 = \frac{(T_3 - T_2)(T_2 - T_1)}{AT_2}$ $C_2 = \frac{T_1T_3}{AT_2}$
		$A = R_1 + R_2$ $T_2 = \left(\frac{R_1R_2}{R_1 + R_2}\right) C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = \frac{AT_2^2}{T_1T_2 + T_2T_3 - T_1T_3}$ $R_2 = \frac{A(T_3 - T_2)(T_2 - T_1)}{T_1T_2 + T_2T_3 - T_1T_3}$ $C_1 = \frac{T_1T_3}{AT_2}$ $C_2 = \frac{(T_1T_2 + T_2T_3 - T_1T_3)^2}{AT_2(T_3 - T_2)(T_2 - T_1)}$
		$A = R_1$ $T_2 = R_2(C_1 + C_2)$ $T_1T_3 = R_1R_2C_1C_2$ $T_1 + T_3 = R_1C_1 + R_2C_2 + R_2C_1$	$R_1 = A$ $R_2 = \frac{A(T_3 - T_2)(T_2 - T_1)}{(T_1 + T_3 - T_2)^2}$ $C_1 = \frac{T_1 + T_3 - T_2}{A}$ $C_2 = \frac{T_1T_3(T_1 + T_3 - T_2)}{A(T_3 - T_2)(T_2 - T_1)}$

Table 3: Transfer Impedances for Operational Amplifiers* (Sec. 1.5) (*Continued*)

Transfer-impedance function	Network	Relations	Inverse relations
$A\left[\frac{1+PT_2}{(1+PT_1)(1+PT_3)}\right]$ $T_2 \leq T_1 \leq T_3$		$A = 2R_1 + \frac{R_1^2}{R_2}$ $T_1 = R_1C_1$ $T_2 = \left(\frac{R_1R_2}{R_1+2R_2}\right)(C_1+C_2)$ $T_3 = R_1C_2$	$R_1 = \frac{AT_2}{(T_1+T_3)}$ $R_2 = \frac{AT_2^2}{(T_1+T_3)(T_1+T_3-2T_2)}$ $C_1 = \frac{T_1(T_1+T_3)}{AT_2}$ $C_2 = \frac{T_3(T_1+T_3)}{AT_2}$
$A\left[\frac{1+PT_2}{(1+PT_1)(1+PT_3)}\right]$ $T_1 \leq T_3 \leq T_2$		$A = R_1 + R_2$ $T_1 = R_1C_1$ $T_2 = \frac{R_1R_2}{R_1+R_2}(2C_1+C_2)$ $T_3 = R_2C_1$	$R_1 = \frac{AT_1}{(T_1+T_3)}$ $R_2 = \frac{AT_3}{(T_1+T_3)}$ $C_1 = \frac{(T_1+T_3)}{A}$ $C_2 = \frac{(T_1+T_3)}{A\left(\frac{T_2}{T_3}+\frac{T_2}{T_1}-2\right)}$

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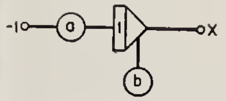
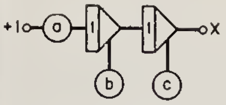
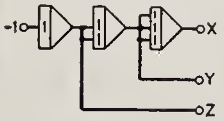
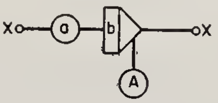
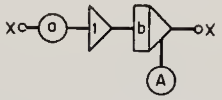
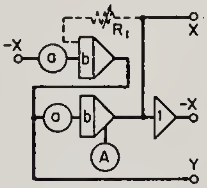
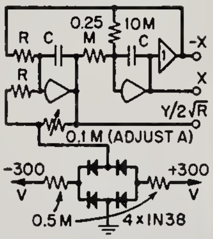
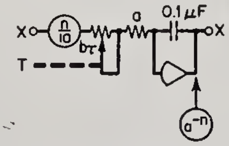
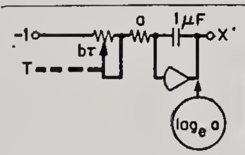
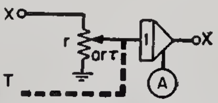
* Reproduced by permission of the McGraw-Hill Publishing Company from F. R. Bradley and R. McCoy, Driftless D-c Amplifier, *Electronics*, April, 1952. This table was developed by S. Godet of the Reeves Instrument Corporation, New York City.

Table 4: Examples of Functions of the Time τ Generated as Solutions of Simple Differential Equations (Sec. 6.6)

No.	Machine equation	Circuit or block diagram	Remarks
4.1	$X = a\tau + b$		Ramp function if source is switched
4.2	$X = \frac{1}{2}a\tau^2 - b\tau + c$		Generation of terms for power series
4.3	$X = \tau^3$ $Y = -\tau^2$ $Z = \tau$		
4.4	$X = Ae^{-ab\tau}$ $(ab > 0)$		
4.5	$X = Ae^{ab\tau}$ $(ab > 0)$		
4.6*a*	$X = A \cos \omega\tau$ $Y = A \sin \omega\tau$ $\left(\omega = ab = \frac{2}{C\sqrt{R}}\right)$		Sine-wave generator; generation of circles and ellipses on plotting board
4.6*b**		 R C 0.25 M 10M C -X R X Y/2√R 0.1 M (ADJUST A) -300 V +300 V 0.5 M 4×IN38	
4.7	$X = -\frac{n}{P(a + b\tau)}X$ $= (a + b\tau)^{-n}$ $(0 < n \le 10, a \ge 0, b = 1)$		*a* and $b\tau$ are in megohms; T = const.τ. See also Table 5
4.8	$X = \log_e (a + b\tau)$ $(a \ge 0, b = 1)$		
4.9	$X = -\frac{a}{P}\tau X = Ae^{-\frac{a}{2}\tau^2}$ $(A > 0, a > 0)$		Potentiometer-loading errors must be corrected as in Sec. 6.9

* Howe and Leite, Low-Frequency Oscillator, *Rev. Sci. Instr.*, **24**: 901, 1953.

Table 5: Generation of Functions with Linear Potentiometers (Sec. 6.9; see also Fig. 6.44)

r_L may be a load resistance. *Errors are computed for perfect resistors and potentiometers, negligible source impedances, and infinite-gain amplifiers.*

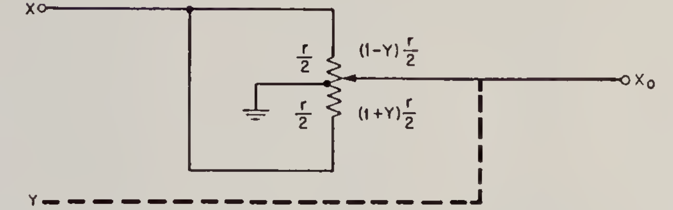
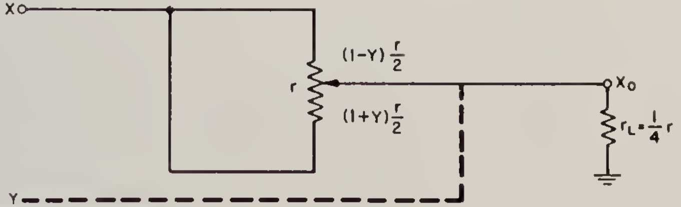
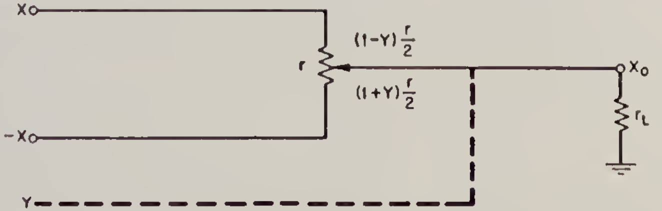
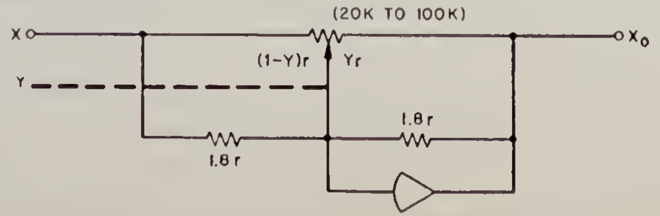
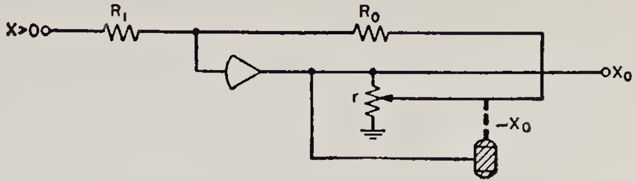
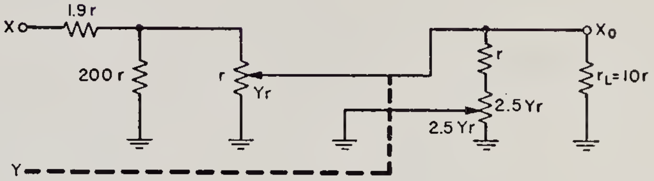
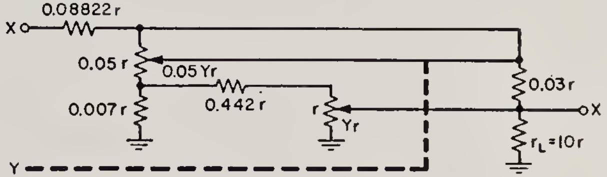
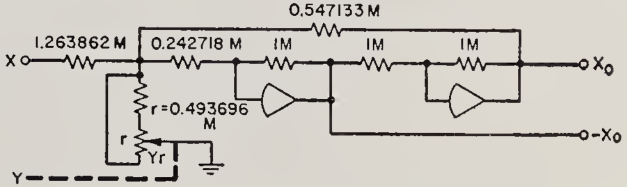
No.	Machine equation	Circuit or block diagram	Remarks
5.1	$X_o = X\|Y\|$	 X $\frac{r}{2}$ $\frac{r}{2}$ $(1-Y)\frac{r}{2}$ $(1+Y)\frac{r}{2}$ X_0 Y	Subject to loading error, which may be corrected by loading the servo follow-up potentiometer (Sec. 6.9)
5.2	$X_o = X\frac{1}{2 - Y^2} \approx \frac{X}{2} \sec Y$	 X r $(1-Y)\frac{r}{2}$ $(1+Y)\frac{r}{2}$ X_0 $r_L = \frac{1}{4}r$ Y	Error ≈ $\frac{1}{24}Y^4$
5.3*a*	$X_o = X\frac{2Y}{3 - Y^2} \approx \frac{2}{3} X \tan Y$ for $r_L = 0.5r$	 X r $(1-Y)\frac{r}{2}$ $(1+Y)\frac{r}{2}$ X_0 r_L $-X$ Y	Angular error within ±0.04° for $\|Y\| \leq \frac{1}{2}$*
5.3*b*	$X_o = X\frac{1.792Y}{2.792 - Y^2} \approx \frac{1.792}{2.792} X \tan Y$ for $r_L = 0.448r$		Angular error within ±0.1° for $\|Y\| \leq 1$*
5.4	$X_o = -X\frac{(2.8 - Y)Y}{(1.8 + Y)(1 - Y)}$ $\approx -X \tan \frac{\pi}{2} Y$	 (20K TO 100K) X Y $(1-Y)r$ Yr X_0 1.8r 1.8r	Angular error within 0.034° for $0 \leq Y < \frac{\pi}{2}$†

Table 5: Generation of Functions with Linear Potentiometers (*Continued*)

No.	Machine equation	Circuit or block diagram	Remarks
5.5	$X_o \approx -\sqrt{\frac{R_o}{R_1}}X \qquad X > 0$	 X>0 R₁ R_o r −X_o X_o	$r \ll R_o$. One obtains a similar circuit by applying $Y = X_o$ to the follow-up potentiometer in Fig. 6.8*a*, but erratic servo performance may result
5.6*a*	$X_o = 0.2X \sin \frac{\pi}{2} Y$	 X 1.9r 200r r Yr r 2.5Yr 2.5Yr X_o r_L=10r Y	Error less than 1% for $0 \leq Y \leq \frac{\pi}{2}$‡
5.6*b*		 X 0.08822r 0.05r 0.05Yr 0.007r 0.442r r Yr 0.03r X r_L=10r Y	Error less than 0.4% for $0 \leq Y \leq \frac{\pi}{2}$‡
5.7	$X_o = 0.242718\ X \sin \frac{\pi}{2} Y$	 X 1.263862 M 0.242718 M 0.547133 M 1M 1M 1M r=0.493696 M r Yr X_o −X_o Y	Error less than 0.004% for $0 \leq Y \leq \frac{\pi}{2}$‡

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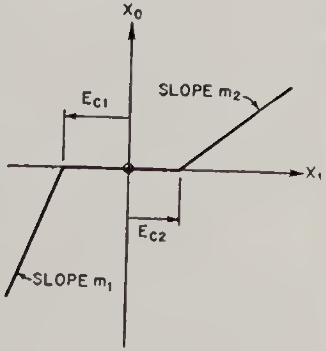
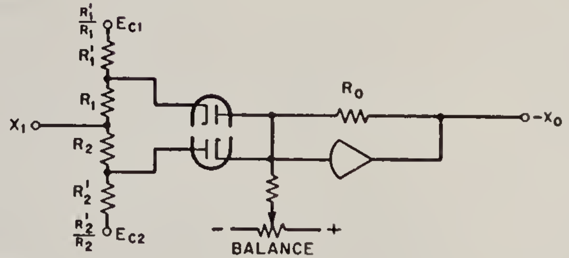
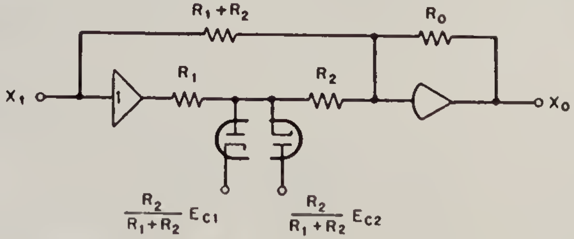
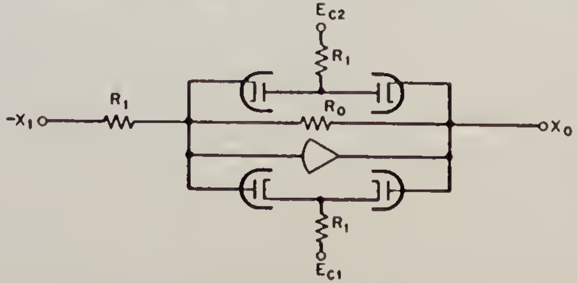
* Hofstadter, R., A Simple Potentiometer Circuit for Production of the Tangent Function, *Rev. Sci. Instr.*, **17**: 298, 1946; see also L. A. Nettleton and F. E. Dole, *op. cit.*; Greenwood *et al.*, *op. cit.*; Chance *et al.*, *op. cit.*

† Seay, P. A., An Accurate Tangent Computing Circuit, *REAC Newsletter*, August, 1954 (Reeves Instrument Corporation, New York City).

‡ *Project TYPHOON Summary Progress Report No. 4*, Vol. 2, RCA Research Laboratories, Princeton, N.J. (unclassified).

Table 6: Nonlinear Circuits for Special Applications (see also Secs. 3.3, 3.4, 3.9, and 6.7)

All formulas are derived for negligible source impedance and infinite amplifier gain.

No.	Application	Transfer characteristic, or function generated	Circuit or block diagram	Remarks
6.1*a*	Representation of dead space (inert zone) in springs, control characteristics, etc. (see also Figs. 6.23*c* and 6.24*f*). With large values of R_o, these circuits are used for the representation of a pair of stops in the manner of Fig. 3.4		 $\frac{R_1'}{R_1}$ E_{C1} R_1' R_1 X_1 R_2 R_2' $\frac{R_2'}{R_2}$ E_{C2} R_0 $-X_0$ - + BALANCE	$m_1 = \frac{R_o}{R_1}$ $m_2 = \frac{R_o}{R_2}$
6.1*b*			 R_1+R_2 R_0 X_1 1 R_1 R_2 X_0 $\frac{R_2}{R_1+R_2}E_{C1}$ $\frac{R_2}{R_1+R_2}E_{C2}$	$m_1 = m_2 = \frac{R_o}{R_1 + R_2}$ Use in older computers with permanently installed shunt limiters. Requires accurate resistance matching
6.1*c*			 E_{C2} R_1 $-X_1$ R_1 R_0 X_0 R_1 E_{C1}	$m_1 = m_2 = \frac{R_o}{R_1}(E_{C2} = -E_{C1})$ Easy to adjust

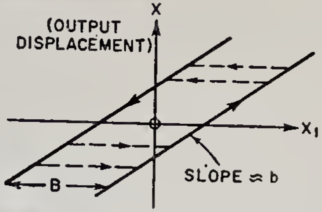
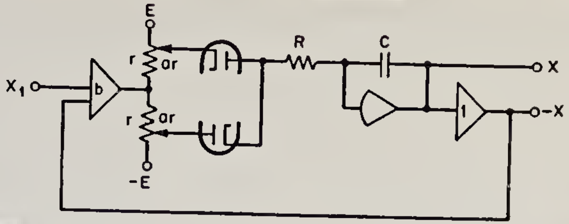
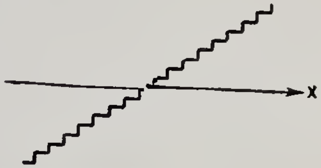
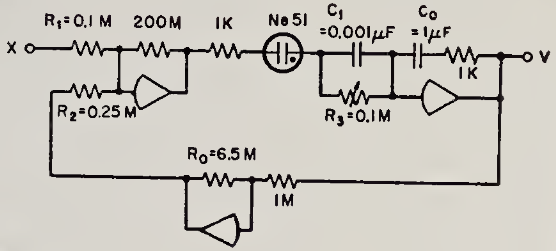
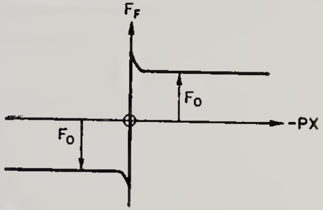
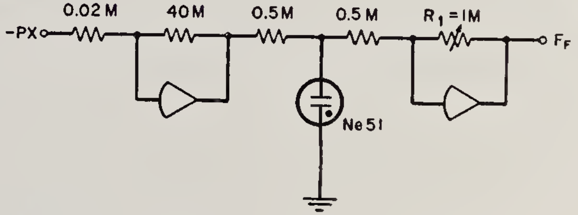
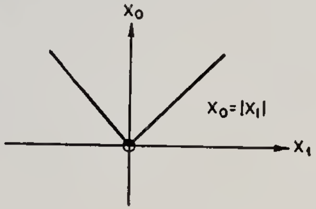
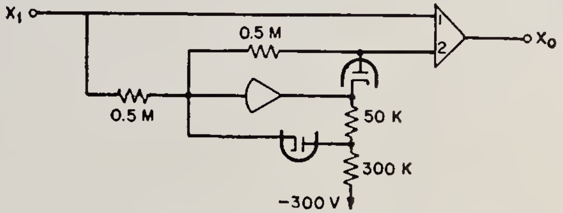
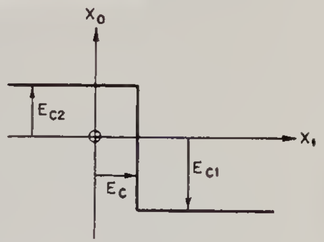
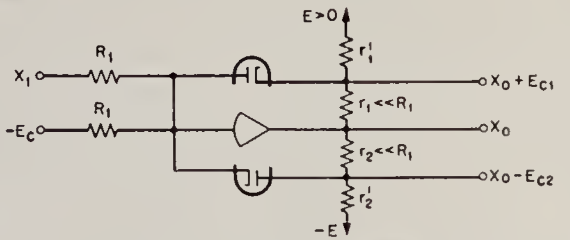
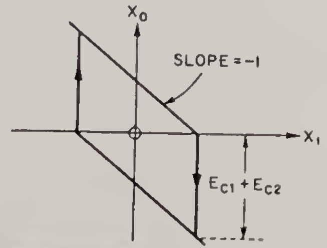
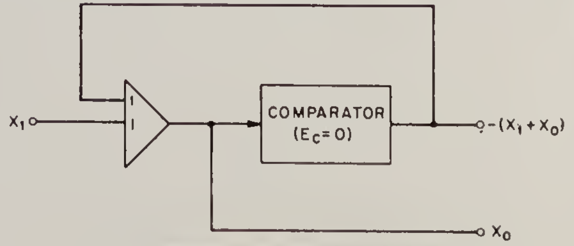
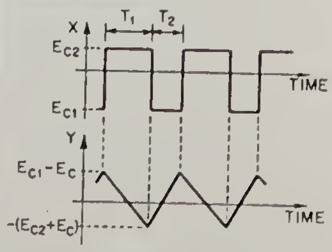
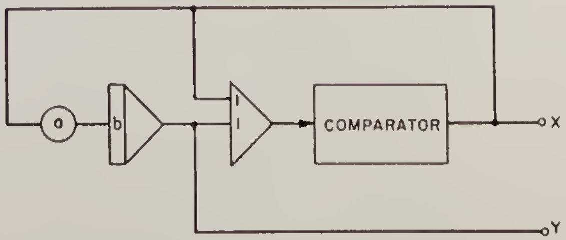
6.2	Approximate representation of **backlash** in gears, linkages, process-control error detectors, etc. (assuming inelastic stops; see also Sec. 3.3)		 E r ar R C X X₁ b r ar 1 -X -E	$B \approx \dfrac{2\,aE}{b(1-a)}$ $(ar + R)C \ll 1$ Typical values: $r = R = 50$ K $C = 1\ \mu$f $b = 1$ $E = 200$ v
6.3	Approximate representation of **granularity** (*e.g.*, potentiometer granularity)[1,2]		 R₁=0.1 M 200M 1K Ne 51 C₁ =0.001μF C₀ =1μF X 1K V R₂=0.25 M R₃=0.1M R₀=6.5 M 1M	Average slope $= \dfrac{R_2}{R_oR_1}$ Step height $\approx \dfrac{R_2}{20}$ R_3C_o controls the step rise time and thus the permissible stepping rate. For a more accurate representation, a pair of thyratrons or diodes may replace the neon bulb[2]
6.4	Approximate representation of **starting friction and coulomb friction** (see Fig. 3.3*f* for a more accurate representation)[1]		 0.02 M 40M 0.5M 0.5 M R₁ =1M -PX F_F Ne 51	$F_o \approx 130R_o$ volts
6.5	**Precision absolute-value device** (linear full-wave rectifier)[3]		 X₁ 0.5 M 1 2 X₀ 0.5 M 50 K 300 K -300 V	Obtain $-\|X_1\|$ by reversing diodes and bias voltage (see also Fig. 6.27)

Table 6: Nonlinear Circuits for Special Applications (*Continued*)

No.	Application	Transfer characteristic, or function generated	Circuit or block diagram	Remarks
6.6	Comparator circuit[3] (also used to represent coulomb friction)		 E>0 r'1 X1 R1 -EC R1 X0 + EC1 r1<<R1 X0 r2<<R1 X0 - EC2 r'2 -E	Useful in many switching applications Drives relays (see also Secs. 6.7 and 7.7) A small feedback capacitor may be required for stability
6.7	Bistable multivibrator[3]			
6.8	Free-running multivibrator[3]			$T_1 = \frac{1}{ab}\left(1 + \frac{E_{C1}}{E_{C2}}\right)$ $T_2 = \frac{1}{ab}\left(1 + \frac{E_{C2}}{E_{C1}}\right)$

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[1] GEDA Simulation Study of a Carrier-type Instrument Servomechanism, *Report GER*-5779, Goodyear Aircraft Corporation, Akron, Ohio, April 21, 1954.

[2] Meneley, C. A., and C. D. Morrill, Application of Electronic Differential Analyzers to Engineering Problems, *Proc. IRE*, **41**: 1487, 1953.

[3] Morrill, C. D., and R. V. Baum, Diode Limiters Simulate Mechanical Phenomena, *Electronics*, **25**: 122, November, 1952.

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